Chapter 4

Frequency Analysis of Signals

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Frequency Analysis of Signals

- The **Fourier transform** is a mathematical tool that is useful in the analysis and design of LTI systems
- These signal representations basically involve the decomposition of the signals in terms of sinusoidal (or complex exponential) components
- With such a decomposition, a signal is said to be represented in the frequency domain
- Most signals of practical interest can be decomposed into a sum of sinusoidal signal components
  - For the class of periodic signals, such a decomposition is called a **Fourier series**
  - For the class of finite energy signals, the decomposition is called the **Fourier transform**

The Father of Fourier Transform

Joseph Fourier

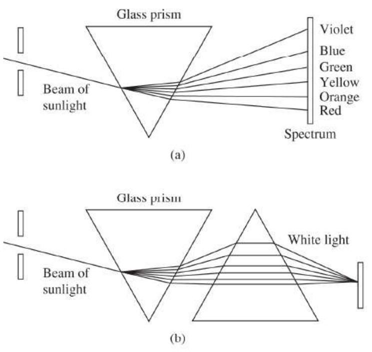
lived from 1768 to 1830

Fourier studied the mathematical theory of heat conduction. He established the partial differential equation governing heat diffusion and solved it by using infinite series of trigonometric functions.

Find out more at:
http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Fourier.html

Frequency Analysis of Continuous-Time Signals

- A prism can be used to break up white light (sunlight) into the colors of the rainbow.



Analysis

Synthesis

(a) Analysis and (b) synthesis of the white light (sunlight) using glass prisms.

Frequency Analysis of Continuous-Time Signals

- Frequency analysis of a signal involves the resolution of the signal into its frequency (sinusoidal) components
- Signal waveforms are basically functions of time
- The role of the prism is played by the **Fourier analysis** tools that we will develop: the **Fourier series** and the **Fourier transform**
- Examples of periodic signals encountered in practice are square waves, rectangular waves, triangular waves, sinusoids and complex exponentials

The Fourier Series for Continuous – Time Periodic Signals

- The basic mathematical representation of periodic signals is the **Fourier series**, which is a **linear weighted sum** of harmonically related sinusoids or complex exponentials

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t}$$

A periodic signal

  - Fundamental period: $T_p = \frac{1}{F_0}$

- F_0 determines the fundamental period of $x(t)$ and the coefficients $\{c_k\}$ specify the shape of the waveform

The Fourier Series for Continuous – Time Periodic Signals

- Suppose that we are given a periodic signal $x(t)$ with period T_p. We can represent the periodic signal, called a **Fourier series**
- The fundamental frequency F_0 is selected to be the reciprocal of the given period T_p
- To determine the expression for the coefficients $\{c_k\}$, we first multiply both sides of

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t}$$

by the complex exponential $e^{-j2\pi F_0 l t}$

where l is an integer

The Fourier Series for Continuous – Time Periodic Signals

- Signal period, from $t_0 + T_p$. Where t_0 is an arbitrary but mathematically convenient starting value. Thus we obtain

$$\int_{t_0}^{t_0+T_p} x(t) e^{-j2\pi l F_0 t} dt = \int_{t_0}^{t_0+T_p} e^{-j2\pi l F_0 t} \left(\sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t} \right) dt$$

- We interchange the order of the summation and integration and combine the two exponentials. Hence

$$\sum_{k=-\infty}^{\infty} c_k \int_{t_0}^{t_0+T_p} e^{j2\pi F_0 (k-l)t} dt = \sum_{k=-\infty}^{\infty} c_k \left[\frac{e^{j2\pi F_0 (k-l)t}}{j2\pi F_0 (k-l)} \right]_{t_0}^{t_0+T_p}$$

$$= \sum_{k=-\infty}^{\infty} c_k T_p \delta(k-l) = c_l T_p$$

The Fourier Series for Continuous – Time Periodic Signals

- We can get $\int_{t_0}^{t_0+T_p} x(t)e^{-j2\pi lF_0 t}dt = c_l T_p$
- Therefore $c_l = \frac{1}{T_p}\int_{t_0}^{t_0+T_p} x(t)e^{-j2\pi lF_0 t}dt$
- The integral for the Fourier series coefficients will be written as $c_l = \frac{1}{T_p}\int_{T_p} x(t)e^{-j2\pi lF_0 t}dt$
- That is, the signal $x(t)$ and its Fourier series representation

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kF_0 t}$$

are equal at every value of t

The Fourier Series for Continuous – Time Periodic Signals

- Dirichlet conditions:
  - The signal $x(t)$ has a finite number of discontinuities in any period
  - The signal $x(t)$ contains a finite number of maxima and minima during any period
  - The signal $x(t)$ is absolutely integrable in any period, that is

$$\int_{T_p} |x(t)|dt < \infty$$

  - All periodic signals of practical interest satisfy these conditions

The Fourier Series for Continuous – Time Periodic Signals

- The weaker condition: signal has finite energy in one period,

$$\int_{T_p} |x(t)|^2 dt < \infty$$

guarantees that the energy in the difference signal

$$e(t) = x(t) - \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kF_0 t}$$

is zero, although $x(t)$ and its Fourier series may not be equal for all value of t

- Frequency Analysis of Continuous-Time Periodic Signals

Synthesis equation	$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kF_0 t}$
Analysis equation	$c_k = \frac{1}{T_p} \int_{T_p} x(t) e^{-j2\pi kF_0 t} dt$

The Fourier Series for Continuous – Time Periodic Signals

- In general, the Fourier coefficients c_k are complex valued.
- If the periodic signal is real, c_k and c_{-k} are complex conjugates.
- As a result, if $c_k = |c_k| e^{j\theta_k}$

  then $c_{-k} = |c_k| e^{-j\theta_k}$

- Consequently, the Fourier series may also be represented in the form

$$x(t) = c_0 + 2\sum_{k=1}^{\infty} |c_k| \cos(2\pi kF_0 t + \theta_k)$$

where c_0 is real valued when $x(t)$ is real

The Fourier Series for Continuous – Time Periodic Signals

- Finally, we should indicate that yet another form for the Fourier series can be obtained by expanding the cosine function above page as

$$\cos(2\pi kF_0 t + \theta_k) = \cos(2\pi kF_0 t)\cos(\theta_k) - \sin(2\pi kF_0 t)\sin(\theta_k)$$

- Then, we can obtain

$$x(t) = a_0 + \sum_{k=1}^{\infty}(a_k \cos(2\pi kF_0 t) - b_k \sin(2\pi kF_0 t))$$

  - where

$$a_0 = c_0, \quad a_k = 2|c_k|\cos\theta_k, \quad b_k = 2|c_k|\sin\theta_k$$

Power Density Spectrum of Periodic Signals

- A periodic signal has infinite energy and a finite average power, which is given as

$$P_x = \frac{1}{T_p}\int_{T_p} |x(t)|^2 \, dt$$

- If we take the complex conjugate and substitute for $x^*(t)$, we obtain

$$P_x = \frac{1}{T_p}\int_{T_p} x(t)\sum_{k=-\infty}^{\infty} c_k^* e^{-j2\pi kF_0 t} dt = \sum_{k=-\infty}^{\infty} c_k^* \left[\frac{1}{T_p}\int_{T_p} x(t)e^{-j2\pi kF_0 t} dt\right] = \sum_{k=-\infty}^{\infty} |c_k|^2$$

Power Density Spectrum of Periodic Signals

- Therefore, we have established the relation

$$P_x = \frac{1}{T_p}\int_{T_p} |x(t)|^2\, dt = \sum_{k=-\infty}^{\infty} |c_k|^2$$

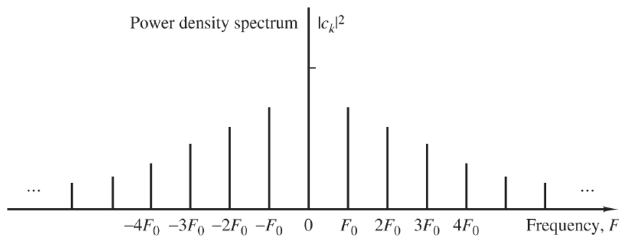
which is called **Parseval's relation** for power signals.

