Chapter 7

The Discrete Fourier Transform: Its Properties and Applications

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Introduction

- Frequency analysis of discrete-time signals is usually and most conveniently performed on a digital signal processor.
- To perform frequency analysis on a discrete-time signal, we convert the time-domain sequence to an equivalent frequency-domain representation.

$$\{x(n)\} \xrightarrow{F} X(\omega)$$

Frequency-Domain Sampling: The Discrete Fourier Transform

- Before we introduce the DFT, we consider the sampling of the Fourier transform of an aperiodic discrete-time sequence.
- Thus, we establish the relationship between the sampled Fourier transform and the DFT.

Frequency-Domain Sampling and Reconstruction of Discrete-Time Signals

- We recall that aperiodic finite-energy signals have continuous spectra.
- Let us consider such an aperiodic discrete-time signal $x(n)$ with Fourier transform

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

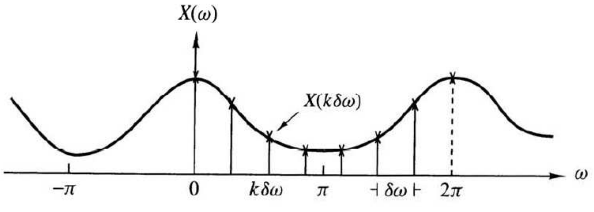
Frequency-Domain Sampling and Reconstruction of Discrete-Time Signals

- Suppose that we sample $X(\omega)$ periodically in frequency at a spacing of $\delta\omega$ radius between successive samples.

- Since $X(\omega)$ is periodic with period 2π, only samples in the fundamental frequency range are necessary.

Frequency-Domain Sampling and Reconstruction of Discrete-Time Signals

- For convenience, we take *N* equidistant samples in the interval $0 \le \omega < 2\pi$ with spacing $\delta\omega = 2\pi / N$, as shown below:
  - Frequency-domain sampling of the Fourier transform



Frequency-Domain Sampling and Reconstruction of Discrete-Time Signals

- First, we consider the selection of *N*, the number of samples in the frequency domain.
- If we evaluate $X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$ at $\omega = 2\pi k / N$
- We obtain

$$X(\frac{2\pi}{N}k) = \sum_{n=-\infty}^{\infty} x(n)e^{-j2\pi kn/N}, \; k = 0,1,\ldots,N-1$$

Frequency-Domain Sampling and Reconstruction of Discrete-Time Signals

- The summation above equation can be subdivided into an infinite number of summations, where each sum contains *N* terms. Thus

$$X(\frac{2\pi}{N}k) = \ldots + \sum_{n=-N}^{-1} x(n)e^{-j2\pi kn/N} + \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}$$

$$+ \sum_{n=N}^{2N-1} x(n)e^{-j2\pi kn/N} + \ldots$$

$$= \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x(n)e^{-j2\pi kn/N}$$

Frequency-Domain Sampling and Reconstruction of Discrete-Time Signals

- If we change the index in the inner summation from *n* to *n - lN* and interchange the order of the summation, we obtain the result

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1}\left[\sum_{l=-\infty}^{\infty} x(n-lN)\right]e^{-j2\pi kn/N} \text{ for } k = 0,1,2,\ldots,N-1.$$

Frequency-Domain Sampling and Reconstruction of Discrete-Time Signals

- The signal $x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN)$ obtained by the peoiodic repetition of $x(n)$ every *N* samples, is clearly periodic with fundamental period *N*.
- It can be expanded in a Fourier series as

$$x_p(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N},\ n = 0,1,\ldots,N-1$$

Frequency-Domain Sampling and Reconstruction of Discrete-Time Signals

- With Fourier coefficients

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x_p(n)e^{-j2\pi kn/N},\ k = 0,1,\dots,N-1$$

- We can conclude that

$$c_k = \frac{1}{N}X(\frac{2\pi}{N}k),\ k = 0,1,\dots,N-1$$

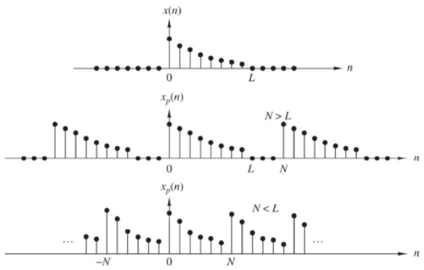
$$x_p(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(\frac{2\pi}{N}k)e^{j2\pi kn/N},\ n = 0,1,\dots,N-1$$

Frequency-Domain Sampling and Reconstruction of Discrete-Time Signals

- The relationship in above equation provides the reconstruction of the periodic signal $x_p(n)$ from the samples of the spectrum $X(\omega)$.
- However, it does not imply that we can recover $X(\omega)$ or $x(n)$ from the samples.
- To accomplish this, we need to consider the relationship between $x_p(n)$ and $x(n)$.

Frequency-Domain Sampling and Reconstruction of Discrete-Time Signals

- Since $x_p(n)$ is the periodic extension of $x(n)$. It is clear that $x(n)$ can be recovered from $x_p(n)$ if there is no **aliasing** in the time domain, that is, if $x(n)$ is time-limited to less than the period *N* of $x_p(n)$.



Original signal

No aliasing

Aliasing

Frequency-Domain Sampling and Reconstruction of Discrete-Time Signals

- We conclude that the spectrum of an aperiodic discrete-time signal with finite duration *L* can be exactly recovered from its samples at frequencies

$$\omega_k = 2\pi k / N, \text{ if } N \geq L$$

- The procedure is to compute $x_p(n),\ n = 0, 1, \ldots, N-1$

then $$x(n) = \begin{cases} x_p(n), & 0 \leq n \leq N-1 \\ 0, & \text{elsewhere} \end{cases}$$

and finally, $X(\omega)$ can be computed.

Frequency-Domain Sampling and Reconstruction of Discrete-Time Signals

- As in the case of continuous-time signals, it is possible to express the spectrum $X(\omega)$ directly in terms of its samples $X(2\pi k/N),\ k=0,1,\ldots,N-1$. To derive such an interpolation formula for $X(\omega)$, we assume that $N \ge L$.

- Since $x(n) = x_p(n)$ for $0 \le n \le N-1$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(\frac{2\pi}{N}k)e^{j2\pi kn/N},\ 0 \le n \le N-1$$

Frequency-Domain Sampling and Reconstruction of Discrete-Time Signals

- If we use $X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$ and substitute for $x(n)$, we obtain

$$X(\omega) = \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{k=0}^{N-1} X(\frac{2\pi}{N}k)e^{j2\pi kn/N}\right]e^{-j\omega n}$$

$$= \sum_{k=0}^{N-1} X(\frac{2\pi}{N}k)\left[\frac{1}{N}\sum_{n=0}^{N-1} e^{-j(\omega-2\pi k/N)n}\right]$$

Frequency-Domain Sampling and Reconstruction of Discrete-Time Signals

- The inner summation term in the brackets of above represents the basic **interpolation function** shifted by $2\pi k / N$ in frequency. Indeed, if we define

$$P(\omega) = \frac{1}{N}\sum_{n=0}^{N-1} e^{-j\omega n} = \frac{1}{N}\frac{1-e^{-j\omega N}}{1-e^{-j\omega}}$$

$$= \frac{\sin(\omega N / 2)}{N\sin(\omega / 2)} e^{-j\omega(N-1)/2}$$

Frequency-Domain Sampling and Reconstruction of Discrete-Time Signals

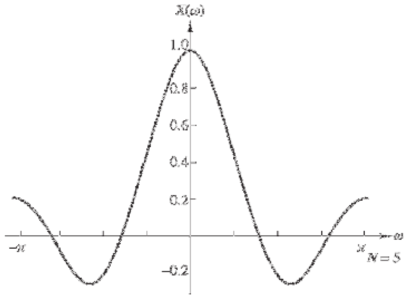
- Then, we can obtain

$$X(\omega) = \sum_{k=0}^{N-1} X(\frac{2\pi}{N}k)P(\omega - \frac{2\pi}{N}k),\ N \geq L$$

- The interpolation function $P(\omega)$ is not the familiar $(\sin\theta)/\theta$ but instead, it is a periodic counterpart of it, and it is due to the periodic nature of $X(\omega)$.

Frequency-Domain Sampling and Reconstruction of Discrete-Time Signals

- The phase shift reflects the fact that the signal $x(n)$ is a causal, finite-duration sequence of length *N*. The function $(\sin \omega N / 2) / (N \sin(\omega / 2))$ is plotted for *N* = 5.



Frequency-Domain Sampling and Reconstruction of Discrete-Time Signals

- We observe that the function $P(\omega)$ has the property

$$P(\frac{2\pi}{N}k) = \begin{cases} 1, & k = 0 \\ 0, & k = 1, 2, \ldots, N-1 \end{cases}$$

- Consequently, the interpolation formula gives exactly the sample values $X(2\pi k / N)$ for $\omega = 2\pi k / N$. At all other frequencies, the formula provides a properly weighted linear combination of the original spectral samples.

