Chapter 3

z-Transform

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The Direct *z*-Transform

- A discrete-time version of Laplace Transform
- For a given sequence *x*(*n*), its (direct) *z*-transform *X*(*z*) is defined as:

$$X(z) \equiv \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

where *z* = *Re*(*z*) + *j Im*(*z*) is a complex variable

- For convenience, the z-transform of a signal x(n0 is denoted by

$$X(z) \equiv Z\{x(n)\}$$

- Whereas the relationship between x(n) and X(z) is indicated by

$$x(n) \overset{z}{\longleftrightarrow} X(z)$$

- The **region of convergence (ROC)** of X(z) is the set of all values of z of which X(z) attains a finite value

ROC of the *z*-Transform

- Express the complex variable *z* in polar form as

$$z = re^{j\theta}$$

Where $r = |z|$ and $\theta = \measuredangle z$, then X(z) can be expressed as

$$X(z)\big|_{z=re^{j\theta}} = \sum_{n=-\infty}^{\infty} x(n) r^{-n} e^{-j\theta n}$$

- In the ROC of *X*(*z*), |*X*(z)| < ∞. But

$$|X(z)| = \left| \sum_{n=-\infty}^{\infty} x(n) r^{-n} e^{-j\theta n} \right|$$

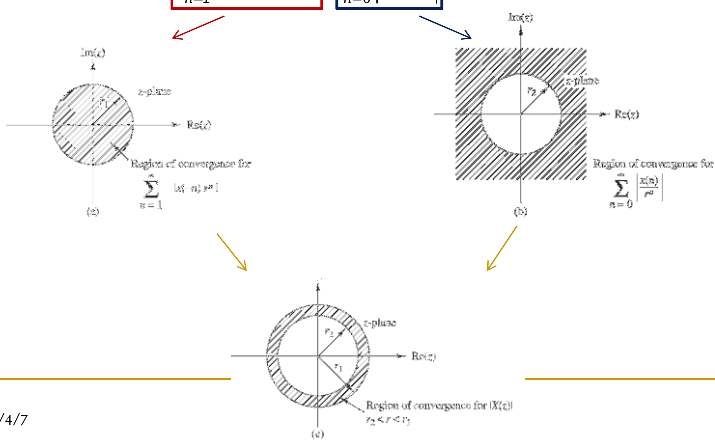
$$\le \sum_{n=-\infty}^{\infty} \left| x(n) r^{-n} e^{-j\theta n} \right| = \sum_{n=-\infty}^{\infty} \left| x(n) r^{-n} \right|$$

- Hence |*X*(z)| is finite if $x(n)r^{-n}$ is absolutely summable
- The problem of finding the ROC for *X*(*z*) is equivalent to determining the range of *r* for which $x(n)r^{-n}$ is absolutely summable

ROC of the *z*-Transform

$$|X(z)| \le \sum_{n=-\infty}^{-1} \left| x(n) r^{-n} \right| + \sum_{n=0}^{\infty} \left| x(n) r^{-n} \right|$$

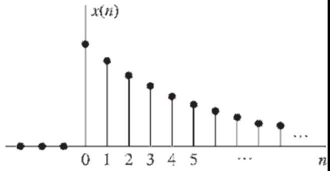
$$\le \sum_{n=1}^{\infty} \left| x(-n) r^{n} \right| + \sum_{n=0}^{\infty} \left| \frac{x(n)}{r^n} \right|$$



The Direct *z*-Transform

- **Example** – Determine the z-Transform $X(z)$ of the causal sequence $x[n] = \alpha^n u(n)$ and its ROC

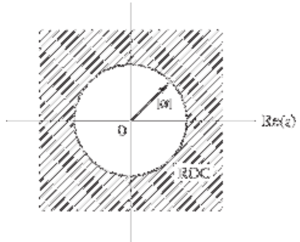
$$X(z) = \sum_{n=-\infty}^{\infty} \alpha^n u(n) z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n}$$



- The above power series converges to

$$X(z) = \frac{1}{1-\alpha z^{-1}}, \qquad \text{for } \left|\alpha z^{-1}\right| < 1$$

- ROC is the annular region $|z| > |\alpha|$



The Direct *z*-Transform

- **Example** – Consider the anti-causal sequence

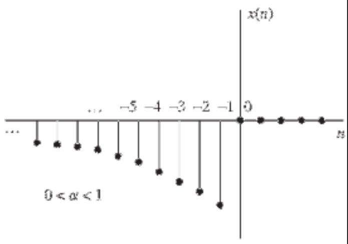
$$x(n) = -\alpha^n u(-n-1) = \begin{cases} 0, & n \geq 0 \\ -\alpha^n, & n \leq -1 \end{cases}$$

