3.2

(a)

$$
\begin{aligned}
X(z) &= \sum_{n} x(n) z^{-n} \\
&= \sum_{n=0}^{\infty} (1+n) z^{-n} \\
&= \sum_{n=0}^{\infty} z^{-n} + \sum_{n=0}^{\infty} n z^{-n} \\
\text{But } \sum_{n=0}^{\infty} z^{-n} &= \frac{1}{1-z^{-1}} \text{ ROC: } |z| > 1 \\
\\
\text{and } \sum_{n=0}^{\infty} n z^{-n} &= \frac{z^{-1}}{(1-z^{-1})^2} \text{ ROC: } |z| > 1 \\
\text{Therefore, } X(z) &= \frac{1-z^{-1}}{(1-z^{-1})^2} + \frac{z^{-1}}{(1-z^{-1})^2} \\
&= \frac{1}{(1-z^{-1})^2}
\end{aligned}
$$

(b)

$$\begin{aligned}
X(z) &= \sum_{n=0}^{\infty}(a^n + a^{-n})z^{-n} \\
&= \sum_{n=0}^{\infty} a^n z^{-n} + \sum_{n=0}^{\infty} a^{-n} z^{-n} \\
\text{But } \sum_{n=0}^{\infty} a^n z^{-n} &= \frac{1}{1-az^{-1}} \text{ ROC: } |z| > |a| \\
\text{and } \sum_{n=0}^{\infty} a^{-n} z^{-n} &= \frac{1}{(1-\frac{1}{a}z^{-1})^2} \text{ ROC: } |z| > \frac{1}{|a|} \\
\text{Hence, } X(z) &= \frac{1}{1-az^{-1}} + \frac{1}{1-\frac{1}{a}z^{-1}} \\
&= \frac{2-(a+\frac{1}{a})z^{-1}}{(1-az^{-1})(1-\frac{1}{a}z^{-1})} \text{ ROC: } |z| > \text{ max } (|a|, \frac{1}{|a|})
\end{aligned}$$

(c)

$$\begin{aligned}
X(z) &= \sum_{n=0}^{\infty}(-\frac{1}{2})^n z^{-n} \\
&= \frac{1}{1+\frac{1}{2}z^{-1}}, |z| > \frac{1}{2}
\end{aligned}$$

(d)

$$\begin{aligned}
X(z) &= \sum_{n=0}^{\infty} na^n sinw_0nz^{-n} \\
&= \sum_{n=0}^{\infty} na^n \left[\frac{e^{jw_0n} - e^{-jw_0n}}{2j}\right] z^{-n} \\
&= \frac{1}{2j}\left[\frac{ae^{jw_0}z^{-1}}{(1-ae^{jw_0}z^{-1})^2} - \frac{ae^{-jw_0}z^{-1}}{(1-ae^{-jw_0}z^{-1})^2}\right] \\
&= \frac{\left[az^{-1} - (az^{-1})^3\right] sinw_0}{(1-2acosw_0z^{-1} + a^2z^{-2})^2}, |z| > a
\end{aligned}$$

(e)

$$\begin{aligned}
X(z) &= \sum_{n=0}^{\infty} na^n cosw_0nz^{-n} \\
&= \sum_{n=0}^{\infty} na^n \left[\frac{e^{jw_0n} + e^{-jw_0n}}{2}\right] z^{-n}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\left[\frac{ae^{jw_0}z^{-1}}{(1-ae^{jw_0}z^{-1})^2}+\frac{ae^{-jw_0}z^{-1}}{(1-ae^{-jw_0}z^{-1})^2}\right] \\
&= \frac{\left[az^{-1}+(az^{-1})^3\right]sinw_0-2a^2z^{-2}}{(1-2acosw_0z^{-1}+a^2z^{-2})^2}, \quad |z|>a
\end{aligned}
$$

(f)

$$
\begin{aligned}
X(z) &= A\sum_{n=0}^{\infty} r^n cos(w_0n+\phi)z^{-n} \\
&= A\sum_{n=0}^{\infty} r^n\left[\frac{e^{jw_0n}e^{j\phi}+e^{-jw_0n}e^{-j\phi}}{2}\right]z^{-n} \\
&= \frac{A}{2}\left[\frac{e^{j\phi}}{1-re^{jw_0}z^{-1}}+\frac{e^{-j\phi}}{1-re^{-jw_0}z^{-1}}\right] \\
&= A\left[\frac{cos\phi-rcos(w_0-\phi)z^{-1}}{1-2rcosw_0z^{-1}+r^2z^{-2}}\right], \quad |z|>r
\end{aligned}
$$