Frequency-Domain Sampling and Reconstruction of Discrete-Time Signals

- **Example**: Consider the signal $x(n) = a^n u(n),\ 0 < a < 1$ the spectrum of this signal is sampled at frequencies

$$\omega_k = 2\pi k / N \text{ for } k = 0,1,\ldots, N-1.$$

- Determine the reconstructed spectra for $a = 0.8$ when *N*=5 and *N*=50.
- **Solution**: The Fourier transform of the sequence $x(n)$ is

$$X(\omega) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \frac{1}{1 - ae^{-j\omega}}$$

Frequency-Domain Sampling and Reconstruction of Discrete-Time Signals

- Suppose that we sample $X(\omega)$ at *N* equidistant frequencies $\omega_k = 2\pi k / N,\ k = 0,1,\ldots, N-1.$ Thus we obtain the spectral samples

$$X(\omega k) \equiv X(\frac{2\pi k}{N}) = \frac{1}{1 - ae^{-j2\pi k/N}},\ k = 0,1,\ldots, N-1$$

- The periodic sequence $x_p(n)$, corresponding to the frequency samples $X(2\pi k / N),\ k = 0,1,\ldots, N-1$ can be obtained.

Frequency-Domain Sampling and Reconstruction of Discrete-Time Signals

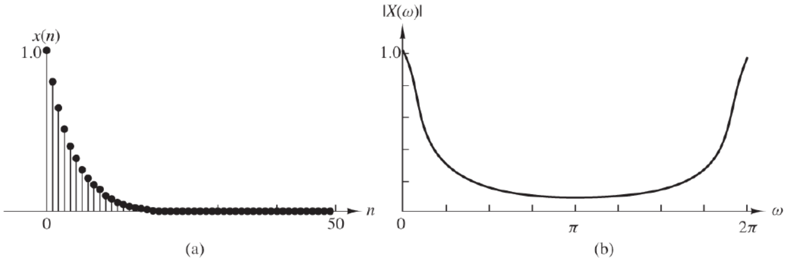
- Hence $$x_p(n)=\sum_{l=-\infty}^{\infty}x(n-lN)=\sum_{l=-\infty}^{0}a^{n-lN}$$

$$=a^n\sum_{l=0}^{\infty}a^{lN}=\frac{a^n}{1-a^N},\quad 0\le n\le N-1$$

where the factor $1/(1-a^N)$ represents the effect of aliasing. Since $0<a<1$, the aliasing error tends toward zero as $N\to\infty$.

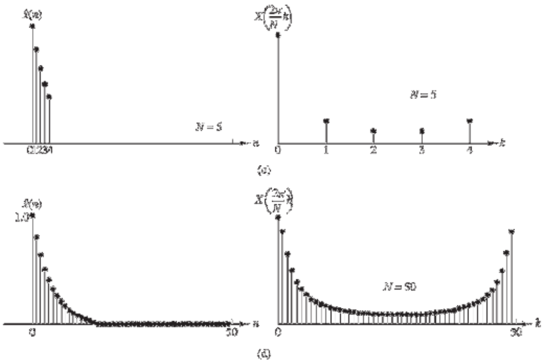
Frequency-Domain Sampling and Reconstruction of Discrete-Time Signals

- For $a=0.8$, the sequence $x(n)$ and its spectrum $X(\omega)$ are shown below (a)(b):



Frequency-Domain Sampling and Reconstruction of Discrete-Time Signals

- The aliased sequences $x_p(n)$ for $N=5$ and $N=50$ and the corresponding spectral samples are shown below (c)(d).



Frequency-Domain Sampling and Reconstruction of Discrete-Time Signals

- We note that the aliasing effects are negligible for *N*=50.
- If we define the aliased finite-duration sequence $x(n)$ as

$$\hat{x}(n)=\begin{cases} x_p(n), & 0\le n\le N-1 \\ 0, & \text{otherwise} \end{cases}$$

- Then its Fourier transform is

$$\hat{X}(\omega)=\sum_{n=0}^{N-1}\hat{x}(n)e^{-j\omega n}=\sum_{n=0}^{N-1}x_p(n)e^{-j\omega N}=\frac{1}{1-a^N}\cdot\frac{1-a^N e^{-j\omega n}}{1-ae^{-j\omega}}$$

Frequency-Domain Sampling and Reconstruction of Discrete-Time Signals

- Note that although $\hat{X}(\omega) \neq X(\omega)$, the sample values at $\omega_k = 2\pi k / N$ are identical. That is,

$$\hat{X}(\frac{2\pi}{N}k) = \frac{1}{1-a^N} \cdot \frac{1-a^N}{1-ae^{-j2\pi kN}} = X(\frac{2\pi}{N}k)$$

The Discrete Fourier Transform (DFT)

- The development in the preceding section is concerned with the frequency-domain sampling of an aperiodic finite-energy sequence $x(n)$.

- In general, the equally spaced frequency samples

$$X(2\pi k / N), \ k = 0,1,\ldots, N-1,$$

do not uniquely represent the original sequence $x(n)$ when $x(n)$ has infinite duration.

The Discrete Fourier Transform (DFT)

- Instead, the frequency samples

$$X(2\pi k / N),\ k = 0, 1, \ldots, N-1,$$

correspond to a periodic sequence $x_p(n)$ of period N, where $x_p(n)$ is an aliased version of $x(n)$, as indicated by the relation in the preceding equation, that is,

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN).$$

The Discrete Fourier Transform (DFT)

- When the sequence $x(n)$ has a finite duration of length $L \le N$, then $x_p(n)$ is simply a periodic repetition of $x(n)$, where $x_p(n)$ over a single period is given as

$$x_p(n) = \begin{cases} x(n), & 0 \le n \le L-1 \\ 0, & L \le n \le N-1 \end{cases}$$

- Consequently, the frequency samples

$$X(2\pi k / N),\ k = 0, 1, \ldots, N-1,$$

uniquely represent the finite-duration sequence $x(n)$.

The Discrete Fourier Transform (DFT)

- Since $x(n) \equiv x_p(n)$ over a single period (padded by N-L zeros), the original finite-duration sequence $x(n)$ can be obtained from the frequency samples $\{X(2\pi k / N)\}$ by means of the formula

$$x_p(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(\frac{2\pi}{N}k)e^{j2\pi kn/N}, \quad n = 0,1,\ldots,N-1.$$

The Discrete Fourier Transform (DFT)

- Note that *zero padding* does not provide any additional information about the spectrum $X(\omega)$ of the sequence $\{x(n)\}$.

- In summary, a finite-duration sequence $x(n)$ of length *L* has a Fourier transform

$$X(\omega) = \sum_{n=0}^{L-1} x(n)e^{-j\omega n}, \quad 0 \le \omega \le 2\pi$$

- Where the upper and lower indices in the summation reflect the fact that $x(n) = 0$ outside the range $0 \le n \le L-1$.

The Discrete Fourier Transform (DFT)

- When we sample $X(\omega)$ at equally spaced frequencies

  $$\omega_k = 2\pi k / N,\ k = 0,1,2,\ldots,N-1,\ \text{where N} \geq \text{L}$$

  the resultant samples are

  $$X(k) \equiv X(\frac{2\pi k}{N}) = \sum_{n=0}^{L-1} x(n)e^{-j2\pi kn/N}$$

  $$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N},\ k = 0,1,2,\ldots,N-1$$

  where for convenience, the upper index in the sum has been increased from *L-1* to *N-1* since *x(n)=0* for $n \geq L$.

The Discrete Fourier Transform (DFT)

- The relation

  $$X(k) \equiv X(\frac{2\pi k}{N}) = \sum_{n=0}^{L-1} x(n)e^{-j2\pi kn/N}$$

  $$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N},\ k = 0,1,2,\ldots,N-1$$

  is called the **discrete Fourier transform (DFT)** of x(n).

The Discrete Fourier Transform (DFT)

- To summarize, the formulas for the DFT and IDFT are

DFT: $$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}, \ k = 0, 1, 2, \ldots, N-1$$

IDFT: $$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N}, \ \mathrm{n} = 0, 1, 2, \ldots, N-1$$

The Discrete Fourier Transform (DFT)

- **Example**: A finite-duration sequence of length *L* is given

as $$x(n) = \begin{cases} 1, & 0 \le n \le L-1 \\ 0, & \text{otherwise} \end{cases}$$

Determine the *N*-point DFT of this sequence for $N \ge L$.

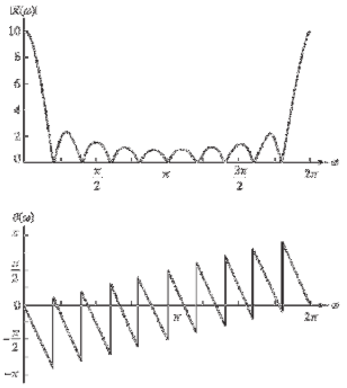
- **Solution**: The Fourier transform of this sequence is

$$X(\omega) = \sum_{n=0}^{L-1} x(n) e^{-j\omega n} = \sum_{n=0}^{L-1} e^{-j\omega n}$$

$$= \frac{1 - e^{-j\omega L}}{1 - e^{-j\omega}} = \frac{\sin(\omega L/2)}{\sin(\omega/2)} e^{-j\omega(L-1)/2}$$

The Discrete Fourier Transform (DFT)

- The magnitude and phase of $X(\omega)$ are illustrated in the below for *L=10*.