- Its z-transform is given by

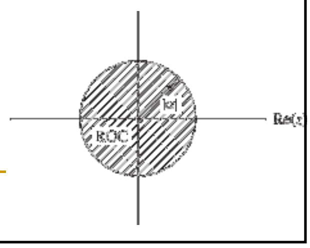
$$X(z) = \sum_{n=-\infty}^{-1} -\alpha^n z^{-n} = -\sum_{l=1}^{\infty} \alpha^{-l} z^{l}$$

$$= -\alpha^{-1} z \sum_{l=0}^{\infty} \alpha^{-l} z^{l} = -\frac{\alpha^{-1} z}{1-\alpha^{-1} z}$$

$$= \frac{1}{1-\alpha z^{-1}}, \quad \text{for } \left|\alpha^{-1} z\right| < 1$$



- ROC is the annular region $|z| < |\alpha|$



z-Transform

- **Note:** the z-Transforms of two sequences $\alpha^n u(n)$ and $-\alpha^n u(-n-1)$ are identical even though the two parent sequences are different
- Only way a unique sequence can be associated with a z-transform is by specifying its ROC

z-Transform

- **Example** – the finite energy sequence

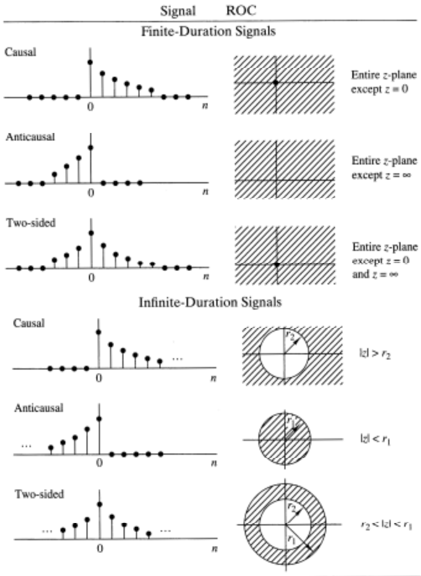
$$h_{LP}(n) = \frac{\sin \omega_c n}{\pi n}, \qquad -\infty < n < \infty$$

has a DTFT given by

$$H_{LP}\left(e^{j\omega}\right) = \begin{cases} 1, & 0 \le |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

which converges in the mean-square sense
- However, $h_{LP}[n]$ does not have a z-transform as it is not absolutely summable for any value of r

Characteristic Families of Signals with Their Corresponding ROCs



The Inverse z-Transform

- The z-transform $X(z)$ is given by

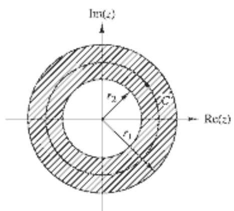
$$X(z) = \sum_{k=-\infty}^{\infty} x(k) z^{-k}$$

- Multiply both side by z^{n-1} and integrate both sides over a closed contour within the ROC of $X(z)$

$$\oint_C X(z) z^{n-1} dz = \oint_C \sum_{k=-\infty}^{\infty} x(k) z^{n-k-1} dz$$

- Interchange the order of integration and summation:

$$\oint_C X(z) z^{n-1} dz = \sum_{k=-\infty}^{\infty} x(k) \oint_C z^{n-k-1} dz$$



- According to the **Cauchy integral theorem**

$$\frac{1}{2\pi j} \oint_C z^{n-k-1} dz = \begin{cases} 1, & k = n \\ 0, & k \neq n \end{cases}$$

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

The Inverse z-Transform (from IDTFT)

- **General Expression**: Recall that, for $z = re^{j\omega}$, the z-transform $X(z)$ given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} x(n) r^{-n} e^{-j\omega n}$$

is merely the DTFT of the modified sequence $x(n)r^{-n}$

- Accordingly, the inverse DTFT is thus given by

$$x(n) r^{-n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} X\left(re^{j\omega}\right) e^{j\omega n} d\omega$$

- By making a change of variable $z = re^{j\omega}$, the previous equation can be converted into a contour integral given by

$$x(n) = \frac{1}{2\pi j} \oint_{C'} X(z) z^{n-1} dz$$

where C' is a counterclockwise contour of integration defined by $|z| = r$

Properties of the z-Transform

- **Linearity** - If

$$x_1(n) \overset{z}{\longleftrightarrow} X_1(z)$$

and

$$x_2(n) \overset{z}{\longleftrightarrow} X_2(z)$$

then

$$x(n) = a_1 x_1(n) + a_2 x_2(n) \overset{z}{\longleftrightarrow} X(z) = a_1 X_1(z) + a_2 X_2(z)$$