- Suppose that $x(t)$ consists of a single complex exponential $x(t) = c_k e^{j2\pi k F_0 t}$
- In this case, all the Fourier series coefficients except c_k are zero
- The average power in the signal is

$$P_x = |c_k|^2$$

Power Density Spectrum of Periodic Signals

- $|c_k|^2$ represents the power in the k-th harmonic component of the signal
- Hence the total average power in the periodic signal is simply the sum of the average powers in all the harmonics.
- Power density spectrum of a continuous-time periodic signal



Power Density Spectrum of Periodic Signals

- The Fourier series coefficients $\{c_k\}$ are complex valued, that is, they can be represented as

$$c_k = |c_k| e^{j\theta_k}$$

where $\theta_k = \measuredangle c_k$

- If the periodic signal is real valued, the Fourier series coefficients $\{c_k\}$ satisfy the condition

$$c_{-k} = c_k^*$$

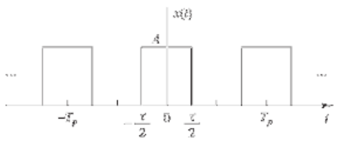
- Consequently $|c_k|^2 = |c_k^*|^2$

Power Density Spectrum of Periodic Signals

- The total average power can be expressed as

$$P_x = c_0^2 + 2\sum_{k=1}^{\infty} |c_k|^2 = a_0^2 + \frac{1}{2}\sum_{k=1}^{\infty}(a_k^2 + b_k^2)$$

- **Example**: Determine the Fourier series and the power density spectrum of the rectangular pulse train signal.



Continuous-time periodic train of rectangular pulses.

Power Density Spectrum of Periodic Signals

- **Solution**: The signal is periodic with fundamental period T_p and, clearly, satisfies the Dirichlet conditions. So the signal can be represented in the Fourier series
- As $x(t)$ is an even signal (i.e., $x(t) = x(-t)$), it is convenient to select the integration interval from $-T_p/2$ to $T_p/2$

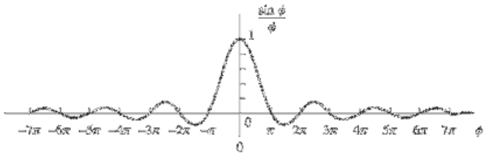
$$c_0 = \frac{1}{T_p}\int_{-\frac{T_p}{2}}^{\frac{T_p}{2}} x(t)dt = \frac{1}{T_p}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} A dt = \frac{A\tau}{T_p}$$

- The term c_0 represents the average value (DC component) of the signal $x(t)$. For $k \neq 0$ we have

$$c_k = \frac{1}{T_p}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} Ae^{-j2\pi kF_0 t}dt = \frac{A\tau}{T_p}\frac{\sin \pi kF_0\tau}{\pi kF_0\tau}, \qquad k = \pm 1, \pm 2, \ldots$$

Power Density Spectrum of Periodic Signals

- The function $\frac{\sin\phi}{\phi}$

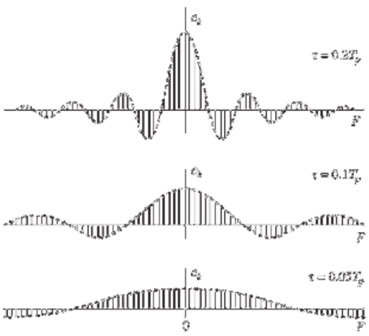


The function (sin ϕ)/ϕ.

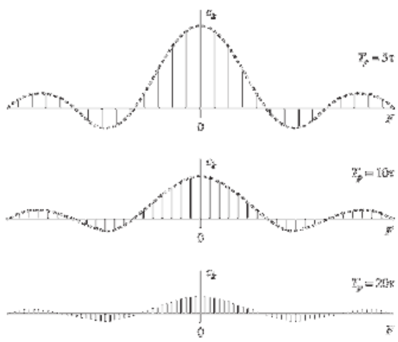
- The power density spectrum for the rectangular pulse train is

$$|c_k|^2 = \begin{cases} \left(\frac{A\tau}{T_p}\right)^2, & k = 0 \\ \left(\frac{A\tau}{T_p}\right)^2 \left(\frac{\sin \pi kF_0\tau}{\pi kF_0\tau}\right)^2, & k = \pm 1, \pm 2, \ldots \end{cases}$$

Power Density Spectrum of Periodic Signals



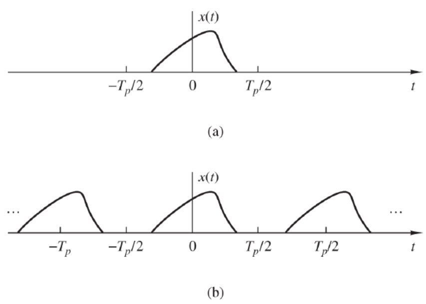
Fourier coefficients of the rectangular pulse train when T_p is fixed and the pulse width τ varies.



Fourier coefficient of a rectangular pulse train with fixed pulse width τ and varying period T_p.

The Fourier Transform for Continuous-Time Aperiodic Signals

- Consider an aperiodic signal $x(t)$ with finite duration. From this aperiodic signal, we can create a periodic signal $x_p(t)$ with period T_p, as shown below



(a) Aperiodic signal $x(t)$ and (b) periodic signal $x_p(t)$ constructed by repeating $x(t)$ with a period T_p.

The Fourier Transform for Continuous-Time Aperiodic Signals

- Clearly, $x_p(t) = x(t)$ in the limit as $T_p \to \infty$, that is
$$x(t) = \lim_{T_p \to \infty} x_p(t)$$
- This interpretation implies that we should be able to obtain the spectrum of $x(t)$ from the spectrum of $x_p(t)$ simply by taking the limit as $T_p \to \infty$
- We begin with the Fourier series representation of $x_p(t)$
$$x_p(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kF_0 t} \qquad c_k = \frac{1}{T_p}\int_{-\frac{T_p}{2}}^{\frac{T_p}{2}} x_p(t) e^{-j2\pi kF_0 t} dt$$
- Since $x_p(t) = x(t)$ for $-T_p/2 \le t \le T_p/2$
$$c_k = \frac{1}{T_p}\int_{-\frac{T_p}{2}}^{\frac{T_p}{2}} x(t) e^{-j2\pi kF_0 t} dt$$

The Fourier Transform for Continuous-Time Aperiodic Signals

- Consequently, the limits on the integral above can be replaced by $-\infty$ and ∞. Hence
$$c_k = \frac{1}{T_p}\int_{-\infty}^{\infty} x(t) e^{-j2\pi kF_0 t} dt$$
- Let us now define a function $X(F)$, called the **Fourier transform** of $x(t)$, as
$$X(F) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi Ft} dt$$
- $X(F)$ is a function of the continuous variable F
- The Fourier coefficients c_k can be expressed in terms of $X(F)$
$$c_k = \frac{1}{T_p} X(kF_0) \quad \text{or equivalently} \quad T_p c_k = X(kF_0) = X\left(\frac{k}{T_p}\right)$$

The Fourier Transform for Continuous-Time Aperiodic Signals

- From above, we can obtain

$$x_p(t) = \frac{1}{T_p}\sum_{k=-\infty}^{\infty} X\left(\frac{k}{T_p}\right)e^{j2\pi kF_0 t}$$

- We define $\Delta F = \frac{1}{T_p}$, then $x_p(t) = \sum_{k=-\infty}^{\infty} X\left(k\Delta F\right)e^{j2\pi k\Delta Ft}\Delta F$

- The limit as T_p approaches infinity, $x_p(t)$ reduces to $x(t)$

$$\lim_{T_p\to\infty} x_p(t) = x(t) = \lim_{\Delta F\to 0}\sum_{k=-\infty}^{\infty} X\left(k\Delta F\right)e^{-j2\pi k\Delta Ft}\Delta F$$

- This integral relationship yields $x(t)$ when $X(F)$ is known, and it is called the **inverse Fourier transform**

$$x(t) = \int_{-\infty}^{\infty} X\left(F\right)e^{j2\pi Ft}dF$$

The Fourier Transform for Continuous-Time Aperiodic Signals

- Frequency Analysis of Continuous-Time Aperiodic Signals

Synthesis equation (inverse transform)	$x(t) = \int_{-\infty}^{\infty} X\left(F\right)e^{j2\pi Ft}dF$
Analysis equation (direct transform)	$X(F) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi Ft}dt$

- Let $\Omega = 2\pi F$, since $dF = d\Omega/2\pi$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X\left(\Omega\right)e^{j\Omega t}d\Omega$$

$$X(\Omega) = \int_{-\infty}^{\infty} x\left(t\right)e^{-j\Omega t}dt$$

The Fourier Transform for Continuous-Time Aperiodic Signals

- Dirichlet conditions:
  - The signal $x(t)$ has a finite number of discontinuities.
  - The signal $x(t)$ has a finite number of maxima and minima.
  - The signal $x(t)$ is absolutely integrable, that is