Magnitude

Phase

The Discrete Fourier Transform (DFT)

- The *N*-point DFT of x(n) is simply $X(\omega)$ evaluated at the set of *N* equally spaced frequencies

$$\omega_k = 2\pi k / N,\ k = 0,1,\ldots, N-1.$$

- Hence

$$X(k) = \frac{1-e^{-j2\pi kL/N}}{1-e^{-j2\pi k/N}},\ k = 0,1,\ldots,N-1$$
$$= \frac{\sin(\pi kL/N)}{\sin(\pi k/N)} e^{-j\pi k(L-1)/N}$$

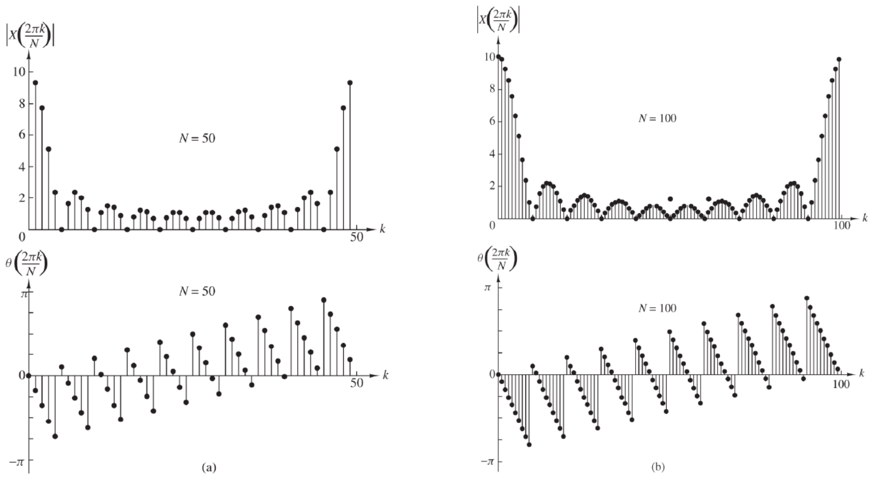
The Discrete Fourier Transform (DFT)

- If N is selected such that N=L, then the DFT becomes

$$X(k)=\begin{cases} L, & k=0 \\ 0, & k=1,2,\ldots,L-1 \end{cases}$$

The Discrete Fourier Transform (DFT)

N=50 **N=100**



The DFT as a Linear Transformation

- The formulas for the DFT and IDFT may be expressed as

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \quad k = 0, 1, 2, \ldots, N-1$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn}, \quad \mathrm{n} = 0, 1, 2, \ldots, N-1$$

$$\text{where } W_N = e^{-j2\pi/N}$$

which is an N^{th} root of unity.

The DFT as a Linear Transformation

- We note that the computation of each point of the DFT can be accomplished by *N* complex multiplications and (*N-1*) complex additions.
- Hence the *N*-point DFT values can be computed in a total of N^2 complex multiplications and *N*(*N-1*) complex additions.
- It is instructive to view the DFT and IDFT as linear transformations on sequences $\{x(n)\}$ and $\{X(k)\}$, respectively.

The DFT as a Linear Transformation

- Let us define an N-point vector $\mathbf{x}_N$ of the signal sequence $x(n)$, n=0,1,…,N-1, an N-point vector $\mathbf{X}_N$ of frequency samples, and an $N\times N$ matrix $\mathbf{W}_N$ as

$$\mathrm{x}_N = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix},\ \mathrm{X}_N = \begin{bmatrix} \mathrm{X}(0) \\ \mathrm{X}(1) \\ \vdots \\ \mathrm{X}(N-1) \end{bmatrix},\ \mathrm{W}_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ \vdots & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

- With these definitions, the N-point DFT may be expressed in matrix forms as $\mathrm{X}_N = \mathrm{W}_N \mathrm{x}_N$.
- Where W_N is the matrix of the linear transformation.

The DFT as a Linear Transformation

- We observe that W_N is a symmetric matrix.
- If we assume that the inverse of W_N exists, then

$$\mathrm{x}_N = \mathrm{W}_N^{-1} \mathrm{X}_N$$

- But this is just an expression for the IDFT.
- In fact, the IDFT can be expressed in matrix form as

$$\mathrm{x}_N = \frac{1}{N} \mathrm{W}_N^{*} \mathrm{X}_N$$

The DFT as a Linear Transformation

- Where $\mathbf{W}_N^*$ denotes the complex conjugate of the matrix $\mathbf{W}_N$. Then we can conclude that

$$\mathbf{W}_N^{-1} = \frac{1}{N}\mathbf{W}_N^*$$

- Which, in turn, implies that

$$\mathbf{W}_N \mathbf{W}_N^* = N\mathbf{I}_N$$

- Where $\mathbf{I}_N$ is an $N \times N$ identity matrix.

The DFT as a Linear Transformation

- Therefore, the matrix $\mathbf{W}_N$ in the transformation is an orthogonal (unitary) matrix.
- Furthermore, its inverse exists and is given as $\mathbf{W}_N^* / N$.
- Of course, the existence of the inverse of $\mathbf{W}_N$ was established previously from our derivation of the IDFT.

The DFT as a Linear Transformation

- The DFT and IDFT are computational tools that play a very important role in many digital signal processing applications, such as frequency analysis (spectrum analysis) of signals, power spectrum estimation, and linear filtering.
- The importance of the DFT and IDFT in such practical applications is due to a large extent to the existence of computationally efficient algorithms, known collectively as fast Fourier transform (FFT) algorithms, for computing the DFT and IDFT.

Relationship of the DFT to Other Transforms

- **Relationship to the Fourier series coefficients of a periodic sequence**
- A periodic sequence $\{x_p(n)\}$ with fundamental period *N* can be represented in a Fourier series of the form

$$x_p(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}, \quad -\infty < n < \infty$$

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x_p(n) e^{-j2\pi kn/N}, \ k = 0,1,..,N-1$$

- Where c_k is the Fourier series coefficients.

Relationship of the DFT to Other Transforms

- **Relationship to the Fourier transform of an aperiodic sequence**

- If $x(n)$ is an aperiodic finite energy sequence with Fourier transform $X(\omega)$, which is sampled at *N* equally spaced frequencies $\omega_k = 2\pi k / N,\ k = 0,1,\dots,N-1,$ the spectral components

$$X(k) = X(\omega)|_{\omega=2\pi k/N} = \sum_{n=-\infty}^{\infty} x(n)e^{-j2\pi kn/N},\ k = 0,1,..,N-1$$

are the DFT coefficients of the periodic sequence of period *N*.

Relationship of the DFT to Other Transforms

- **Relationship to the Fourier transform of an aperiodic sequence**

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN)$$

- Thus $x_p(n)$ is determined by aliasing $\{x(n)\}$ over the interval $0 \le n \le N-1$. The finite-duration sequence

$$\hat{x}(n) = \begin{cases} x_p(n), & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

Relationship of the DFT to Other Transforms

- **Relationship to the Fourier transform of an aperiodic sequence**
- Bears no resemblance to the original sequence $\{x(n)\}$, unless $x(n)$ is of finite duration and length $L \le N$, in which case $x(n) = \hat{x}(n),\ 0 \le n \le N-1$
- Only in this case will the IDFT of $\{X(k)\}$ yield the original sequence $\{x(n)\}$.

Relationship of the DFT to Other Transforms

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- Only in this case will the IDFT of $\{X(k)\}$ yield the original sequence $\{x(n)\}$.

Relationship of the DFT to Other Transforms

- **Relationship to the z-transform**

- Consider a sequence $x(n)$ having the z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

with an ROC that includes the unit circle.

Relationship of the DFT to Other Transforms

- **Relationship to the z-transform**

- If $X(z)$ is sampled at the *N* equally spaced points on the unit circle $z_k = e^{j2\pi k/N}$, $k = 0, 1, 2, ..., N-1$, we obtain

$$X(k) \equiv X(z)|_{z=e^{j2\pi nk/N}}, \; k = 0, 1, .., N-1$$

$$= \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi nk/N}$$

Relationship of the DFT to Other Transforms

- **Relationship to the Fourier series coefficients of a continuous-time signal**

- Suppose that $x_a(t)$ is a continuous-time periodic signal with fundamental period $T_p = 1/F_0$. The signal can be expressed in a Fourier series

$$x_a(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k t F_0}$$

- Where $\{c_k\}$ are the Fourier coefficients.

Relationship of the DFT to Other Transforms

- **Relationship to the Fourier series coefficients of a continuous-time signal**

- If we sample $x_a(t)$ at a uniform rate $F_s = N/T_p = 1/T$, we obtain the discrete-time sequence

$$x(n) \equiv x_a(nt) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 n T} = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kn/N}$$

$$= \sum_{k=0}^{N-1} \left[\sum_{l=-\infty}^{\infty} c_{k-lN} \right] e^{j2\pi kn/N}$$

Relationship of the DFT to Other Transforms

- **Relationship to the Fourier series coefficients of a continuous-time signal**

- It is clear the above equation is in the form of an IDFT formula, where

$$X(k) = N\sum_{l=-\infty}^{\infty} c_{k-lN} \equiv N\tilde{c}_k \quad \text{and} \quad \tilde{c}_k = \sum_{l=-\infty}^{\infty} c_{k-lN}$$

- Thus the $\{\tilde{c}_k\}$ sequence is an aliased version of the sequence $\{c_k\}$.

Properties of the DFT

- The DFT as a set of *N* samples $\{X(k)\}$ of the Fourier transform $X(\omega)$ for a finite-duration sequence $\{x(n)\}$ of length $L \le N$.