(g)

$$
\begin{aligned}
X(z) &= \sum_{n=1}^{\infty}\frac{1}{2}(n^2+n)(\frac{1}{3})^{n-1}z^{-n} \\
\text{But} \sum_{n=1}^{\infty} n(\frac{1}{3})^{n-1}z^{-1} &= \frac{(\frac{1}{3})3z^{-1}}{(1-\frac{1}{3}z^{-1})^2}=\frac{z^{-1}}{(1-\frac{1}{3}z^{-1})^2} \\
\sum_{n=1}^{\infty} n^2(\frac{1}{3})^{n-1}z^{-n} &= \frac{z^{-1}+\frac{1}{3}z^{-2}}{(1-\frac{1}{3}z^{-1})^3} \\
\text{Therefore, } X(z) &= \frac{1}{2}\left[\frac{z^{-1}}{(1-\frac{1}{3}z^{-1})^2}+\frac{z^{-1}+\frac{1}{3}z^{-2}}{(1-\frac{1}{3}z^{-1})^3}\right] \\
&= \frac{z^{-1}}{(1-\frac{1}{3}z^{-1})^3}, \quad |z|>\frac{1}{3}
\end{aligned}
$$

(h)

$$
\begin{aligned}
X(z) &= \sum_{n=0}^{\infty}(\frac{1}{2})^n z^{-n}-\sum_{n=10}^{\infty}(\frac{1}{2})^n z^{-n} \\
&= \frac{1}{1-\frac{1}{2}z^{-1}}-\frac{(\frac{1}{2})^{10}z^{-10}}{1-\frac{1}{2}z^{-1}} \\
&= \frac{1-(\frac{1}{2}z^{-1})^{10}}{1-\frac{1}{2}z^{-1}}, \quad |z|>\frac{1}{2}
\end{aligned}
$$

The pole-zero patterns are as follows:
(a) Double pole at $z=1$ and a zero at $z=0$.
(b) Poles at $z=a$ and $z=\frac{1}{a}$. Zeros at $z=0$ and $z=\frac{1}{2}(a+\frac{1}{a})$.
(c) Pole at $z=-\frac{1}{2}$ and zero at $z=0$.
(d) Double poles at $z=ae^{jw_0}$ and $z=ae^{-jw_0}$ and zeros at $z=0$, $z=\pm a$.
(e) Double poles at $z=ae^{jw_0}$ and $z=ae^{-jw_0}$ and zeros are obtained by solving the quadratic

$$acosw_0z^2-2a^2z+a^3cosw_0=0.$$

(f) Poles at $z=re^{jw_0}$ and $z=ae^{-jw_0}$ and zeros at $z=0$, and $z=rcos(w_0-\phi)/cos\phi$.
(g) Triple pole at $z=\frac{1}{3}$ and zeros at $z=0$ and $z=\frac{1}{3}$. Hence there is a pole-zero cancellation so

that in reality there is only a double pole at $z = \frac{1}{3}$ and a zero at $z = 0$.
(h) $X(z)$ has a pole of order 9 at $z = 0$. For nine zeros which we find from the roots of

$$\begin{aligned} 1 - (\frac{1}{2}z^{-1})^{10} &= 0 \\ \text{or, equivalently, } (\frac{1}{2})^{10} - z^{10} &= 0 \\ \text{Hence, } z_n &= \frac{1}{2}e^{\frac{j2\pi n}{10}}, n = 1, 2, \ldots, k. \end{aligned}$$

Note the pole-zero cancellation at $z = \frac{1}{2}$.