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty$$

The Fourier Transform for Continuous-Time Aperiodic Signals

- The third condition follows easily from the definition of the Fourier transform

$$|X(F)| = \left| \int_{-\infty}^{\infty} x(t) e^{-j2\pi Ft} dt \right| \le \int_{-\infty}^{\infty} |x(t)| dt$$

- Hence $|X(F)| < \infty$
- A weaker condition for the existence of the Fourier transform is that $x(t)$ has finite energy; that is,

$$\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$$

The Fourier Transform for Continuous-Time Aperiodic Signals

- Note that if a signal $x(t)$ is absolutely integrable, it will also have finite energy. That is, if

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty$$

then $E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$

- For example, the signal $x(t) = \frac{\sin 2\pi F_0 t}{\pi t}$ is a square integrable but is not absolutely integrable. This signal has the Fourier transform

$$X(F) = \begin{cases} 1, & |F| \le F_0 \\ 0, & |F| > F_0 \end{cases}$$

Energy Density Spectrum of Aperiodic Signals

- Let *x(t)* be any finite energy signal with Fourier transform *X(F)*. Its energy is

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

- To express *X(F)* as follow

$$E_x = \int_{-\infty}^{\infty} x(t) x^*(t) dt = \int_{-\infty}^{\infty} x(t) dt \left[\int_{-\infty}^{\infty} X^*(F) e^{-j2\pi Ft} dF \right]$$

$$= \int_{-\infty}^{\infty} X^*(F) dF \left[\int_{-\infty}^{\infty} x(t) e^{-j2\pi Ft} dt \right]$$

$$= \int_{-\infty}^{\infty} |X(F)|^2 dF$$

Parseval's relation for aperiodic signal

Energy Density Spectrum of Aperiodic Signals

- Therefore, we conclude that

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(F)|^2 dF$$

- Polar form $X(F) = |X(F)| e^{j\Theta(F)}$
- Magnitude $|X(F)|$, phase spectrum $\Theta(F) = \measuredangle X(F)$
- Energy density spectrum

$$S_{xx}(F) = |X(F)|^2$$

- Finally, as in the case of Fourier series, it is easily shown that if the signal $x(t)$ is real, then

$$|X(-F)| = |X(F)|, \quad \measuredangle X(-F) = -\measuredangle X(F)$$

- We obtain $S_{xx}(-F) = S_{xx}(F)$

The Fourier Series for Discrete-Time Periodic Signals

- A periodic sequence $x(n)$ with period N, that is,

$$x(n) = x(n+N), \quad \forall n$$

- The Fourier series representation for $x(n)$ consists of N harmonically related exponential functions

$$e^{j\frac{2\pi kn}{N}}, \quad k = 0, 1, \ldots, N-1$$

- And is expressed as $x(n) = \sum_{k=0}^{N-1} c_k e^{\frac{j2\pi kn}{N}}$
- The $\{c_k\}$ are the coefficients in the series representation.

The Fourier Series for Discrete-Time Periodic Signals

- To derive the expression for the Fourier coefficients, we use the following formula:

$$\sum_{n=0}^{N-1} e^{j\frac{2\pi kn}{N}} = \begin{cases} N, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & otherwise \end{cases}$$

- The geometric summation formula

$$\sum_{n=0}^{N-1} a^n = \begin{cases} N, & a = 1 \\ \frac{1-a^N}{1-a}, & a \neq 1 \end{cases}$$

The Fourier Series for Discrete-Time Periodic Signals

- The expression for the Fourier coefficients c_k can be obtained by multiplying both sides of

$$x(n) = \sum_{k=0}^{N-1} c_k e^{\frac{j2\pi kn}{N}}$$

by the exponential $e^{-j\frac{2\pi ln}{N}}$ and summing the product from n=0 to $n = N - 1$. Thus

$$\sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi ln}{N}} = \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} c_k e^{j\frac{2\pi(k-l)n}{N}}$$

The Fourier Series for Discrete-Time Periodic Signals

- If we perform the summation over *n* first, we obtain

$$\sum_{n=0}^{N-1} e^{j\frac{2\pi(k-l)n}{N}} = \begin{cases} N, & k-1=0, \pm N, \pm 2N, \ldots \\ 0, & otherwise \end{cases}$$

- Hence $c_l = \frac{1}{N}\sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi ln}{N}}, \quad l = 0,1,\ldots,N-1$
- Frequency Analysis of Discrete-Time Periodic Signals

Synthesis equation	$x(n) = \sum_{k=0}^{N-1} c_k e^{j\frac{2\pi kn}{N}}$	Discrete-time Fourier Series (DTFS)
Analysis equation	$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi kn}{N}}$	

Power Density Spectrum of Periodic Signals

- The average power of a discrete-time periodic signal with period *N* was defined as

$$P_x = \frac{1}{N}\sum_{n=0}^{N-1} |x(n)|^2$$

- We shall now derive an expression for P_x in terms of the Fourier coefficient $\{c_x\}$. We have

$$P_x = \frac{1}{N}\sum_{n=0}^{N-1} x(n)x^*(n) = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\left(\sum_{k=0}^{N-1} c_k^* e^{-j\frac{2\pi kn}{N}}\right)$$

- Then $P_x = \sum_{k=0}^{N-1} c_k^* \left[\frac{1}{N}\sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi kn}{N}}\right] = \sum_{k=0}^{N-1} |c_k|^2 = \frac{1}{N}\sum_{n=0}^{N-1} |x(n)|^2$
- If we are interested in the energy of the sequence $x(n)$ over a single period, it implies that $E_N = \sum_{n=0}^{N-1} |x(n)|^2 = N\sum_{k=0}^{N-1} |c_k|^2$

Power Density Spectrum of Periodic Signals

- If the signal $x(n)$ is real, then, we can easily show that

$$c_k^* = c_{-k}$$

- Or equivalently $|c_{-k}| = |c_k|$ Even symmetry

$-\measuredangle c_{-k} = \measuredangle c_k$ Odd symmetry

- We obtain

$$|c_k| = |c_{N-k}|$$
$$\measuredangle c_k = -\measuredangle c_{N-k}$$

- More specifically, we have

$$|c_0| = |c_N|, \quad \measuredangle c_0 = -\measuredangle c_N = 0$$
$$|c_1| = |c_{N-1}|, \quad \measuredangle c_1 = -\measuredangle c_{N-1}$$
$$|c_{N/2}| = |c_{N/2}|, \quad \measuredangle c_{N-2} = 0, \quad \text{if } N \text{ is even}$$
$$|c_{(N-1)/2}| = |c_{(N+1)/2}|, \quad \measuredangle c_{(N-1)/2} = -\measuredangle c_{(N+1)/2}, \quad \text{if } N \text{ is odd}$$

Power Density Spectrum of Periodic Signals

- The Fourier series can also be expressed in the alternative forms

$$x(n) = c_0 + 2\sum_{k=1}^{L} |c_k| \cos\left(\frac{2\pi}{N}kn + \theta_k\right)$$
$$= a_0 + \sum_{k=1}^{L}\left(a_k \cos\frac{2\pi}{N}kn - b_k \sin\frac{2\pi}{N}kn\right)$$

- Where $a_0 = c_0$, $a_k = 2|c_k|\cos\theta_k$, $b_k = 2|c_k|\sin\theta_k$

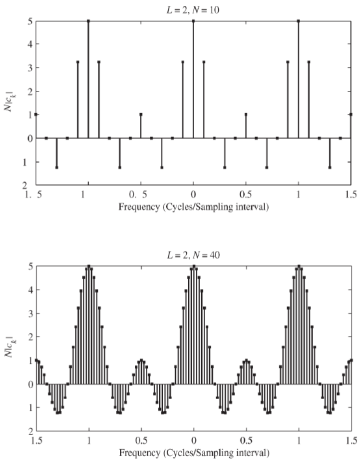
$$L = \begin{cases} N/2 & \text{If } N \text{ is even} \\ (N-1)/2 & \text{If } N \text{ is odd} \end{cases}$$

The Fourier Transform of Discrete-Time Aperiodic Signals

- The Fourier transform of a finite-energy discrete-time signal $x(n)$ is defined as

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

- $X(\omega)$ represents the frequency content of the signal $x(n)$
- $X(\omega)$ is a decomposition of $x(n)$ into its frequency components.