- The sampling of $X(\omega)$ occurs at the *N* equally spaced frequencies $\omega_k = 2\pi k / N,\ k = 0,1,2,\ldots,N-1.$

DFT: $X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn},\ k = 0,1,2,\ldots,N-1$

IDFT: $x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) W_N^{-kn},\ \mathrm{n} = 0,1,2,\ldots,N-1$

where $W_N = e^{-j2\pi/N}$

$$x(n) \xleftrightarrow[N]{DFT} X(k)$$

Periodicity, Linearity, and Symmetry Properties

- **Periodicity**. If $x(n)$ and $X(k)$ are an *N*-point DFT pair, then

$$x(n+N) = x(n) \quad \forall n$$
$$X(k+N) = X(k) \quad \forall k$$

- **Linearity**. If $x_1(n) \xleftrightarrow[N]{DFT} X_1(k)$ and $x_2(n) \xleftrightarrow[N]{DFT} X_2(k)$

  then for any real-valued or complex-valued constants a_1 and a_2 ,

$$a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow[N]{DFT} a_1 X_1(k) + a_2 X_2(k)$$

Periodicity, Linearity, and Symmetry Properties

- **Circular Symmetries of a Sequence**. As we have seen, the *N*-point DFT of a finite duration sequence $x(n)$, of length $L \le N$, is equivalent to the *N*-point DFT of a periodic sequence $x_p(n)$, of period *N*, which is obtained by periodically extending $x(n)$, that is

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN)$$

Periodicity, Linearity, and Symmetry Properties

- **Circular Symmetries of a Sequence**. Suppose that we shift the periodic sequence $x_p(n)$ by k units to the right. Thus we obtain another periodic sequence

$$x_p^{'}(n) = x_p(n-k) = \sum_{l=-\infty}^{\infty} x(n-k-lN)$$

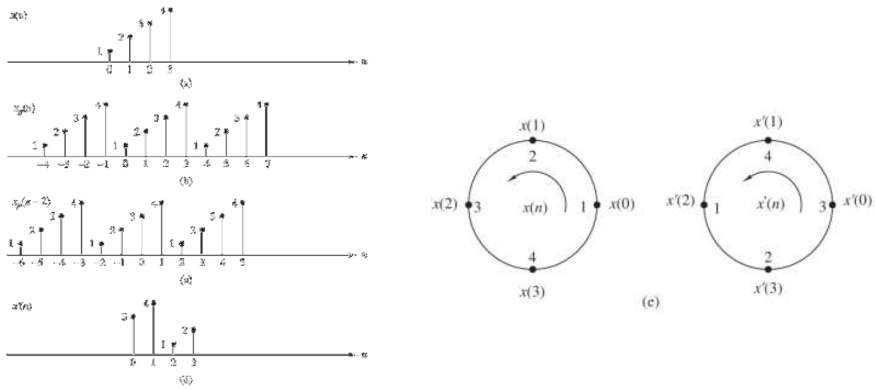
- The finite-duration sequence

$$x'(n) = \begin{cases} x_p^{'}(n), & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

is related to the original sequence $x(n)$ by a circular shift.

Periodicity, Linearity, and Symmetry Properties

- **Circular Symmetries of a Sequence**. This relationship is illustrated as below for N=4.



Periodicity, Linearity, and Symmetry Properties

- **Circular Symmetries of a Sequence**.

- In general, the circular shift of the sequence can be represented as the index modulo *N*. thus we can write

$$x'(n) = x(n-k, \text{ modulo } N)$$
$$= x((n-k))_N$$

- Time reversal of *N*-point sequence

$$x((-n))_N = x(N-n), \; 0 \le n \le N-1$$

Periodicity, Linearity, and Symmetry Properties

- **Circular Symmetries of a Sequence**. An equivalent definition of even and odd sequences for the associated periodic sequence $x_p(n)$ is given as follows

Even: $x_p(n) = x_p(-n) = x_p(N-n)$

Odd: $x_p(n) = -x_p(-n) = -x_p(N-n)$

- If the periodic sequence is complex valued, we have

Conjugate even: $x_p(n) = x_p^*(N-n)$

Conjugate odd: $x_p(n) = -x_p^*(N-n)$

Periodicity, Linearity, and Symmetry Properties

- **Circular Symmetries of a Sequence**. These relationships suggest that we decompose the sequence $x_p(n)$ as

$$x_p(n) = x_{pe}(n) + x_{po}(n)$$

where

$$x_{pe}(n) = \frac{1}{2}[x_p(n) + x_p^*(N-n)]$$

$$x_{po}(n) = \frac{1}{2}[x_p(n) - x_p^*(N-n)]$$

Periodicity, Linearity, and Symmetry Properties

- **Symmetry properties of the DFT**. The sequences can be expressed as

$$x(n) = x_R(n) + jx_I(n),\ 0 \le n \le N-1$$
$$X(k) = X_R(k) + jX_I(k),\ 0 \le k \le N-1$$

- We can obtain

$$X_R(k) = \sum_{n=0}^{N-1}\left[x_R(n)\cos\frac{2\pi kn}{N} + x_I(n)\sin\frac{2\pi kn}{N}\right]$$
$$X_I(k) = -\sum_{n=0}^{N-1}\left[x_R(n)\sin\frac{2\pi kn}{N} - x_I(n)\cos\frac{2\pi kn}{N}\right]$$

Periodicity, Linearity, and Symmetry Properties

- **Symmetry properties of the DFT**.
- Similarly,

$$x_R(n) = \frac{1}{N}\sum_{k=0}^{N-1}\left[X_R(k)\cos\frac{2\pi kn}{N} - X_I(k)\sin\frac{2\pi kn}{N}\right]$$

$$x_I(k) = \frac{1}{N}\sum_{k=0}^{N-1}\left[X_R(k)\sin\frac{2\pi kn}{N} + X_I(k)\cos\frac{2\pi kn}{N}\right]$$

Periodicity, Linearity, and Symmetry Properties

- **Real-valued sequences**. If the sequence $x(n)$ is real, it follows

$$X(N-k) = X^*(k) = X(-k)$$

- Consequently,

$$|X(N-k)| = |X(k)|$$
$$\angle X(N-k) = -\angle X(k)$$

Periodicity, Linearity, and Symmetry Properties

- **Real and even sequences**. If $x(n)$ is real and even, that is

$$x(n) = x(N-n),\ \ 0 \le n \le N-1$$

- And $X_I(k) = 0$. Hence the DFT reduces to

$$X(k) = \sum_{n=0}^{N-1} x(n)\cos\frac{2\pi kn}{N},\ \ 0 \le k \le N-1$$

- IDFT $$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\cos\frac{2\pi kn}{N},\ \ 0 \le n \le N-1$$

Periodicity, Linearity, and Symmetry Properties

- **Real and odd sequences**. If $x(n)$ is real and odd, that is

$$x(n) = -x(N-n),\ \ 0 \le n \le N-1$$

- And $X_R(k) = 0$. Hence the DFT reduces to

$$X(k) = -j\sum_{n=0}^{N-1} x(n)\sin\frac{2\pi kn}{N},\ \ 0 \le k \le N-1$$

- IDFT $$x(n) = j\frac{1}{N}\sum_{k=0}^{N-1} X(k)\sin\frac{2\pi kn}{N},\ \ 0 \le n \le N-1$$

Periodicity, Linearity, and Symmetry Properties

- **Purely imaginary sequences.** $x(n) = jx_I(n)$

$$X_R(k) = \sum_{n=0}^{N-1} x_I(n) \sin\frac{2\pi kn}{N}$$

$$X_I(k) = \sum_{n=0}^{N-1} x_I(n) \cos\frac{2\pi kn}{N}$$

- We observe that $X_R(k)$ is odd and $X_I(k)$ is even.

Periodicity, Linearity, and Symmetry Properties

N-Point Sequence $x(n)$, $0 \le n \le N-1$	N-Point DFT
$x(n)$	$X(k)$
$x^*(n)$	$X^*(N-k)$
$x^*(N-n)$	$X^*(k)$
$x_R(n)$	$X_{ce}(k) = \frac{1}{2}[X(k) + X^*(N-k)]$
$jx_I(n)$	$X_{co}(k) = \frac{1}{2}[X(k) - X^*(N-k)]$
$x_{ce}(n) = \frac{1}{2}[x(n) + x^*(N-n)]$	$X_R(k)$
$x_{co}(n) = \frac{1}{2}[x(n) - x^*(N-n)]$	$jX_I(k)$
Real Signals	
Any real signal	$X(k) = X^*(N-k)$
$x(n)$	$X_R(k) = X_R(N-k)$
	$X_I(k) = -X_I(N-k)$
	$\|X(k)\| = \|X(N-k)\|$
	$\angle X(k) = -\angle X(N-k)$
$x_{ce}(n) = \frac{1}{2}[x(n) + x(N-n)]$	$X_R(k)$
$x_{co}(n) = \frac{1}{2}[x(n) - x(N-n)]$	$jX_I(k)$

Multiplication of Two DFTs and Circular Convolution

- Suppose that we have two finite-duration sequences of length *N*, $x_1(n)$ and $x_2(n)$. Their respective *N*-point DFTs are

$$X_1(k) = \sum_{n=0}^{N-1} x_1(n) e^{-j2\pi nk/N}, \quad k = 0, 1, \ldots, N-1$$

$$X_2(k) = \sum_{n=0}^{N-1} x_2(n) e^{-j2\pi nk/N}, \quad k = 0, 1, \ldots, N-1$$