3.5

$$x_1(n) = \begin{cases} (\frac{1}{3})^n, & n \geq 0 \\ (\frac{1}{2})^{-n}, & n < 0 \end{cases}$$

$$\begin{aligned}
X_1(z) &= \sum_{n=0}^{\infty} (\frac{1}{3})^n z^{-n} + \sum_{n=-\infty}^{-1} (\frac{1}{2})^{-n} z^{-n} \\
&= \frac{1}{1-\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{2}z} - 1 \\
&= \frac{\frac{5}{6}}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{2}z)} \\
X_2(z) &= \sum_{n=0}^{\infty} (\frac{1}{2})^n z^{-n} \\
&= \frac{1}{1-\frac{1}{2}z^{-1}}, \frac{1}{2} < |z| < 2 \\
\text{Then,} Y(z) &= \frac{-2}{1-\frac{1}{3}z^{-1}} + \frac{\frac{10}{3}}{1-\frac{1}{2}z^{-1}} + \frac{\frac{-4}{3}}{1-2z^{-1}} \\
\text{Hence,} y(n) &= \begin{cases} -2(\frac{1}{3})^n + \frac{10}{3}(\frac{1}{2})^n, & n \geq 0 \\ \frac{4}{3}(2)^n, & n < 0 \end{cases}
\end{aligned}$$

3.9

(a)

$$x_1(n) = \begin{cases} x(\frac{n}{2}), & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

$$\begin{aligned} X_1(z) &= \sum_{n=-\infty}^{\infty} x_1(n) z^{-n} \\ &= \sum_{n=-\infty}^{\infty} x(\frac{n}{2}) z^{-n} \\ &= \sum_{k=-\infty}^{\infty} x(k) z^{-2k} \\ &= X(z^2) \end{aligned}$$

(b)

$$\begin{aligned} x_2(n) &= x(2n) \\ X_2(z) &= \sum_{n=-\infty}^{\infty} x_2(n) z^{-n} \\ &= \sum_{n=-\infty}^{\infty} x(2n) z^{-n} \\ &= \sum_{k=-\infty}^{\infty} x(k) z^{-\frac{k}{2}} \\ &= \sum_{k=-\infty}^{\infty} \left[\frac{x(k) + (-1)^k x(k)}{2} \right] z^{-\frac{k}{2}}, k \text{ even} \\ &= \frac{1}{2} \sum_{k=-\infty}^{\infty} x(k) z^{-\frac{k}{2}} + \frac{1}{2} \sum_{k=-\infty}^{\infty} x(k) (-z^{\frac{1}{2}})^{-k} \\ &= \frac{1}{2} \left[X(\sqrt{z} + X(-\sqrt{z}) \right] \end{aligned}$$

3.11

(a)

$$
\begin{aligned}
x_1(n) &= \frac{1}{4}(\frac{1}{4})^{n-1}u(n-1) \\
\Rightarrow X_1(z) &= \frac{(\frac{1}{4})z^{-1}}{1-\frac{1}{4}z^{-1}}, |z| > \frac{1}{4} \\
x_2(n) &= \left[1+(\frac{1}{2})^n\right]u(n) \\
\Rightarrow X_2(z) &= \frac{1}{1-z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}}, |z| > 1 \\
Y(z) &= X_1(z)X_2(z) \\
&= \frac{-\frac{4}{3}}{1-\frac{1}{4}z^{-1}} + \frac{\frac{1}{3}}{1-z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}} \\
y(n) &= \left[-\frac{4}{3}(\frac{1}{4})^n + \frac{1}{3} + (\frac{1}{2})^n\right]u(n)
\end{aligned}
$$

(b)

$$
\begin{aligned}
x_1(n) &= u(n) \\
\Rightarrow X_1(z) &= \frac{1}{1-z^{-1}}, \\
x_2(n) &= \delta(n) + (\frac{1}{2})^n u(n) \\
\Rightarrow X_2(z) &= 1 + \frac{1}{1-\frac{1}{2}z^{-1}} \\
Y(z) &= X_1(z)X_2(z) \\
&= \frac{3}{1-z^{-1}} - \frac{1}{1-\frac{1}{2}z^{-1}} \\
y(n) &= \left[3-(\frac{1}{2})^n\right]u(n)
\end{aligned}
$$

(c)

$$\begin{aligned}
x_1(n) &= (\frac{1}{2})^n u(n) \\
\Rightarrow X_1(z) &= \frac{1}{1-\frac{1}{2}z^{-1}}, \\
x_2(n) &= cos\pi n u(n) \\
\Rightarrow X_2(z) &= \frac{1+z^{-1}}{1+2z^{-1}+z^{-2}} \\
Y(z) &= X_1(z)X_2(z) \\
&= \frac{1+z^{-1}}{(1-\frac{1}{2}z^{-1})(1+2z^{-1}+z^{-2})}
\end{aligned}$$

$$\begin{aligned}
&= \frac{A(1+z^{-1})}{1+2z^{-1}+z^{-2}} + \frac{B}{1-\frac{1}{2}z^{-1}} \\
A &= \frac{2}{3}, B = \frac{1}{3} \\
y(n) &= \left[\frac{2}{3}cos\pi n + \frac{1}{3}(\frac{1}{2})^n\right] u(n)
\end{aligned}$$