The Fourier Transform of Discrete-Time Aperiodic Signals

- $X(\omega)$ is periodic with period 2π, that is

$$X(\omega + 2\pi k) = \sum_{n=-\infty}^{\infty} x(n)e^{-j(\omega+2\pi k)n} = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}e^{-j2\pi kn}$$

$$= \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} = X(\omega)$$

- Hence $X(\omega)$ is periodic with period 2π
- Let us evaluate the sequence $x(n)$ from $X(\omega)$. Thus we have

$$\int_{-\pi}^{\pi} X(\omega)e^{j\omega m}d\omega = \int_{-\pi}^{\pi}\left[\sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}\right]e^{j\omega m}d\omega$$

- This interchange can be made if the series

$$X_N(\omega) = \sum_{n=-N}^{N} x(n)e^{-j\omega n}$$

- converges uniformly to $X(\omega)$ as $N \to \infty$

The Fourier Transform of Discrete-Time Aperiodic Signals

- We can interchange the order of summation and integration

$$\int_{-\pi}^{\pi} e^{j\omega(m-n)} d\omega = \begin{cases} 2\pi, & m = n \\ 0, & m \neq n \end{cases}$$

- Consequently,

$$\sum_{n=-\infty}^{\infty} x(n) \int_{-\pi}^{\pi} e^{j\omega(m-n)} d\omega = \begin{cases} 2\pi x(m), & m = n \\ 0, & m \neq n \end{cases}$$

- The desired result that

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

Synthesis equation (inverse transform)	$x(n) = \frac{1}{2\pi} \int_{2\pi} X(\omega) e^{j\omega n} d\omega$
Analysis equation (direct transform)	$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$

Convergence of the Fourier Transform

- We assume that the series $X_N(\omega) = \sum_{n=-N}^{N} x(n) e^{-j\omega n}$ converges uniformly to. By uniform convergence we mean that for each ω

$$\lim_{N\to\infty} \left\{ \sup_{\omega} |X(\omega) - X_N(\omega)| \right\} = 0$$

- Uniform convergence is guaranteed if *x*(*n*) is absolutely summable. If $\sum_{n=-\infty}^{\infty} |x(n)| < \infty$

  then $|X(\omega)| = \left| \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \right| \leq \sum_{n=-\infty}^{\infty} |x(n)| < \infty$

- Some sequences are not absolutely summable, but they are square summable. That is, they have finite energy

$$E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2 < \infty$$

Convergence of the Fourier Transform

- This leads to a mean-square convergence condition:

$$E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2 < \infty$$

the energy in the error $X(\omega) - X_N(\omega)$ tends toward zero

$$\lim_{N\to\infty} \int_{-\pi}^{\pi} |X(\omega) - X_N(\omega)|^2 \, d\omega = 0$$

- **Example** - Suppose that

$$X(\omega) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & \omega_c \le |\omega| \le \pi \end{cases}$$

- $X(\omega)$ is periodic with period 2π
- The inverse transform of $X(\omega)$ results in the sequence

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega = \frac{\sin \omega_c n}{\pi n}, \qquad n \ne 0$$

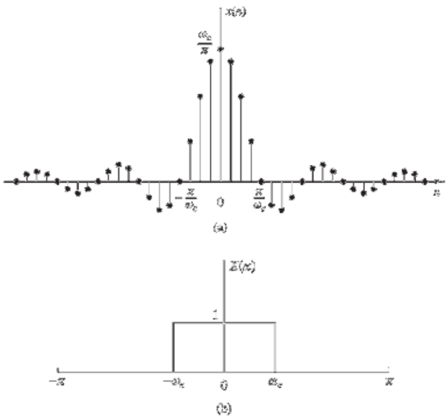
Convergence of the Fourier Transform

- For $n = 0$, we have

$$x(0) = \frac{1}{2\pi}\int^{\omega_c} -\omega_c d\omega = \frac{\omega_c}{\pi}$$

- Hence

$$x(n) = \begin{cases} \dfrac{\omega_c}{\pi}, & n = 0 \\ \dfrac{\omega_c}{\pi}\dfrac{\sin \omega_c n}{\omega_c n}, & n \ne 0 \end{cases}$$



- The sequence $\{x(n)\}$ is expressed as

$$x(n) = \frac{\sin \omega_c n}{\pi n}, \qquad -\infty < n < \infty$$

Convergence of the Fourier Transform

- Note, the above he sequence $\{x(n)\}$ is not absolutely summable. Hence the infinite series
$$\sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$$
does not converge uniformly for all ω
- To elaborate on this point, let us consider the finite sum
$$X_N(\omega) = \sum_{n=-N}^{N} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$$
- Right figures shows the function $X_N(\omega)$ for several values of N.

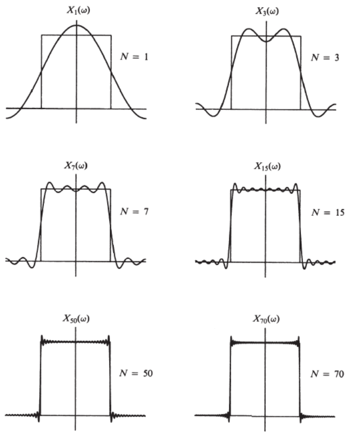


Illustration of convergence of the Fourier transform and the Gibbs phenomenon at the point of discontinuity.

Energy Density Spectrum of Aperiodic Signals

- The energy of a discrete-time signal $x(n)$ is defined as
$$E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2$$
- The energy E_x in terms of the spectral characteristic $X(\omega)$, first we have
$$E_x = \sum_{n=-\infty}^{\infty} x^*(n)x(n) = \sum_{n=-\infty}^{\infty} x(n)\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(\omega)e^{-j\omega n} d\omega\right]$$
- Then
$$E_x = \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(\omega)\left[\sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}\right] d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 \, d\omega$$

Energy Density Spectrum of Aperiodic Signals

- The energy relation between $x(n)$ and $X(\omega)$ is

$$E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 \, d\omega$$

  this is Parseval's relation for discrete-time aperiodic signals with finite energy.

- The spectrum $X(\omega)$ is, in general, a complex-valued function of frequency. It may be expressed as

$$X(\omega) = |X(\omega)| e^{j\Theta(\omega)}$$

where $\Theta(\omega) = \measuredangle X(\omega)$ phase spectrum

Energy Density Spectrum of Aperiodic Signals

- As in the case of continuous-time signals, the quantity

$$S_{xx}(\omega) = |X(\omega)|^2$$

  represents the distribution of energy as a function of frequency, namely the **energy density spectrum** of $x(n)$
- Clearly, $S_{xx}(\omega)$ does not contain any phase information
- If $x(n)$ is real, then it easily follows that

$$X^*(\omega) = X(-\omega)$$

- or equivalently $|X(-\omega)| = |X(\omega)|$ even symmetry

  $\measuredangle X(-\omega) = -\measuredangle X(\omega)$ odd symmetry
- It also follows that $S_{xx}(-\omega) = S_{xx}(\omega)$ even symmetry

Energy Density Spectrum of Aperiodic Signals

- **Example** - Determine and sketch the energy density spectrum $S_{xx}(\omega)$ of the signal

$$x(n) = a^n u(n), \qquad -1 < a < 1$$

- **Solution**:
  - Since $|a| < 1$, the sequence $x(n)$ is absolutely summable, as can be verified by applying the geometric summation formula,

$$\sum_{n=-\infty}^{\infty} |x(n)| = \sum_{n=0}^{\infty} |a|^n = \frac{1}{1-|a|} < \infty$$

  - Hence the Fourier transform of $x(n)$ exists and is obtained. Thus

$$x(\omega) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n$$

Energy Density Spectrum of Aperiodic Signals

- Since $|ae^{-j\omega}| = |a| < 1$, use of the geometric summation formula again yields

$$x(\omega) = \frac{1}{1-ae^{-j\omega}}$$

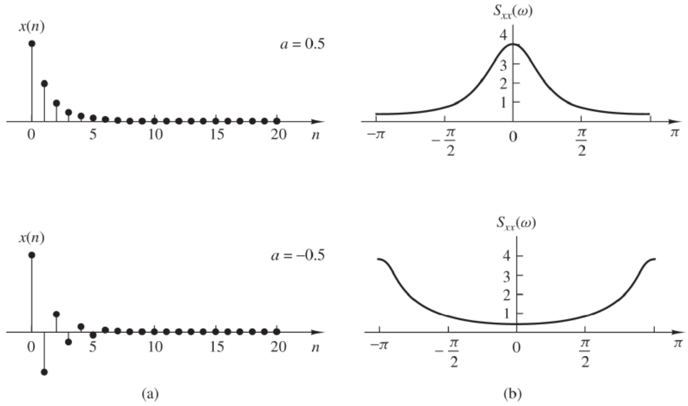
- The energy density spectrum is given by

$$S_{xx}(\omega) = |X(\omega)|^2 = X(\omega)X(\omega) = \frac{1}{(1-ae^{-j\omega})(1-ae^{j\omega})}$$

- Equivalently $S_{xx}(\omega) = \dfrac{1}{1-2a\cos\omega + a^2}$

- Note that $S_{xx}(-\omega) = S_{xx}(\omega)$

Energy Density Spectrum of Aperiodic Signals



(a) Sequence $x(n) = (\frac{1}{2})^n u(n)$ and $x(n) = (-\frac{1}{2})^n u(n)$; (b) their energy density spectra.