- If we multiply the two DFTs together, the result is a DFT, say $X_3(k)$, of a sequence $x_3(n)$ of length *N*.

$$X_3(k) = X_1(k) X_2(k), \quad k = 0, 1, \ldots, N-1$$

Multiplication of Two DFTs and Circular Convolution

- The IDFT of $\{X_3(k)\}$ is

$$x_3(m) = \frac{1}{N} \sum_{k=0}^{N-1} X_3(k) e^{j2\pi km/N} = \frac{1}{N} \sum_{k=0}^{N-1} X_1(k) X_2(k) e^{j2\pi km/N}$$

- Suppose that we substitute for $X_1(k)$ and $X_2(k)$ using the DFTs, thus we obtain

$$x_3(m) = \frac{1}{N} \sum_{k=0}^{N-1} \left[\sum_{n=0}^{N-1} x_1(n) e^{-j2\pi kn/N} \right] \left[\sum_{l=0}^{N-1} x_2(l) e^{-j2\pi kn/N} \right] e^{j2\pi km/N}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \left[\sum_{k=0}^{N-1} e^{j2\pi k(m-n-l)/N} \right]$$

Multiplication of Two DFTs and Circular Convolution

- The inner sum in the brackets in the above equation has the form

$$\sum_{k=0}^{N-1} a^k = \begin{cases} N, & a=1 \\ \dfrac{1-a^N}{1-a}, & a \neq 1 \end{cases}$$

  where a is defined as $a = e^{j2\pi(m-n-l)/N}$

- We observe that $a=1$ when $m-n-l$ is a multiple of *N*.

Multiplication of Two DFTs and Circular Convolution

- On the other hand, $a^N = 1$ for any value of $a \neq 0$.
- Consequently, the above equation reduces to

$$\sum_{k=0}^{N-1} a^k = \begin{cases} N, & l = m-n+pN = ((m-n))_N, \text{ p an integer} \\ 0, & \text{otherwise} \end{cases}$$

- Then we obtain the desired expression for $x_3(m)$ in the form

$$x_3(m) = \sum_{n=0}^{N-1} x_1(n)x_2((m-n))_N, \; m = 0,1,\ldots,N-1$$

Multiplication of Two DFTs and Circular Convolution

- **Circular convolution**. If

$$x_1(n) \xleftrightarrow[N]{DFT} X_1(k) \text{ and } x_2(n) \xleftrightarrow[N]{DFT} X_2(k)$$

then $$x_1(n) \, \textcircled{N} \, x_2(n) \xleftrightarrow[N]{DFT} X_1(k) X_2(k)$$

where $x_1(n) \, \textcircled{N} \, x_2(n)$ denotes the circular convolution of the sequence $x_1(n)$ and $x_2(n)$.

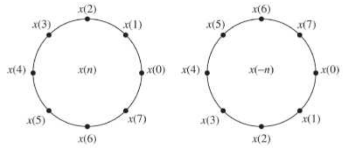
Additional DFT Properties

- **Time reversal of a sequence**. If

$$x(n) \xleftrightarrow[N]{DFT} X(k)$$

then $$x((-n))_N = x(N-n) \xleftrightarrow[N]{DFT} X((-k))_N = X(N-k)$$

- Hence reversing the *N*-point sequence in time is equivalent to reversing the DFT values.



Additional DFT Properties

- **Circular time shift of a sequence**. If

$$x(n) \xleftrightarrow[N]{DFT} X(k)$$

then $$x((n-l))_N \xleftrightarrow[N]{DFT} X(k)e^{-j2\pi kl/N}$$

- **Circular frequency shift**. If

$$x(n) \xleftrightarrow[N]{DFT} X(k)$$

$$x(n)e^{j2\pi ln/N} \xleftrightarrow[N]{DFT} X((k-l))_N$$

Additional DFT Properties

- **Complex-conjugate properties**. If

$$x(n) \xleftrightarrow[N]{DFT} X(k)$$

then $$x^*(n) \xleftrightarrow[N]{DFT} X^*((-k))_N = X^*(N-k)$$

- The IDFT of $X^*(k)$ is

$$\frac{1}{N}\sum_{k=0}^{N-1} X^*(k)e^{j2\pi kn/N} = \left[\frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi k(N-n)/N}\right]$$

therefore, $x^*((-n))_N = x^*(N-n) \xleftrightarrow[N]{DFT} X^*(k)$

Additional DFT Properties

- **Circular correlation**. If

$$x(n) \xleftrightarrow[N]{DFT} X(k) \text{ and } y(n) \xleftrightarrow[N]{DFT} Y(k)$$

then $$\tilde{r}_{xy}(l) \xleftrightarrow[N]{DFT} \tilde{R}_{xy}(k) = X(k)Y^*(k)$$

where $\tilde{r}_{xy}(l)$ is the (unnormalized) circular crosscorrelation sequence, defined as

$$\tilde{r}_{xy}(l) = \sum_{n=0}^{N-1} x(n)y^*((n-l))_N$$

Additional DFT Properties

- **Multiplication of two sequences**. If

$$x_1(n) \xleftrightarrow[N]{DFT} X_1(k) \text{ and } x_2(n) \xleftrightarrow[N]{DFT} X_2(k)$$

then $$x_1(n)x_2(n) \xleftrightarrow[N]{DFT} \frac{1}{N} X_1(k) \, \text{Ⓝ} \, X_2(k)$$

- **Parseval's Theorem**. If

$$x(n) \xleftrightarrow[N]{DFT} X(k) \text{ and } y(n) \xleftrightarrow[N]{DFT} Y(k)$$

then $$\sum_{n=0}^{N-1} x(n)y^*(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)$$

Linear Filtering Methods based on the DFT

- The DFT provides a discrete frequency representation of a finite-duration sequence in the frequency domain, it explore its use as a computational tool for linear system analysis and, especially, for linear filtering.
- We have already established that a system with frequency response $H(\omega)$, when excited with an input signal that has a spectrum $X(\omega)$, possesses an output spectrum $Y(\omega) = X(\omega)H(\omega)$.
- The output sequence $y(n)$ is determined from its spectrum via the inverse Fourier transform.

Use of the DFT in Linear Filtering

- Suppose that we have a finite-duration sequence $x(n)$ of length *L* which excites an FIR filter of length *M*. Without loss of generality, let

$$x(n) = 0, \quad n < 0 \text{ and } n \geq L$$

$$h(n) = 0, \quad n < 0 \text{ and } n \geq M$$

  where $h(n)$ is the impulse response of the FIR filter.

- The output sequence

$$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k)$$

Use of the DFT in Linear Filtering

- The frequency-domain equivalent to the above is

$$Y(\omega) = X(\omega)H(\omega)$$

- If

$$Y(k) \equiv Y(\omega)|_{\omega=2\pi k/N},\ k = 0,1,\dots,N-1$$
$$= X(\omega)H(\omega)|_{\omega=2\pi k/N},\ k = 0,1,\dots,N-1$$

then

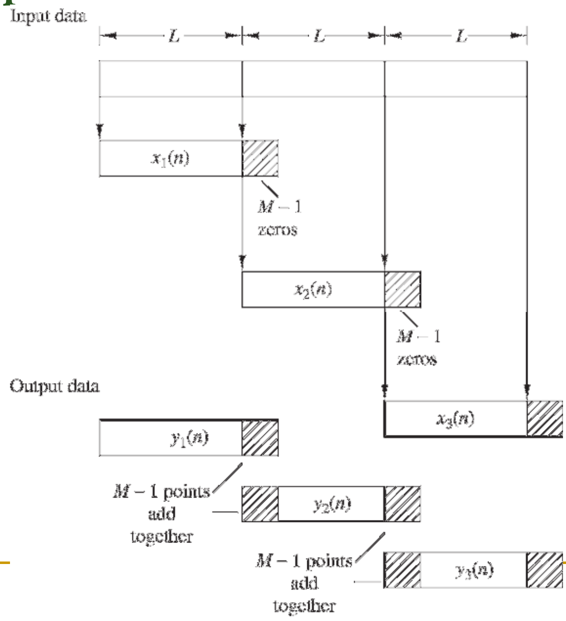
$$Y(k) = X(k)H(k),\ k = 0,1,\dots,N-1$$

where $\{X(k)\}$ and $\{H(k)\}$ are the *N*-point DFTs of the corresponding sequences $x(n)$ and $h(n)$.

Filtering of Long Data Sequences

- In practical applications involving linear filtering of signals, the input sequence $x(n)$ is often a very long sequence. This is especially true in some real-time signal processing applications concerned with signal monitoring and analysis.
- Since linear filtering performed via the DFT involves operations on a block of data, which by necessity must be limited in size due to limited memory of a digital computer, a long input signal sequence must be segmented to fixed-size blocks prior to processing.