- **Example** – Determine the z-Transform and the ROC of

$$x(n) = \left[3\left(2^n\right) - 4\left(3^n\right)\right] u(n)$$

Ans:

$$x_1(n) = 2^n u(n) \overset{z}{\longleftrightarrow} X_1(z) = \frac{1}{1-2z^{-1}}, \quad \text{ROC: } |z| > 2$$

$$x_2(n) = 3^n u(n) \overset{z}{\longleftrightarrow} X_2(z) = \frac{1}{1-3z^{-1}}, \quad \text{ROC: } |z| > 3$$

$$X(z) = \frac{3}{1-2z^{-1}} - \frac{4}{1-3z^{-1}}, \quad \text{ROC: } |z| > 3$$

Properties of the *z*-Transform

- **Example** - Consider the two-sided sequence

$$x[n] = \alpha^n u[n] - b^n u[-n-1]$$

- Let $x[n] = \alpha^n u[n]$ and $y[n] = -b^n u[-n-1]$ with $X_1(z)$ and $X_2(z)$ denoting, respectively, their *z*-transforms
- Now

$$X_1(z) = \frac{1}{1-\alpha z^{-1}}, \quad |z| > |\alpha|$$

  and

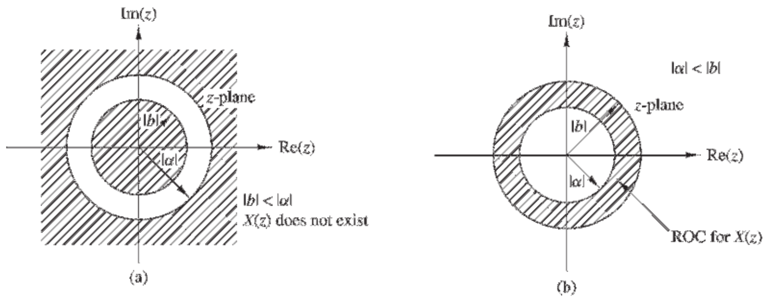
$$X_2(z) = \frac{1}{1-bz^{-1}}, \quad |z| < |b|$$

- Using the linearity property we arrive at

$$X(z) = X_1(z) + X_2(z) = \frac{1}{1-\alpha z^{-1}} + \frac{1}{1-bz^{-1}}$$

- The ROC of $X(z)$ is given by the overlap regions of $|z| > |\alpha|$ and $|z| < |b|$

Properties of the *z*-Transform



Properties of the *z*-Transform

- **Example** - Determine the *z*-transform and its ROC of the causal sequence

$$x[n] = (\cos \omega_0 n) u(n)$$

- We can express $x[n] = v[n] + v^*[n]$ where

$$v[n] = \frac{1}{2} e^{j\omega_0 n} u[n] = \frac{1}{2} \alpha^n u[n]$$

- The *z*-transform of v[n] is given by

$$V(z) = \frac{1}{2} \cdot \frac{1}{1-\alpha z^{-1}} = \frac{1}{2} \cdot \frac{1}{1-e^{j\omega_0} z^{-1}}, \quad |z| > |\alpha| = 1$$

- Using the conjugation property we obtain the *z*-transform of $v^*[n]$ as

$$V^*(z^*) = \frac{1}{2} \cdot \frac{1}{1-\alpha^* z^{-1}} = \frac{1}{2} \cdot \frac{1}{1-e^{-j\omega_0} z^{-1}}, \quad |z| > |\alpha| = 1$$

Properties of the *z*-Transform

- Using the linearity property we get

$$X(z) = V(z) + V^*(z^*) = \frac{1}{2}\left(\frac{1}{1-e^{j\omega_0} z^{-1}} + \frac{1}{1-e^{-j\omega_0} z^{-1}}\right)$$

or,

$$X(z) = \frac{1-(\cos \omega_0) z^{-1}}{1-(2\cos \omega_0) z^{-1} + z^{-2}}, \quad |z| > 1$$

- Therefore, we obtain

$$x(n) = (\cos \omega_0 n) u(n) \xleftrightarrow{z} X(z) = \frac{1-(\cos \omega_0) z^{-1}}{1-2(\cos \omega_0) z^{-1} + z^{-2}}, \quad \text{ROC: } |z| > 1$$