Relationship of the Fourier Transform to the z-Transform

- The z-transform of a sequence *x*(*n*) is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}, \qquad ROC: \ r_2 < |z| < r_1$$

where $r_2 < |z| < r_1$ is the region of convergence of *X*(*z*)

- Let us express the complex variable *z* in polar form as

$$z = re^{j\omega}$$

where $r = |z|$ and $\omega = \measuredangle z$.

Relationship of the Fourier Transform to the z-Transform

- Then, within the region of convergence of $X(z)$, we can substitute $z = re^{j\omega}$ in above

$$X(z)|_{z=re^{j\omega}} = \sum_{n=-\infty}^{\infty} \left[x(n) r^{-n} \right] e^{-j\omega n}$$

- If $X(z)$ converges for $|z| = 1$, then

$$X(z)|_{z=e^{j\omega}} \equiv X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

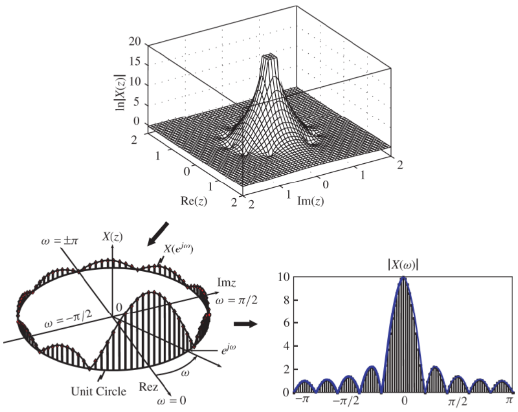
Relationship of the Fourier Transform to the z-Transform

- The existence of the z-transform requires that the sequence

$$\left\{ x(n) r^{-n} \right\}$$

be absolutely summable for some value of r, that is

$$\left| \sum_{n=-\infty}^{\infty} x(n) r^{-n} \right| < \infty$$



relationship between $X(z)$ and $X(\omega)$ for the sequence in Example 4.2.4, with $A = 1$ and $L = 10$

Relationship of the Fourier Transform to the z-Transform

- There are sequences, however, that do not satisfy the requirement in above equation, for example, the sequence

$$x(n) = \frac{\sin \omega_c n}{\pi n}, \qquad -\infty < n < \infty$$

- This sequence does not have a z-transform. Since it has a finite energy, its Fourier transform converges in the mean-square sense to the discontinuous function $X(\omega)$, defined as

$$X(\omega) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

The Fourier Transform of Signals with Poles on the Unit Circle

- There are some aperiodic sequences that are neither absolutely summable nor square summable. Hence their Fourier transforms do not exist. One such sequence is the unit step sequence, which has the z-transform

$$X(z) = \frac{1}{1 - z^{-1}}$$

- Another such sequence is the causal sinusodial signal sequence $x(n) = (\cos \omega_0 n) u(n)$. This sequence has the z-transform

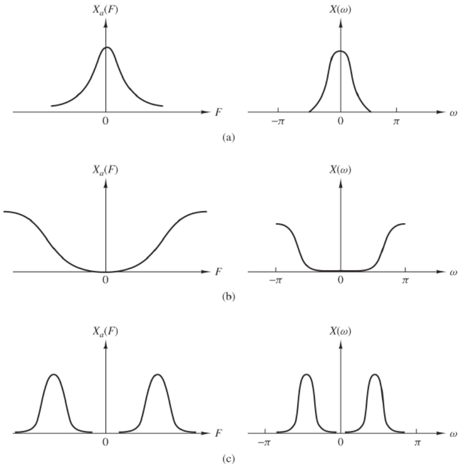
$$X(z) = \frac{1 - z^{-1} \cos \omega_0}{1 - 2z^{-1} \cos \omega_0 + z^{-2}}$$

Frequency-Domain Classification of Signals: The concept of Bandwidth

Low-frequency

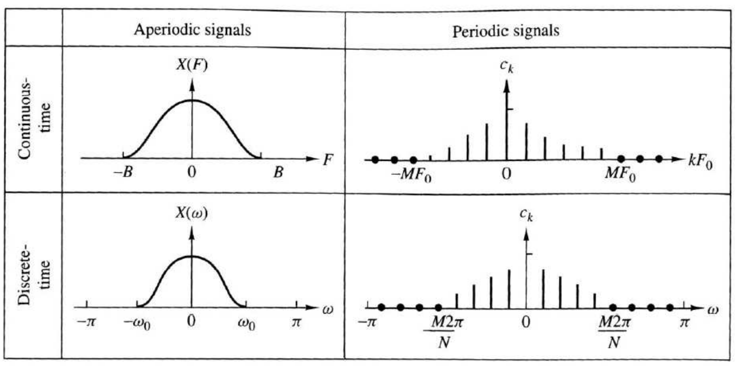
High-frequency

Medium-frequency



Frequency-Domain Classification of Signals: The concept of Bandwidth

- Some examples of bandlimited signals



The Frequency Ranges of Some Natural Signals

- Frequency ranges of some biological signals

Type of Signal	Frequency Range (Hz)
Electroretinogram[a]	0–20
Electronystagmogram[b]	0–20
Pneumogram[c]	0–40
Electrocardiogram (ECG)	0–100
Electroencephalogram (EEG)	0–100
Electromyogram[d]	10–200
Sphygmomanogram[e]	0–200
Speech	100–4000

[a] A graphic recording of retina characteristics.
[b] A graphic recording of involuntary movement of the eyes.
[c] A graphic recording of respiratory activity.
[d] A graphic recording of muscular action, such as muscular contraction.
[e] A recording of blood pressure.

The Frequency Ranges of Some Natural Signals

- Frequency ranges of some seismic signals

Type of Signal	Frequency Range (Hz)
Wind noise	100–1000
Seismic exploration signals	10–100
Earthquake and nuclear explosion signals	0.01–10
Seismic noise	0.1–1

The Frequency Ranges of Some Natural Signals

- Frequency ranges of electromagnetic signals

Type of Signal	Wavelength (m)	Frequency Range (Hz)
Radio broadcast	10^4–10^2	3×10^4–3×10^6
Shortwave radio signals	10^2–10^{-2}	3×10^6–3×10^{10}
Radar, satellite communications, space communications, common-carrier microwave	1–10^{-2}	3×10^8–3×10^{10}
Infrared	10^{-3}–10^{-6}	3×10^{11}–3×10^{14}
Visible light	3.9×10^{-7}–8.1×10^{-7}	3.7×10^{14}–7.7×10^{14}
Ultraviolet	10^{-7}–10^{-8}	3×10^{15}–3×10^{16}
Gamma rays and X rays	10^{-9}–10^{-10}	3×10^{17}–3×10^{18}

Frequency-Domain and Time-Domain Signal Properties

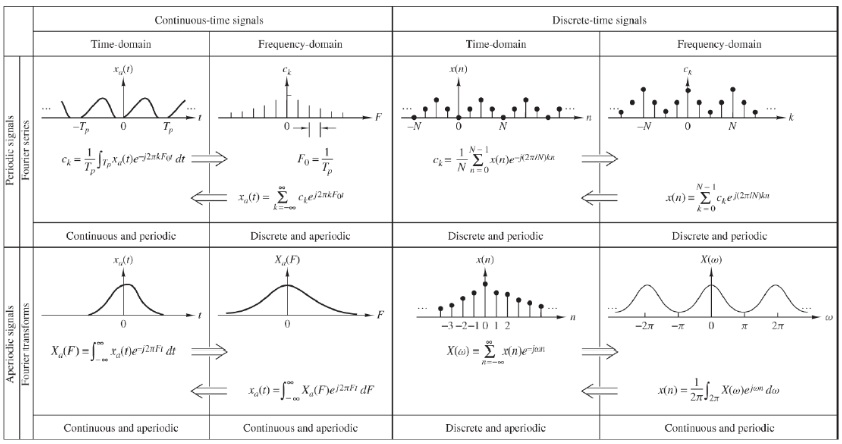
- To summarize, the following frequency analysis tools have been introduced:
  - The Fourier series for continuous-time periodic signals.
  - The Fourier transform for continuous-time aperiodic signals.
  - The Fourier series for discrete-time periodic signals.
  - The Fourier transform for discrete-time aperiodic signals.