Overlap-Add Method



Overlap-Add Method

- We first segment $x(n)$, assumed to be a causal sequence here without any loss of generality, into a set of contiguous finite-length subsequences of length L each:

$$x(n) = \sum_{m=0}^{\infty} x_m(n - mL)$$

where

$$x_m(n) = \begin{cases} x(n+mL), & 0 \le n \le L-1 \\ 0, & \text{otherwise} \end{cases}$$

- Thus we can write

$$y(n) = h(n) * x(n) = \sum_{m=0}^{\infty} y_m(n - mL)$$

where

$$y_m(n) = h(n) * x_m(n)$$

- Since $h(n)$ is of length M and $x_m(n)$ is of length L, $y_m(n)$ is of length $L + M - 1$

Overlap-Add Method

- The desired linear convolution $y(n) = h(n) \ast x(n)$ is broken up into a sum of infinite number of short-length linear convolutions of length $L + M - 1$ each: $y_m(n) = h(n) \ast x_m(n)$
- Consider implementing the following convolutions using the DFT-based method, where now the DFTs (and the IDFT) are computed on the basis of $(L + M - 1)$ points

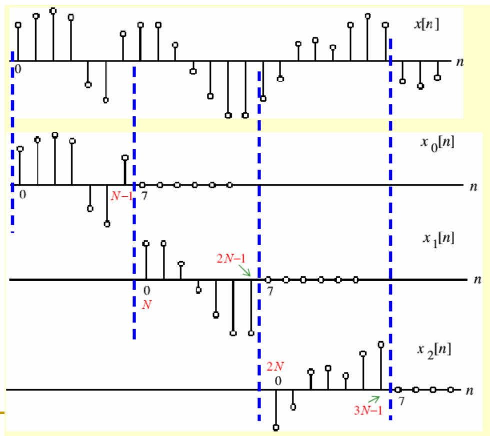
$$y(n) = \sum_{m=0}^{\infty} y_m(n - mL)$$

- The first convolution in the above sum, $y_0(n) = h(n) \ast x_0(n)$, is of length $L + M - 1$ and is defined for $0 \le n \le L + M - 2$
- The second short convolution $y_1[n] = h(n) \ast x_1(n)$, is also of length $L + M - 1$ but is defined for $L \le n \le 3L + M - 2$

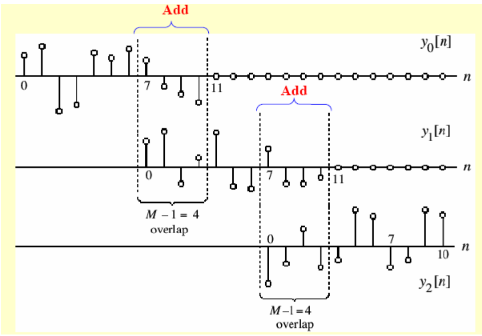
⇨ There is an overlap of samples between these two short linear convolutions

Overlap-Add Method

- In general, there will be an overlap of $M - 1$ samples between the samples of the short convolutions $h(n) \ast x_{r-1}(n)$ and $h(n) \ast x_m(n)$ for $(r - 1)L \le n \le rL + M - 2$



Overlap-Add Method

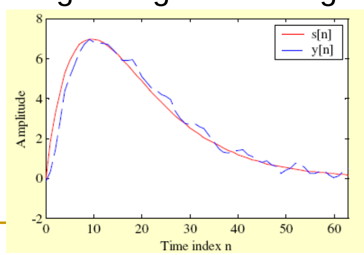


- Therefore, *y*(*n*) obtained by a linear convolution of *x*(*n*) and *h*(*n*) is given by

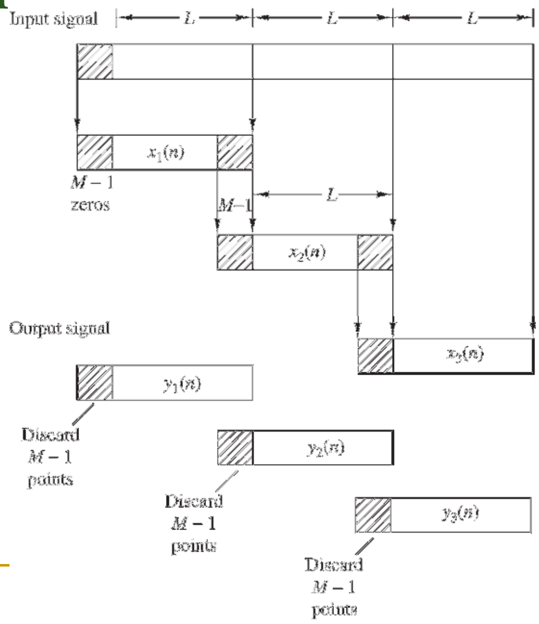
$$y(n) = y_0(n), \qquad 0 \le n \le 6$$
$$y(n) = y_1(n) + y_1(n-7), \quad 7 \le n \le 10$$
$$y(n) = y_1(n-7), \qquad 11 \le n \le 13$$
$$\vdots$$

Overlap-Add Method

- The above procedure is called the **overlap-add method** since the results of the short linear convolutions overlap and the overlapped portions are added to get the correct final result
- The MATLAB function **fftfilt** can be used to implement the above method
- The following illustrates an example of filtering of a noise-corrupted signal using a length-3 moving average filter:



Overlap-Save Method



Overlap-Save Method

- In implementing the overlap-add method using the DFT, we need to compute two ($L + M - 1$)-point DFTs and one ($L + M - 1$)-point IDFT for each short linear convolution
- It is possible to implement the overall linear convolution by performing instead circular convolution of length shorter than ($L + M - 1$)
- To this end, it is necessary **to segment $x(n)$ into overlapping blocks $x_m(n)$**, keep the terms of the circular convolution of $h(n)$ with that corresponds to the terms obtained by a linear convolution of $h(n)$ and $x_m(n)$, and throw away the other parts of the circular convolution

Overlap-Save Method

- To understand the correspondence between the linear and circular convolutions, consider a length-4 sequence $x(n)$ and a length-3 sequence $h(n)$
- Let $y_L(n)$ denote the result of a linear convolution of $x(n)$ with $h(n)$
- The six samples of $y_L(n)$ are given by

$$y_L(0) = h(0)x(0)$$
$$y_L(1) = h(0)x(1) + h(1)x(0)$$
$$y_L(2) = h(0)x(2) + h(1)x(1) + h(2)x(0)$$
$$y_L(3) = h(0)x(3) + h(1)x(2) + h(2)x(1)$$
$$y_L(0) = h(1)x(3) + h(2)x(2)$$
$$y_L(5) = h(2)x(3)$$

Overlap-Save Method

- If we append $h(n)$ with a single zero-valued sample and convert it into a length-4 sequence $h_e(n)$, the 4-point circular convolution $Yc(n)$ of $h_e(n)$ and $x(n)$ is given by

$$y_C(0) = h(0)x(0) + h(1)x(3) + h(2)x(2)$$
$$y_C(1) = h(0)x(1) + h(1)x(0) + h(2)x(3)$$
$$y_C(2) = h(0)x(2) + h(1)x(1) + h(2)x(0)$$
$$y_C(3) = h(0)x(3) + h(1)x(2) + h(2)x(1)$$

- If we compare the expressions for the samples $y_L(n)$ of with those of $y_C(n)$, we observe that the first 2 terms of $y_C(n)$ do not correspond to the first 2 terms of $y_L(n)$, whereas the last 2 terms of $y_C(n)$ are precisely the same as the 3rd and 4th terms of $y_L(n)$, i.e.

$$y_L(0) \neq y_C(0),\ y_L(1) \neq y_C(1),\ y_L(2) = y_C(2),\ y_L(3) = y_C(3)$$

Overlap-Save Method

- **General case**: N-point circular convolution of a length-M sequence $h(n)$ with a length-L sequence $x(n)$ with $N > M$
- First $M - 1$ samples of the circular convolution are incorrect and are rejected
- Remaining $L - M + 1$ samples correspond to the correct samples of the linear convolution of $h(n)$ with $x(n)$
- Now, consider an infinitely long or very long sequence $x(n)$
- Break it up as a collection of smaller length (length-4) overlapping sequences $x_m(n)$ as $x_m(n) = x(n + 2m)$, $0 \le n \le 3$, $0 \le m \le \infty$
- Next, form

$$w_m(n) = h(n) \textcircled{4} x_m(n)$$

Overlap-Save Method

- Or, equivalently,

$$w_m(0) = h(0)x_m(0) + h(1)x_m(3) + h(2)x_m(2)$$
$$w_m(1) = h(0)x_m(1) + h(1)x_m(0) + h(2)x_m(3)$$
$$w_m(2) = h(0)x_m(2) + h(1)x_m(1) + h(2)x_m(0)$$
$$w_m(3) = h(0)x_m(3) + h(1)x_m(2) + h(2)x_m(1)$$

- Computing the above for $m = 0, 1, 2, 3, \ldots,$ and substituting the values of $x_m[n]$ we arrive at

$$w_0(0) = h(0)x(0) + h(1)x(3) + h(2)x(2) \quad \leftarrow \text{Reject}$$
$$w_0(1) = h(0)x(1) + h(1)x(0) + h(2)x(3) \quad \leftarrow \text{Reject}$$
$$w_0(2) = h(0)x(2) + h(1)x(1) + h(2)x(0) = y[2] \quad \leftarrow \text{Save}$$
$$w_0(3) = h(0)x(3) + h(1)x(2) + h(2)x(1) = y[3] \quad \leftarrow \text{Save}$$