Similarly

$$x(n) = (\sin \omega_0 n) u(n) \xleftrightarrow{z} X(z) = \frac{(\sin \omega_0) z^{-1}}{1-2(\cos \omega_0) z^{-1} + z^{-2}}, \quad \text{ROC: } |z| > 1$$

Properties of the *z*-Transform

- **Time Shifting** - If

$$x(n) \overset{z}{\longleftrightarrow} X(z)$$

then

$$x(n-k) \overset{z}{\longleftrightarrow} z^{-k} X(z)$$

And the ROC remains unchanged except for $z = 0$ if $k > 0$ and $z = \infty$ if $k < 0$

- **Example** – Determine the z-Transform of the signal

$$x(n) = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & \text{elsewhere} \end{cases}$$

Ans:

$$x(n) = u(n) - u(n-N)$$

$$X(z) = Z\{u(n)\} - Z\{u(n-N)\} = \left(1 - z^{-N}\right)\frac{1}{1-z^{-1}}, \quad \text{ROC: } |z| > 1$$

Properties of the *z*-Transform

- **Scaling in the *z*-domain** -

If

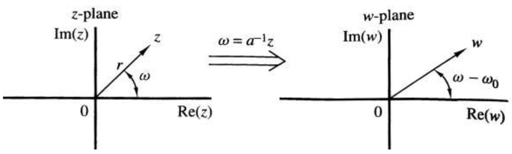
$$x(n) \overset{z}{\longleftrightarrow} X(z), \quad \text{ROC: } r_1 < |z| < r_2$$

then

$$a^n x(n) \overset{z}{\longleftrightarrow} X(a^{-1}z), \quad \text{ROC: } |a| r_1 < |z| < |a| r_2$$

Fro any constant *a*

- **Proof**



$$Z\{a^n x(n)\} = \sum_{n=-\infty}^{\infty} a^n x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\left(a^{-1}z\right)^{-n}$$

$$= X\left(a^{-1}z\right) \qquad \text{ROC: } r_1 < \left|a^{-1}z\right| < r_2 \text{ or ROC: } |a| r_1 < |z| < |a| r_2$$

Properties of the *z*-Transform

- **Example** - Determine the *z*-transform and its ROC of the causal sequence

$$x(n) = r^n(\cos\omega_0 n)u(n)$$

and

$$x(n) = r^n(\sin\omega_0 n)u(n)$$

- Ans:

By applying the scaling property

$$x(n) = r^n(\cos\omega_0 n)u(n) \overset{z}{\longleftrightarrow} X(z) = \frac{1 - r(\cos\omega_0)z^{-1}}{1 - 2r(\cos\omega_0)z^{-1} + r^2 z^{-2}}, \quad \text{ROC: } |z| > r$$

and similarly

$$x(n) = r^n(\sin\omega_0 n)u(n) \overset{z}{\longleftrightarrow} X(z) = \frac{r(\sin\omega_0)z^{-1}}{1 - 2r(\cos\omega_0)z^{-1} + r^2 z^{-2}}, \quad \text{ROC: } |z| > r$$

Properties of the *z*-Transform

- **Time Reversal** -

If

$$x(n) \overset{z}{\longleftrightarrow} X(z), \quad \text{ROC: } r_1 < |z| < r_2$$

then

$$x(-n) \overset{z}{\longleftrightarrow} X(z^{-1}), \quad \text{ROC: } \frac{1}{r_2} < |z| < \frac{1}{r_1}$$

- **Proof**

$$Z\{x(-n)\} = \sum_{n=-\infty}^{\infty} x(-n)z^{-n} = \sum_{l=-\infty}^{\infty} x(l)(z^{-1})^{-l} = X(z^{-1})$$

$$\text{ROC: } r_1 < |z^{-1}| < r_2 \quad \text{or} \quad \text{ROC: } \frac{1}{r_2} < |z| < \frac{1}{r_1}$$

- **Example** -

$$x(n) = u(-n)$$

$$u(n) \overset{z}{\longleftrightarrow} \frac{1}{1 - z^{-1}}, \quad \text{ROC: } |z| > 1 \qquad u(-n) \overset{z}{\longleftrightarrow} \frac{1}{1 - z}, \quad \text{ROC: } |z| < 1$$

Properties of the *z*-Transform

- **Differentiation in the *z*-Domain** -

  If
  $$x(n) \overset{z}{\longleftrightarrow} X(z)$$
  then
  $$nx(n) \overset{z}{\longleftrightarrow} -z\frac{dX(z)}{dz}$$
  ROC remains unchanged