Frequency-Domain and Time-Domain Signal Properties

- Let us briefly summarize the results of the previous sections.
  - Continuous-time signals have aperiodic spectra.
  - Discrete-time signals have periodic spectra.
  - Periodic signals have discrete spectra.
  - Aperiodic finite energy signals have continuous spectra.

Frequency-Domain and Time-Domain Signal Properties

- Summary of analysis and synthesis formulas



Frequency-Domain and Time-Domain Signal Properties

- We observe that there are dualities between the following analysis and synthesis equations:
  - The analysis and synthesis equations of the continuous-time Fourier transform.
  - The analysis and synthesis equations of the discrete-time Fourier series.
  - The analysis equation of the continuous-time Fourier series and the synthesis equation of the discrete-time Fourier transform.
  - The analysis equation of the discrete-time Fourier transform and the synthesis equation of the continuous-time Fourier series.

Properties of the Fourier Transform for Discrete-Time Signals

- The Fourier transform for aperiodic finite-energy discrete-time signals described in the preceding section possesses a number of properties that are very useful in reducing the complexity of frequency analysis problems in many practical applications.

- We develop the important properties of the Fourier transform. Similar properties hold for the Fourier transform of aperiodic finite-energy continuous-time signals.

Properties of the Fourier Transform for Discrete-Time Signals

- For convenience, we adopt the notation

$$X(\omega) \equiv F\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

- For the direct transform (analysis equation) and

$$x(n) \equiv F^{-1}\{X(\omega)\} = \frac{1}{2\pi}\int_{2\pi} X(\omega)e^{j\omega n}d\omega$$

- For the inverse transform (synthesis equation). We also refer to $x(n)$ and $X(\omega)$ as a **Fourier transform pair** and denote this relationship with the notation

$$x(n) \xleftrightarrow{F} X(\omega)$$

Symmetry Properties of the Fourier Transform

- Suppose that both the signal $x(n)$ and its transform $X(\omega)$ are complex-valued functions. Then they can be expressed in rectangular form as

$$x(n) = x_R(n) + jx_I(n)$$
$$X(\omega) = X_R(\omega) + jX_I(\omega)$$

- By substituting above and $e^{-j\omega} = \cos\omega - j\sin\omega$ and separating the real and imaginary parts, we obtain

$$X_R(\omega) = \sum_{n=-\infty}^{\infty}\left[x_R(n)\cos\omega n + x_I(n)\sin\omega n\right]$$

$$X_I(\omega) = -\sum_{n=-\infty}^{\infty}\left[x_R(n)\sin\omega n - x_I(n)\cos\omega n\right]$$

Symmetry Properties of the Fourier Transform

- In a similar manner, by substituting above and

$$e^{j\omega} = \cos\omega + j\sin\omega$$

we obtain

$$x_R(n) = \frac{1}{2\pi}\int_{2\pi}\left[X_R(\omega)\cos\omega n - X_I(\omega)\sin\omega n\right]d\omega$$

$$x_I(n) = \frac{1}{2\pi}\int_{2\pi}\left[X_R(\omega)\sin\omega n + X_I(\omega)\cos\omega n\right]d\omega$$

Symmetry Properties of the Fourier Transform

- **Real signals**
  - If $x(n)$ is real, then $x_R(n) = x(n)$ and $x_I(n) = 0$. Then

$$X_R(\omega) = \sum_{n=-\infty}^{\infty} x(n)\cos\omega n$$

$$X_I(\omega) = -\sum_{n=-\infty}^{\infty} x(n)\sin\omega n$$

  - Since $\cos(-\omega n) = \cos\omega n$ and $\sin(-\omega n) = -\sin\omega n$, it follows from above

$$X_R(-\omega) = X_R(\omega) \qquad \text{(even)}$$

$$X_I(-\omega) = -X_I(\omega) \qquad \text{(odd)}$$

Symmetry Properties of the Fourier Transform

- If we combine above into a single equation, we have

$$X^*(\omega) = X(-\omega)$$

- In this case we say that the spectrum of a real signal has Hermitian symmetry.
- We observe that the magnitude and phase spectra for real signals are

$$|X(\omega)| = \sqrt{X_R^2(\omega) + X_I^2(\omega)}$$

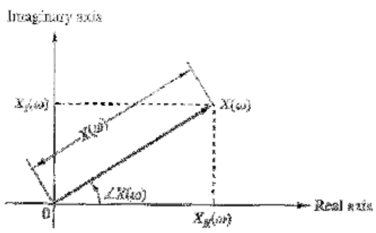
$$\measuredangle X|\omega| = \tan^{-1}\frac{X_I(\omega)}{X_R(\omega)}$$

Symmetry Properties of the Fourier Transform

- The magnitude and phase spectra also possess the symmetry properties

$$|X(\omega)| = |X(-\omega)| \qquad \text{(even)}$$

$$\measuredangle X(-\omega) = -\measuredangle X(\omega) \qquad \text{(odd)}$$



Symmetry Properties of the Fourier Transform

- In the case of the inverse transform of a real-valued signal [i.e., $x(n) = x_R(n)$], which implies that

$$x(n) = \frac{1}{2\pi}\int_{2\pi}\left[X_R(\omega)\cos\omega n - X_I(\omega)\sin\omega n\right]d\omega$$

- Since both products $X_R(\omega)\cos\omega n$ and $X_I(\omega)\sin\omega n$ are even functions of ω , we have

$$x(n) = \frac{1}{\pi}\int_0^{\pi}\left[X_R(\omega)\cos\omega n - X_I(\omega)\sin\omega n\right]d\omega$$

Symmetry Properties of the Fourier Transform

- **Real and even signals**
  - If $x(n)$ is real and even [i.e., $x(-n) = x(n)$], then $x(n)\cos\omega n$ is even and $x(n)\sin\omega n$ is odd. Hence

$$X_R(\omega) = x(0) + 2\sum_{n=1}^{\infty} x(n)\cos\omega n \qquad \text{(even)}$$

$$X_I(\omega) = 0$$

$$x(n) = \frac{1}{\pi}\int_0^{\pi} X_R(\omega)\cos\omega n d\omega$$

  - Thus real and even signals possess real-valued spectra, which, in addition, are even functions of the frequency variable ω .

Symmetry Properties of the Fourier Transform

- **Real and odd signals**
  - If $x(n)$ is real and odd [i.e., $x(-n) = -x(n)$], then $x(n)\cos\omega n$ is odd and $x(n)\sin\omega n$ is even. Hence

$$X_R(\omega) = 0$$

$$X_I(\omega) = -2\sum_{n=1}^{\infty} x(n)\sin\omega n \qquad \text{(odd)}$$

$$x(n) = -\frac{1}{\pi}\int_0^{\pi} X_I(\omega)\sin\omega n d\omega$$

  - Thus real-valued odd signals possess purely imaginary-valued spectra characteristics, which, in addition, are odd functions of the frequency variable ω.