Overlap-Save Method

$w_1(0) = h(0)x(2) + h(1)x(5) + h(2)x(4)$ ← Reject
$w_1(1) = h(0)x(3) + h(1)x(2) + h(2)x(5)$ ← Reject
$w_1(2) = h(0)x(4) + h(1)x(3) + h(2)x(2) = y(4)$ ← Save
$w_1(3) = h(0)x(5) + h(1)x(4) + h(2)x(3) = y(5)$ ← Save

$w_2(0) = h(0)x(4) + h(1)x(7) + h(2)x(6)$ ← Reject
$w_2(1) = h(0)x(5) + h(1)x(4) + h(2)x(7)$ ← Reject
$w_2(2) = h(0)x(6) + h(1)x(5) + h(2)x(4) = y(6)$ ← Save
$w_2(3) = h(0)x(7) + h(1)x(6) + h(2)x(5) = y(7)$ ← Save

- It should be noted that to determine $y(0)$ and $y(1)$, we need to form $x_{-1}(n)$: $x_{-1}(0) = 0$, $x_{-1}(1) = 0$, $x_{-1}(2) = x(0)$, $x_{-1}(3) = x(1)$ and compute $w_{-1}(n) = h(n)$ ④ $x_{-1}(n)$ for $0 \le n \le 3$, reject $w_{-1}(0)$ and $w_{-1}(1)$, and save $w_{-1}(2) = y(0)$, and $w_{-1}(3) = y(1)$

Overlap-Save Method

- **General Case**: Let $h(n)$ be a length-L sequence
- Let $x_m(n)$ denote the m-th section of an infinitely long sequence $x[n]$ of length L and defined by

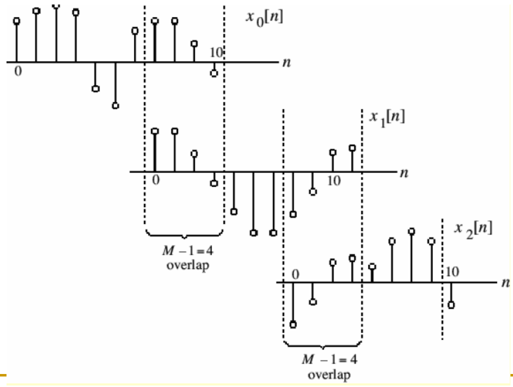
  $x_m(n) = x(n + m(L - m + 1))$, $0 \le n \le L - 1$ with $M < L$
- Let $w_m(n) = h(n) Ⓝ x_m(n)$
- Then, we reject the first $M - 1$ samples of $w_m(n)$ and "abut" the remaining $L - M + 1$ samples of $w_m(n)$ to form $y_L(n)$, the linear convolution of $h(n)$ and $x(n)$
- If $y_m[n]$ denotes the saved portion of $w_m(n)$, i.e.,

$$y(n) = \begin{cases} 0, & 0 \le n \le M-2 \\ w_m(n), & M-1 \le n \le L-2 \end{cases}$$

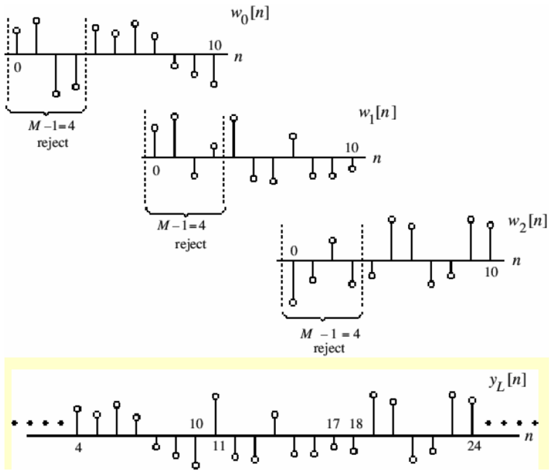
- Then $y_L(n + m(L - M + 1)) = y_m(n)$, $M - 1 \le n \le l - 1$

Overlap-Save Method

- The approach is called **overlap-save method** since the input is segmented into overlapping sections and parts of the results of the circular convolutions are saved and abutted to determine the linear convolution result



Overlap-Save Method



Frequency Analysis of Signals Using the DFT

- Let $\{x(n)\}$ denote the sequence to be analyzed. Limiting the duration of the sequence to *L* samples, in the interval $0 \le n \le L-1$, is equivalent to multiplying $\{x(n)\}$ by a rectangular window $w(n)$ of length *L*. That is

$$\hat{x}(n) = x(n)w(n)$$

where

$$w(n) = \begin{cases} 1, & 0 \le n \le L-1 \\ 0, & \text{otherwise} \end{cases}$$

Frequency Analysis of Signals Using the DFT

- Suppose that the sequence $x(n)$ consists of a signal sinusoid, that is

$$x(n) = \cos \omega_0 n$$

- Then the Fourier transform of the finite-duration sequence $x(n)$ can be expresses as

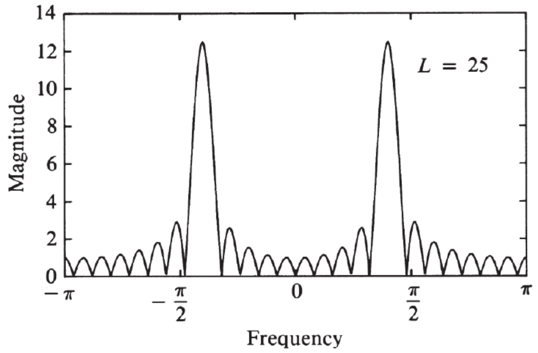
$$\hat{X}(\omega) = \frac{1}{2}[W(\omega - \omega_0) + W(\omega + \omega_0)]$$

where $W(\omega)$ is the Fourier transform of the window sequence, which is

$$W(\omega) = \frac{\sin(\omega L/2)}{\sin(\omega/2)} e^{-j\omega(L-1)/2}$$

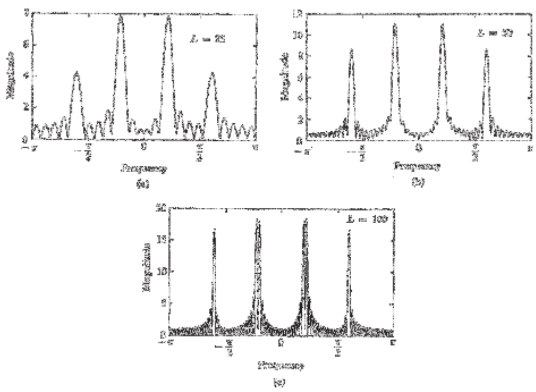
Frequency Analysis of Signals Using the DFT

- Magnitude spectrum for L=25 and N=2048, illustrating the occurrence of leakage.



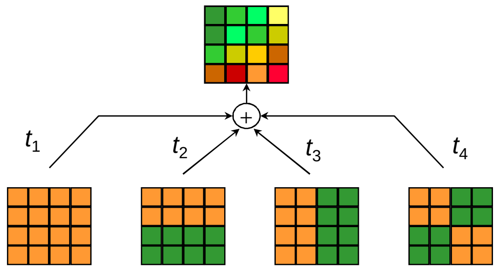
Frequency Analysis of Signals Using the DFT

- $x(n) = \cos \omega_0 n + \cos \omega_1 n + \cos \omega_2 n$

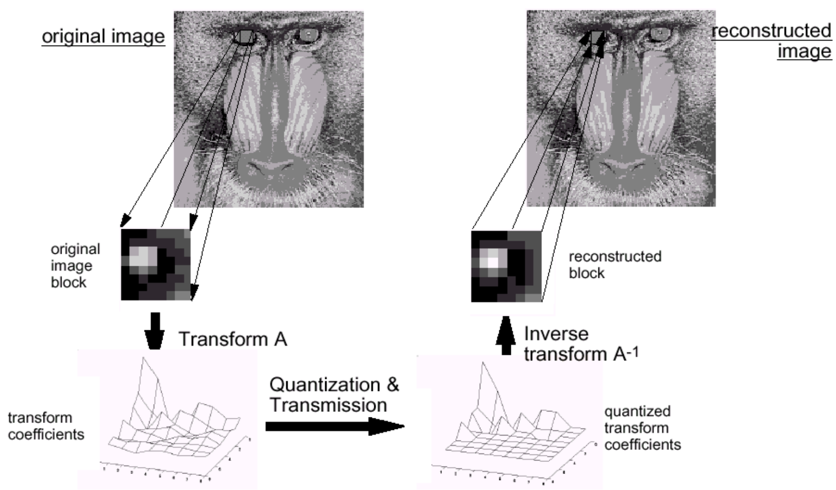


Signal Transform

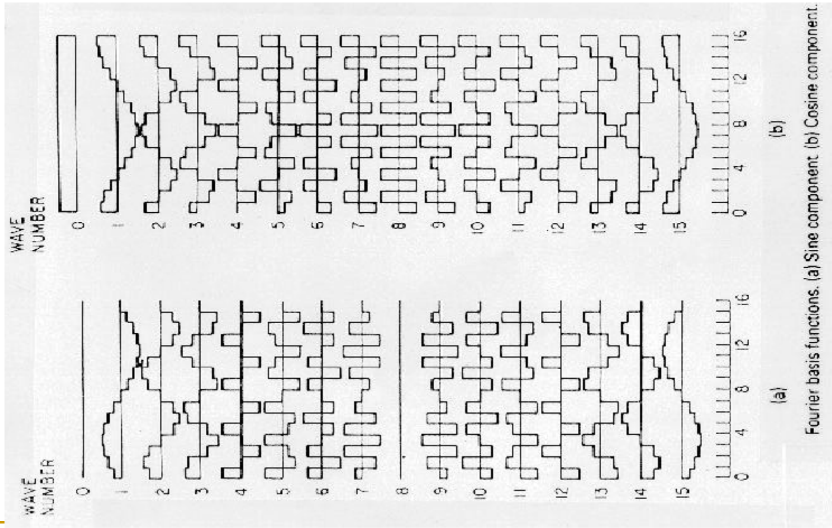
- Motivation:
  - Represent a vector (e.g. a block of image samples) as the superposition of some typical vectors (block patterns)