(d)

$$\begin{aligned}
x_1(n) &= nu(n) \\
\Rightarrow X_1(z) &= \frac{z^{-1}}{(1-z^{-1})^2}, \\
x_2(n) &= 2^n u(n-1) \\
\Rightarrow X_2(z) &= \frac{2z^{-1}}{1-2z^{-1}} \\
Y(z) &= X_1(z)X_2(z) \\
&= \frac{2z^{-2}}{(1-z^{-1})^2(1-2z^{-1})} \\
&= \frac{-2}{1-z^{-1}} - \frac{-2z^{-1}}{(1-z^{-1})^2} + \frac{2}{1-2z^{-1}} \\
y(n) &= \left[-2(n+1) + 2^{n+1}\right] u(n)
\end{aligned}$$

3.14

(a)

$$\begin{aligned} x_1(n) &= r^n sinw_0 nu(n), \qquad 0 < r < 1 \\ X_1(z) &= \frac{rsinw_0 z^{-1}}{1 - 2rcosw_0 z^{-1} + r^2 z^{-2}} \end{aligned}$$

Zero at $z = 0$ and poles at $z = re^{\pm jw_0} = r(cosw_0 \pm jsinw_0)$.

(b)

$$\begin{aligned} X_2(z) &= \frac{z}{(1 - re^{jw_0} z^{-1})(1 - re^{-jw_0} z^{-1})} \\ &= \frac{z}{1 - 2rcosw_0 z^{-1} + r^2 z^{-2}} \end{aligned}$$

(c) $X_1(z)$ and $X_2(z)$ differ by a constant, which can be determined by giving the value of $X_1(z)$ at $z = 1$.

3.18

$$\begin{aligned} X(z) &= \frac{3}{1-\frac{10}{3}z^{-1}+z^{-2}} \\ &= \frac{-\frac{3}{8}}{1-\frac{1}{3}z^{-1}} + \frac{\frac{27}{8}}{1-3z^{-1}} \\ \text{ROC: } \frac{1}{3} < |z| < 3, x(n) &= \frac{3}{8}(\frac{1}{3})^n u(n) - \frac{27}{8}3^n u(-n-1) \end{aligned}$$

3.22

(a)

$$
\begin{aligned}
Y(z)\left[1-0.2z^{-1}\right] &= X(z)\left[1-0.3z^{-1}-0.02z^{-2}\right] \\
\frac{Y(z)}{X(z)} &= \frac{(1-0.1z^{-1})(1-0.2z^{-1})}{1-0.2z^{-1}} \\
&= 1-0.1z^{-1}
\end{aligned}
$$

(b)

$$
\begin{aligned}
Y(z) &= X(z)\left[1-0.1z^{-1}\right] \\
\frac{Y(z)}{X(z)} &= 1-0.1z^{-1}
\end{aligned}
$$

Therefore, (a) and (b) are equivalent systems.

3.25

$$
\begin{aligned}
y(n) &= 0.7y(n-1) - 0.12y(n-2) + x(n-1) + x(n-2) \\
Y(z) &= \frac{z^{-1} + z^{-2}}{1 - 0.7z^{-1} + 0.12z^{-2}} X(z) \\
x(n) &= nu(n) \\
X(z) &= \frac{z^{-1}}{(1-z^{-1})^2} \\
Y(z) &= \frac{z^{-2} + z^{-3}}{(1-z^{-1})^2(1 - \frac{3}{10}z^{-1})(1 - \frac{2}{50}z^{-2})}
\end{aligned}
$$

$\Rightarrow$ System is stable

$$
Y(z) = \frac{4.76z^{-1}}{(1-z^{-1})^2} + \frac{-12.36}{(1-z^{-1})} + \frac{-26.5}{(1-\frac{3}{10}z^{-1})} + \frac{38.9}{(1-\frac{2}{5}z^{-1})}
$$

$$
y(n) = \left[4.76n - 12.36 - 26.5(\frac{3}{10})^n + 38.9(\frac{2}{5})^n\right] u(n)
$$