Symmetry Properties of the Fourier Transform

- **Purely imaginary signals**
  - In this case $x_R(n) = 0$ and $x(n) = jx_I(n)$. Then

$$X_R(\omega) = \sum_{n=-\infty}^{\infty} x_I(n)\sin\omega n \qquad \text{(odd)}$$

$$X_I(\omega) = \sum_{n=-\infty}^{\infty} x_I(n)\cos\omega n \qquad \text{(even)}$$

$$x_I(n) = \frac{1}{\pi}\int_0^{\pi} \left[X_R(\omega)\sin\omega n + X_I(\omega)\cos\omega n\right] d\omega$$

Symmetry Properties of the Fourier Transform

- **Purely imaginary signals**
  - If $x_I(n)$ is odd [i.e., $x_I(-n) = -x_I(n)$], then

$$X_R(\omega) = 2\sum_{n=1}^{\infty} x_I(n)\sin\omega n \qquad \text{(odd)}$$

$$X_I(\omega) = 0$$

$$x_I(n) = \frac{1}{\pi}\int_0^{\pi} X_R(\omega)\sin\omega n\, d\omega$$

Symmetry Properties of the Fourier Transform

- **Purely imaginary signals**
  - Similarly, If $x_I(n)$ is even [i.e., $x_I(-n) = x_I(n)$], then

$$X_R(\omega) = 0$$

$$X_I(\omega) = x_I(0) + 2\sum_{n=1}^{\infty} x_I(n)\cos\omega n \qquad \text{(even)}$$

$$x_I(n) = \frac{1}{\pi}\int_0^{\pi} X_I(\omega)\cos\omega n\, d\omega$$

  - An arbitrary, possibly complex-valued signal $x(n)$ can be decomposed as

$$x(n) = x_R(n) + jx_I(n) = x_R^e(n) + jx_R^o(n) + j\left[x_I^e(n) + x_I^o(n)\right] = x_e(n) + x_o(n)$$

Symmetry Properties of the Fourier Transform

- **Purely imaginary signals**
  - By definition

$$x_e(n) = x_R^e(n) + jx_I^e(n) = \frac{1}{2}\left[x(n) + x^*(-n)\right]$$

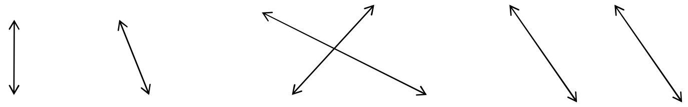
$$x_o(n) = x_R^o(n) + jx_I^o(n) = \frac{1}{2}\left[x(n) - x^*(-n)\right]$$

  - The superscripts *e* and *o* denote the even and odd signal components, respectively. We note that $x_e(n) = x_e(-n)$ and $x_o(-n) = -x_o(n)$.

Symmetry Properties of the Fourier Transform

- **Purely imaginary signals**
  - From above and the Fourier transform properties established, we obtain the following relationships:

$$x(n) = \left[x_R^e(n) + jx_I^e(n)\right] + \left[x_R^o(n) + jx_I^o(n)\right] = x_e(n) + x_o(n)$$



$$X(\omega) = \left[X_R^e(\omega) + jX_I^e(\omega)\right] + \left[X_R^o(\omega) - jX_I^o(\omega)\right] = X_e(\omega) + X_o(\omega)$$

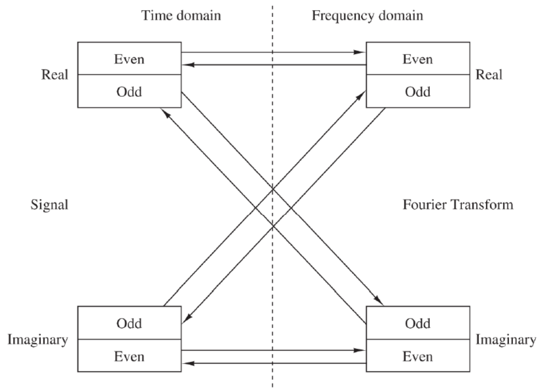
Symmetry Properties of the Fourier Transform

- Symmetry properties of the discrete-time Fourier transform

Sequence	DTFT
$x(n)$	$X(\omega)$
$x^*(n)$	$X^*(-\omega)$
$x^*(-n)$	$X^*(\omega)$
$x_R(n)$	$X_e(\omega) = \frac{1}{2}[X(\omega) + X^*(-\omega)]$
$jx_I(n)$	$X_o(\omega) = \frac{1}{2}[X(\omega) - X^*(-\omega)]$
$x_e(n) = \frac{1}{2}[x(n) + x^*(-n)]$	$X_R(\omega)$
$x_o(n) = \frac{1}{2}[x(n) - x^*(-n)]$	$jX_I(\omega)$
Real Signals	
Any real signal $x(n)$	$X(\omega) = X^*(-\omega)$ $X_R(\omega) = X_R(-\omega)$ $X_I(\omega) = -X_I(-\omega)$ $\lvert X(\omega)\rvert = \lvert X(-\omega)\rvert$ $\measuredangle X(\omega) = -\measuredangle X(-\omega)$
$x_e(n) = \frac{1}{2}[x(n) + x(-n)]$ (real and even)	$X_R(\omega)$ (real and even)
$x_o(n) = \frac{1}{2}[x(n) - x(-n)]$ (real and odd)	$jX_I(\omega)$ (imaginary and odd)

Symmetry Properties of the Fourier Transform

- Summary of symmetry properties of the Fourier transform



Fourier Transform Theorems and Properties

- **Linearity**

If $x_1(n) \overset{F}{\longleftrightarrow} X_1(\omega)$

$x_2(n) \overset{F}{\longleftrightarrow} X_2(\omega)$

then $a_1x_1(n) + a_2x_2(n) \overset{F}{\longleftrightarrow} a_1X_1(\omega) + a_2X_2(\omega)$

Fourier Transform Theorems and Properties

- **Example**: Determine the Fourier transform of the signal

$$x(n) = a^{|n|}, \quad -1 < a < 1$$

- **Solution**: First, we observe that $x(n)$ can be expressed as

$$x(n) = x_1(n) + x_2(n)$$

where $x_1(n) = \begin{cases} a^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$

and $x_2(n) = \begin{cases} a^{-n}, & n < 0 \\ 0, & n \geq 0 \end{cases}$

Fourier Transform Theorems and Properties

- Beginning with the definition of the Fourier transform, we have

$$X_1(\omega) = \sum_{n=-\infty}^{\infty} x_1(n)e^{-j\omega n} = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n$$

- The summation is a geometric series that converges to

$$X_1(\omega) = \frac{1}{1-ae^{-j\omega}}$$

- Provided that

$$\left|ae^{-j\omega}\right| = |a| \cdot \left|e^{-j\omega}\right| = |a| < 1$$

Fourier Transform Theorems and Properties

- Which is a condition that is satisfied in this problem. Similarly, the Fourier transform of $x_2(n)$ is

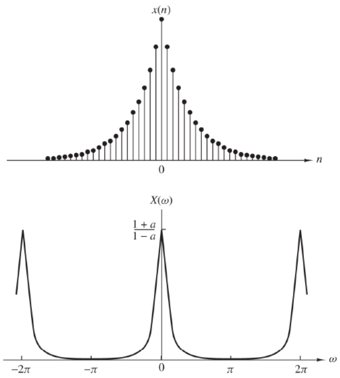
$$X_2(\omega) = \sum_{n=-\infty}^{\infty} x_2(n)e^{-j\omega n} = \sum_{n=-\infty}^{-1} a^{-n}e^{-j\omega n} = \sum_{n=-\infty}^{-1} (ae^{j\omega})^{-n} = \sum_{k=1}^{\infty} (ae^{j\omega})^k = \frac{ae^{j\omega}}{1-ae^{j\omega}}$$

- By combining these two transforms, we obtain the Fourier transform of $x(n)$ in the form

$$X(\omega) = X_1(\omega) + X_2(\omega) = \frac{1-a^2}{1-2a\cos\omega + a^2}$$

Fourier Transform Theorems and Properties

- The figure illustrates $x(n)$ and $X(\omega)$ for the case in which $a = 0.8$.