Transform Coding of a Image

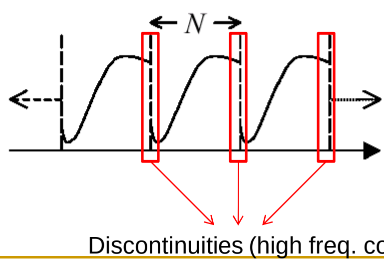


1-D 16-Pont DFT Basis Vectors



Disadvantages of DFT in Signal Coding

- Fourier Transform of a real function results in complex numbers
- May result in artifacts due to discontinuity at the block boundary



From DFT to DCT

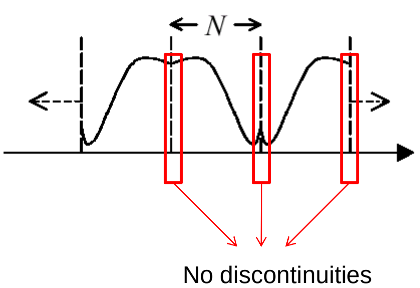
- DFT of any real and symmetric sequence contains only real coefficients corresponding to the cosine terms of the series
- Construct a new symmetric sequence *y(n)* of length *2N* out of *x(n)* of length *N*

$$y(n) = x(n), 0 \le n \le N-1,$$
$$y(n) = x(2N-1-n), N \le n \le 2N-1.$$

- *Y*(*n*) is symmetrical about *n* = *N* - (1/2)

From DFT to DCT

- DCT has a higher compression ration than DFT
  - DCT avoids the generation of spurious spectral components



From DFT to DCT

$$
\begin{aligned}
Y(k) &= \sum_{n=0}^{2N-1} y(n) W_{2N}^{(n+\frac{1}{2})k}, \\
&= \sum_{n=0}^{N-1} y(n) W_{2N}^{(n+\frac{1}{2})k} + \sum_{n=N}^{2N-1} y(n) W_{2N}^{(n+\frac{1}{2})k} \\
&= \sum_{n=0}^{N-1} x(n) W_{2N}^{(n+\frac{1}{2})k} + \sum_{n=N}^{2N-1} x(2N-1-n) W_{2N}^{(n+\frac{1}{2})k} \\
&= \sum_{n=0}^{N-1} x(n) W_{2N}^{(n+\frac{1}{2})k} + \sum_{n=0}^{N-1} x(n) W_{2N}^{[2N-(n+\frac{1}{2})k]} \\
&= \sum_{n=0}^{N-1} 2x(n) \cos \frac{\pi(2n+1)k}{2N},
\end{aligned}
$$

$$0 \le k \le 2N-1, \text{ and } \; W_{2N} = e^{-j\frac{2\pi}{2N}}$$

1-D *N*-Point DCT

$$F(k) = C(k) \sum_{n=0}^{N-1} f(n) \cos\left[\frac{(2n+1)k\pi}{2N}\right],$$

$$k = 0,1,\cdots,N-1,$$

$$f(n) = \sum_{k=0}^{N-1} C(k) F(k) \cos\left[\frac{(2n+1)k\pi}{2N}\right],$$

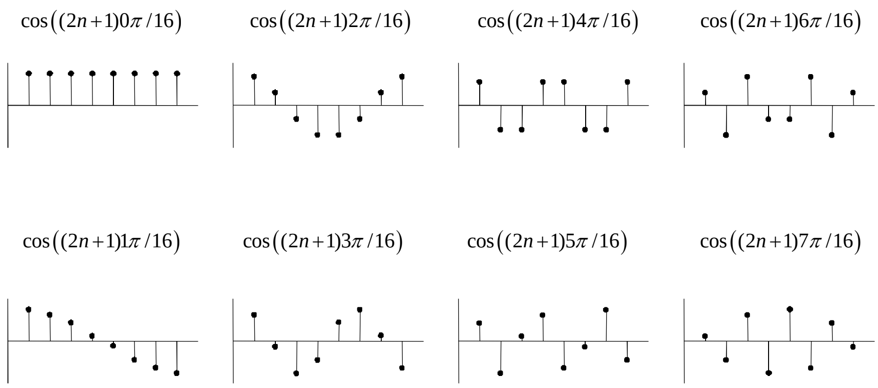
$$n = 0,1,\cdots,N-1,$$

where

$$C(0) = \sqrt{\frac{1}{N}}, \; C(k) = \sqrt{\frac{2}{N}}, \; k = 1,2,\cdots,N-1$$

- The constants are often defined differently

1-D N-Point DCT



Example of 1-D DCT

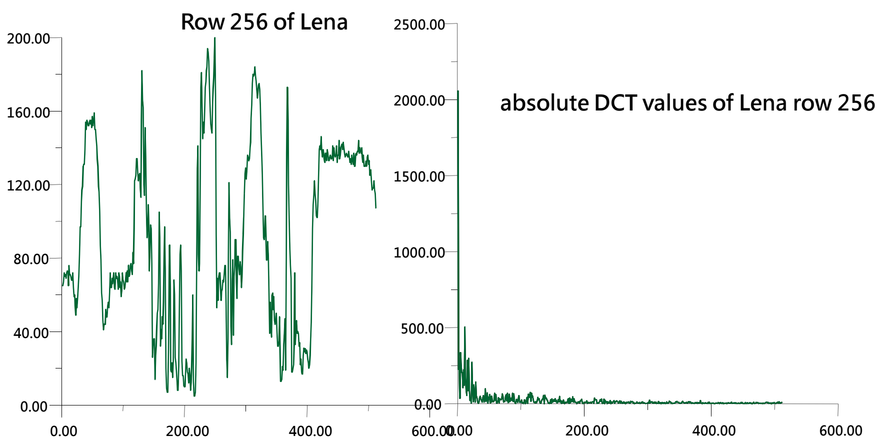
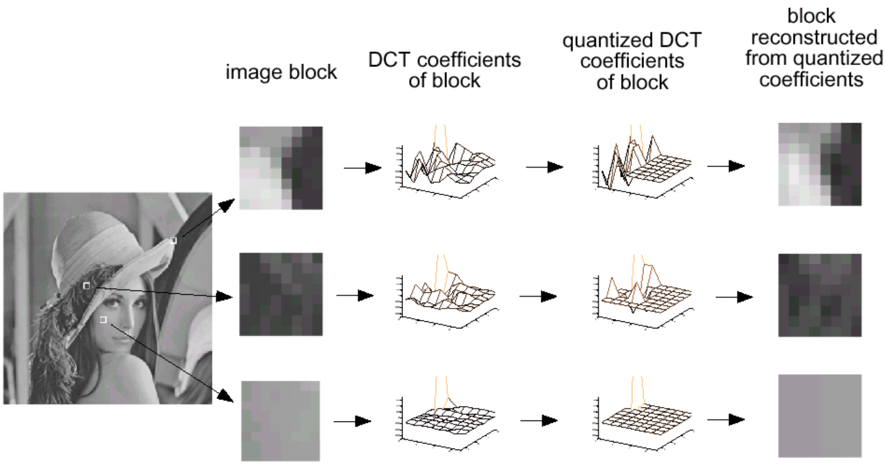


Illustration of Image Coding Using 2-D DCT



DCT-Based Image Coding with Different Quantization Levels

- DCT coding with increasingly coarse quantization, block size 8x8

quantizer step-size for AC coefficient: 25

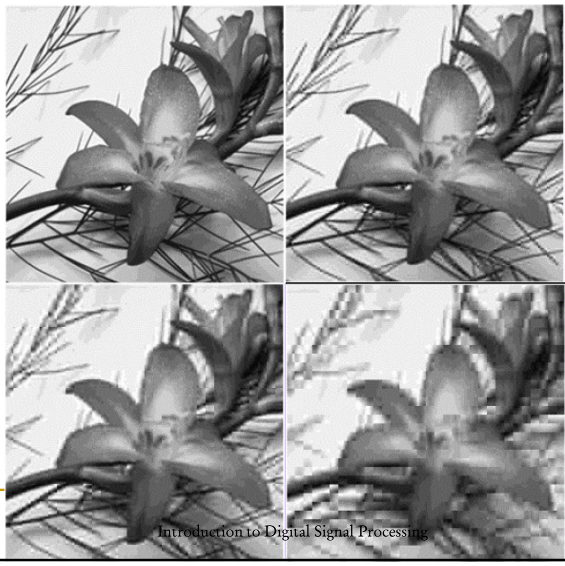
quantizer step-size for AC coefficient: 100

quantizer step-size for AC coefficient: 200

Image Coding with Different Numbers of DCT Coefficients

original

with 16/64 coefficients



with 8/64 coefficients

with 4/64 coefficients

DCT-Based Coding: JPEG