3.28

$$
\begin{aligned}
H(z) &= \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2}} \\
\text{If } a_1^2 - 4a_2 &< 0, \text{ there are two complex poles} \\
p_{1,2} &= \frac{-a_1 \pm j\sqrt{4a_2 - a_1^2}}{2}
\end{aligned}
$$

$$
\begin{aligned}
|p_{1,2}|^2 &= (\frac{a_1}{2})^2 + \left(\frac{\sqrt{4a_2 - a_1^2}}{2}\right)^2 < 1 \\
\Rightarrow a_2 &< 1 \\
\text{If } a_1^2 - a4a_2 &\geq 0, \text{ there are two real poles} \\
p_{1,2} &= \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} \\
\frac{-a_1 + \sqrt{a_1^2 - 4a_2}}{2} &< 1 \text{ and} \\
\frac{-a_1 - \sqrt{a_1^2 - 4a_2}}{2} &> -1 \\
\Rightarrow a_1 - a_2 &< 1 \text{ and} \\
a_1 + a_2 &> 1
\end{aligned}
$$

Refer to fig 3.41-1.

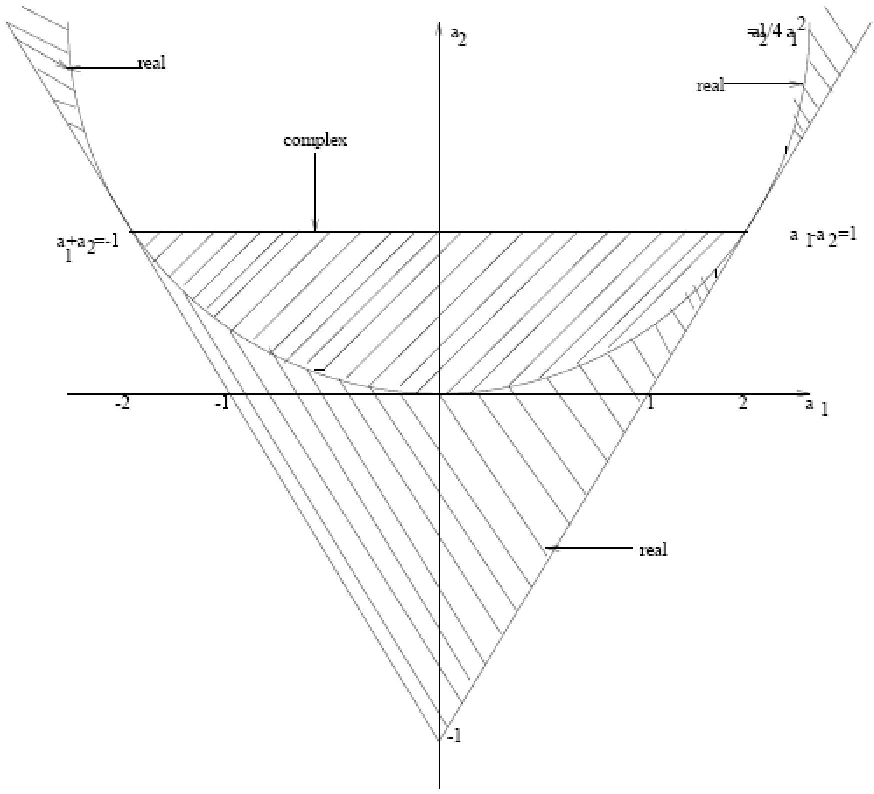

Figure 3.41-1:

3.31

(a)

$$
\begin{aligned}
H(z) &= C\frac{(z-re^{j\Theta})(z-re^{-j\Theta})}{z(z+0.8)} \\
&= C\frac{1-2rcos\Theta z^{-1}+r^2z^{-2}}{(1+0.8z^{1})} \\
H(z)|_{z=1}=1 \Rightarrow C &= \frac{1.8}{1-2rcos\Theta+r^2}=2.77
\end{aligned}
$$

(b) The poles are inside the unit circle, so the system is stable.
(c) $y(n)=-0.8y(n-1)+Cx(n)-1.5\sqrt{3}Cx(n-1)+2.25Cx(n-2)$. Refer to fig 3.46-1.