Fourier Transform Theorems and Properties

- **Time shifting**

If $x(n) \xleftrightarrow{F} X(\omega)$

then $x(n-k) \xleftrightarrow{F} e^{-j\omega k} X(\omega)$

- The proof of this property follows immediately from the Fourier transform of $x(n-k)$ by making a change in the summation index. Thus

$$F\{x(n-k)\} = X(\omega)e^{-j\omega k} = |X(\omega)| e^{j[\measuredangle X(\omega) - \omega k]}$$

Fourier Transform Theorems and Properties

- **Time reversal**

If $x(n) \stackrel{F}{\longleftrightarrow} X(\omega)$

then $x(-n) \stackrel{F}{\longleftrightarrow} X(-\omega)$

- This property can be established by performing the Fourier transformation of $x(-n)$ by making a simple change in the summation index. Thus

$$F\{x(-n)\} = \sum_{l=-\infty}^{\infty} x(l)e^{j\omega l} = X(-\omega)$$

Fourier Transform Theorems and Properties

- If $x(n)$ is real, then we obtain

$$F\{x(-n)\} = X(-\omega) = |X(-\omega)|e^{j\measuredangle X(-\omega)} = |X(\omega)|e^{-j\measuredangle X(\omega)}$$

Fourier Transform Theorems and Properties

- **Convolution theorem**

If $x_1(n) \xleftrightarrow{F} X_1(\omega)$

and $x_2(n) \xleftrightarrow{F} X_2(\omega)$

then $x(n) = x_1(n) * x_2(n) \xleftrightarrow{F} X(\omega) = X_1(\omega) X_2(\omega)$

- To prove above, we recall the convolution formula

$$x(n) = x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k) x_2(n-k)$$

Fourier Transform Theorems and Properties

- By multiplying both side of this equation by the exponential $e^{-j\omega n}$ and summing over all *n*, we obtain

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x_1(n) x_2(n-k) \right] e^{-j\omega n}$$

Fourier Transform Theorems and Properties

- **The correlation theorem**

If $x_1(n) \overset{F}{\longleftrightarrow} X_1(\omega)$

and $x_2(n) \overset{F}{\longleftrightarrow} X_2(\omega)$

then $r_{x_1x_2}(m) \overset{F}{\longleftrightarrow} S_{x_1x_2}(\omega) = X_1(\omega)X_2(-\omega)$

- In this case, we have

$$r_{x_1x_2}(n) = \sum_{k=-\infty}^{\infty} x_1(k)x_2(k-n)$$

Fourier Transform Theorems and Properties

- By multiplying both sides of this equation by the exponential $e^{-j\omega n}$ and summing over all *n*, we obtain

$$S_{x_1x_2}(\omega) = \sum_{n=-\infty}^{\infty} r_{x_1x_2}(n)e^{-j\omega n} = \sum_{n=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty} x_1(k)x_2(k-n)\right]e^{-j\omega n}$$

- The function $S_{x_1x_2}(\omega)$ is called the **cross-energy density spectrum** of the signals $x_1(n)$ and $x_2(n)$.

Fourier Transform Theorems and Properties

- **The Wiener-Khintchine theorem**
  - Let $x(n)$ be a real signal. Then

$$r_{xx}(l) \overset{F}{\longleftrightarrow} S_{xx}(\omega)$$

  - That is, the energy spectral density of an energy signal is the Fourier transform of its autocorrelation sequence.

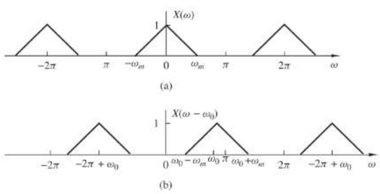
Fourier Transform Theorems and Properties

- **Frequency shifting**

If $\quad x(n) \overset{F}{\longleftrightarrow} X(\omega)$

then $\quad e^{j\omega_0 n}x(n) \overset{F}{\longleftrightarrow} X(\omega-\omega_0)$

- Illustration of the frequency-shifting property of the Fourier transform $\omega_0 \le 2\pi - \omega_m$.

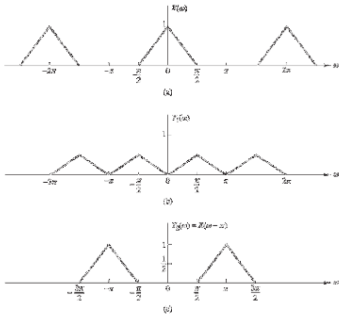


Fourier Transform Theorems and Properties

- **The modulation theorem**

If $x(n) \xleftrightarrow{F} X(\omega)$

then $x(n)\cos\omega_0 n \xleftrightarrow{F} \frac{1}{2}[X(\omega+\omega_0)+X(\omega-\omega_0)]$



Fourier Transform Theorems and Properties

- **Parseval's theorem**

If $x_1(n) \xleftrightarrow{F} X_1(\omega)$

and $x_2(n) \xleftrightarrow{F} X_2(\omega)$

then $$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\omega)X_2^*(\omega)d\omega$$

Fourier Transform Theorems and Properties

- To prove this theorem, we eliminate $X_1(\omega)$ on the right-hand side of above. Thus we have

$$\frac{1}{2\pi}\int_{2\pi}\left[\sum_{n=-\infty}^{\infty} x_1(n)e^{-j\omega n}\right]X_2^*(\omega)d\omega$$
$$=\sum_{n=-\infty}^{\infty} x_1(n)\frac{1}{2\pi}\int_{2\pi} X_2^*(\omega)e^{-j\omega n}d\omega$$
$$=\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n)$$

Fourier Transform Theorems and Properties

- In the special case where $x_2(n)=x_1(n)=x(n)$, Parseval's relation reduces to

$$\sum_{n=-\infty}^{\infty}|x(n)|^2=\frac{1}{2\pi}\int_{2\pi}|X(\omega)|^2 d\omega$$

- Therefore, we conclude that

$$E_x=r_{xx}(0)=\sum_{n=-\infty}^{\infty}|x(n)|^2=\frac{1}{2\pi}\int_{2\pi}|X(\omega)|^2 d\omega=\frac{1}{2\pi}\int_{-\pi}^{\pi} S_{xx}(\omega)d\omega$$

Fourier Transform Theorems and Properties

- **Multiplication of two sequences (Windows theorem)**

If $x_1(n) \overset{F}{\longleftrightarrow} X_1(\omega)$

and $x_2(n) \overset{F}{\longleftrightarrow} X_2(\omega)$

then $x_3(n) \equiv x_1(n)x_2(n) \overset{F}{\longleftrightarrow} X_3(\omega)$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\lambda)X_2(\omega-\lambda)d\lambda$$

Fourier Transform Theorems and Properties

- We begin with the Fourier transform of $x_3(n) = x_1(n)x_2(n)$ and use the formula for the inverse transform, namely,

$$x_1(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\lambda)e^{j\lambda n}d\lambda$$

- Thus, we have

$$X_3(\omega) = \sum_{n=-\infty}^{\infty} x_3(n)e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x_1(n)x_2(n)e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\lambda)e^{j\lambda n}d\lambda\right]x_2(n)e^{-j\omega n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\lambda)d\lambda\left[\sum_{n=-\infty}^{\infty} x_2(n)e^{-j(\omega-\lambda)n}\right]$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\lambda)X_2(\omega-\lambda)d\lambda$$

Fourier Transform Theorems and Properties

- **Differentiation in the frequency domain**

  If $\quad x(n) \overset{F}{\longleftrightarrow} X(\omega)$

  then $\quad nx(n) \overset{F}{\longleftrightarrow} j\dfrac{dX(\omega)}{d\omega}$

- To prove this property, we use the definition of the Fourier transform and differentiae the series term by term with respect to ω. Thus we obtain

$$\frac{dX(\omega)}{d\omega} = \frac{d}{d\omega}\left[\sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}\right] = \sum_{n=-\infty}^{\infty} x(n)\frac{d}{d\omega}e^{-j\omega n} = -j\sum_{n=-\infty}^{\infty} nx(n)e^{-j\omega n}$$

Fourier Transform Theorems and Properties

- Properties of the Fourier transform for discrete-time signals

Property	Time Domain	Frequency Domain
Notation	$x(n)$	$X(\omega)$
	$x_1(n)$	$X_1(\omega)$
	$x_2(n)$	$X_2(\omega)$
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(\omega) + a_2X_2(\omega)$
Time shifting	$x(n-k)$	$e^{-j\omega k}X(\omega)$
Time reversal	$x(-n)$	$X(-\omega)$
Convolution	$x_1(n) * x_2(n)$	$X_1(\omega)X_2(\omega)$
Correlation	$r_{x_1x_2}(l) = x_1(l) * x_2(-l)$	$S_{x_1x_2}(\omega) = X_1(\omega)X_2(-\omega)$
		$= X_1(\omega)X_2^*(\omega)$
		[if $x_2(n)$ is real]
Wiener–Khintchine theorem	$r_{xx}(l)$	$S_{xx}(\omega)$
Frequency shifting	$e^{j\omega_0 n}x(n)$	$X(\omega - \omega_0)$
Modulation	$x(n)\cos\omega_0 n$	$\frac{1}{2}X(\omega + \omega_0) + \frac{1}{2}X(\omega - \omega_0)$
Multiplication	$x_1(n)x_2(n)$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\lambda)X_2(\omega - \lambda)d\lambda$
Differentiation in		
the frequency domain	$nx(n)$	$j\frac{dX(\omega)}{d\omega}$
Conjugation	$x^*(n)$	$X^*(-\omega)$
Parseval's theorem	$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\omega)X_2^*(\omega)d\omega$	

Fourier Transform Theorems and Properties

- Some useful Fourier transform pairs for discrete-time aperiodic signals