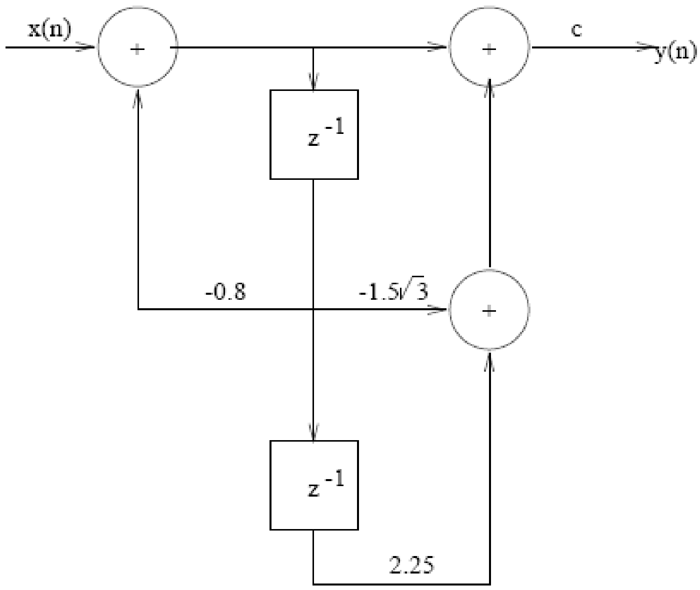

Figure 3.46-1:

3.35

$x(n)$ is causal.
(a)

$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} x(n) z^{-n} \\ lim_{z\to\infty} X(z) &= x(0) \end{aligned}$$

(b)(i) $X(z) = \frac{(z-\frac{1}{2})^4}{(z-\frac{1}{3})^3} \Rightarrow lim_{z\to\infty} X(z) = \infty \Rightarrow x(n)$ is not causal.

(ii) $X(z) = \frac{(1-\frac{1}{2}z^{-2})^2}{1-\frac{1}{3}z^{-1}} \Rightarrow lim_{z\to\infty} X(z) = 1$ Hence $X(z)$ can be associated wih a causal sequence.

(iii) $X(z) = \frac{(z-\frac{1}{3})^2}{(z-\frac{1}{2})^3} \Rightarrow lim_{z\to\infty} X(z) = 0$. Hence $X(z)$ can be associated wih a causal sequence.

3.37

$$s(n) = (\frac{1}{3})^{n-2}u(n+2)$$

(a)

$$\begin{aligned} h(n) &= s(n) - s(n-1) \\ &= (\frac{1}{3})^{n-2}u(n+2) - (\frac{1}{3})^{n-3}u(n+1) \\ &= 3^4\delta(n+2) - 54\delta(n+1) - 18(\frac{1}{3})^n u(n) \\ H(z) &= 81z^2 - 54z + \frac{-18}{1-\frac{1}{3}z^{-1}} \\ &= \frac{81z(z^{-1})}{1-\frac{1}{3}z^{-1}} \end{aligned}$$

$H(z)$ has zeros at $z = 0, 1$ and a pole at $z = \frac{1}{3}$.
(b) $h(n) = 81\delta(n+2) - 54\delta(n+1) - 18(\frac{1}{3})^n u(n)$
(c) The system is not causal, but it is stable since the pole is inside the unit circle

3.39

$$\begin{aligned}
X(z) &= \frac{z^{20}}{(z-\frac{1}{2})(z-2)^5(z+\frac{5}{2})^2(z+3)}, \frac{1}{2} < |z| < 2 \\
x(n) &= \frac{1}{2\pi j}\oint_c \frac{z^{n-1}z^{20}}{(z-\frac{1}{2})(z-2)^5(z+\frac{5}{2})^2(z+3)}dz \\
x(-18) &= \frac{1}{2\pi j}\oint_c \frac{z}{(z-\frac{1}{2})(z-2)^5(z+\frac{5}{2})^2(z+3)}dz \\
&= \frac{\frac{1}{2}}{(\frac{1}{2}-2)^5(\frac{1}{2}+\frac{5}{2})^2(\frac{1}{2}+3)} \\
&= \frac{-\frac{1}{2}}{(\frac{3}{2})^5(3)^2(\frac{7}{2})} \\
&= \frac{-2^5}{(3^7)(7)} \\
&= \frac{-32}{15309}
\end{aligned}$$