- **Example** – Find the *z*-Transform of
  $$x(n) = na^n u(n)$$

- **Example** – Find the inverse *z*-transform of
  $$X(z) = \log\left(1 + az^{-1}\right), \quad \mathrm{ROC}: |z| > |a|$$

Properties of the *z*-Transform

- **Convolution of Two Sequences** -

  If
  $$x_1(n) \overset{z}{\longleftrightarrow} X_1(z) \quad \text{and} \quad x_2(n) \overset{z}{\longleftrightarrow} X_2(z)$$
  then
  $$x(n) = x_1(n) * x_2(n) \overset{z}{\longleftrightarrow} X(z) = X_1(z)X_2(z)$$
  The ROC is the intersection of that for $X_1(z)$ and $X_2(z)$

- **Proof**
  $$x(n) = \sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k)$$
  $$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} = \sum_{n=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k)\right]z^{-n}$$

Intercahnge the order of the summations and apply the time shifting property

$$X(z) = \sum_{k=-\infty}^{\infty} x_1(k)\left[\sum_{n=-\infty}^{\infty} x_2(n-k)z^{-n}\right] = \left[\sum_{k=-\infty}^{\infty} x_1(k)z^{-k}\right]X_2(z) = X_1(z)X_2(z)$$

Properties of the z-Transform

- **Correlation of Two Sequences** -

  If $x_1(n) \overset{z}{\longleftrightarrow} X_1(z)$ and $x_2(n) \overset{z}{\longleftrightarrow} X_2(z)$

  then $r_{x_1x_2}(l) = \sum_{n=-\infty}^{\infty} x_1(n)x_2(n-l) \overset{z}{\longleftrightarrow} R_{x_1x_2}(z) = X_1(z)X_2(z^{-1})$

  The ROC is at least the intersection of that for $X_1(z)$ and $X_2(z^{-1})$

- **Proof**

$$r_{x_1x_2}(l) = x_1(l) * x_2(-l)$$

$$X(z) = Z\{x_1(l)\}Z\{x_2(-l)\} = X_1(z)X_2(z^{-1})$$

- **Example** -

$$x(n) = a^n u(n)$$

$$R_{x_1x_2}(z) = \frac{1}{1-az^{-1}}\frac{1}{1-az} = \frac{1}{1-a\left(z+z^{-1}\right)+a^2}, \quad \text{ROC: } |a| < |z| < \frac{1}{|a|}$$

Properties of the z-Transform

- **Multiplication of Two Sequences** -

  If $x_1(n) \overset{z}{\longleftrightarrow} X_1(z)$ and $x_2(n) \overset{z}{\longleftrightarrow} X_2(z)$

  then $x(n) = x_1(n)x_2(n) \overset{z}{\longleftrightarrow} X(z) = \frac{1}{2\pi j}\oint_C X_1(v)X_2\left(\frac{z}{v}\right)v^{-1}dv$

  where C is the closed contour that encloses the origina and lies withing the ROC common to both $X_1(v)$ and $X_2(v^{-1})$

- **Proof**

$$X(z) = \sum_{n=-\infty}^{\infty} x_1(n)x_2(n)z^{-n}$$

Since

$$x_1(n) = \frac{1}{2\pi j}\oint_C X_1(v)v^{n-1}dv$$

$$X(z) = \frac{1}{2\pi j}\oint_C X_1(v)\left[\sum_{n=-\infty}^{\infty} x_2(n)\left(\frac{z}{v}\right)^{-n}\right]v^{-1}dv$$

$$X_2\left(\frac{z}{v}\right)$$

Properties of the z-Transform

- **Parseval's relation** -

  If $x_1(n) \overset{z}{\longleftrightarrow} X_1(z)$ and $x_2(n) \overset{z}{\longleftrightarrow} X_2(z)$

  then
$$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi j}\oint_C X_1(v)X_2^*\left(\frac{1}{v^*}\right)v^{-1}dv$$

- **The Initial Value Theorem** -
- If x(n) is causal, then
$$x(0) = \lim_{z\to\infty} X(z)$$
- **Proof** -
$$x(z) = \sum_{n=0}^{\infty} x(n)z^{-n} = x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots$$

Properties of the z-Transform

Property	Time Domain	z-Domain	ROC
Notation	$x(n)$	$X(z)$	ROC: $r_2 < \lvert z\rvert < r_1$
	$x_1(n)$	$X_1(z)$	ROC_1
	$x_2(n)$	$X_2(z)$	ROC_2
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(z) + a_2X_2(z)$	At least the intersection of ROC_1 and ROC_2
Time shifting	$x(n-k)$	$z^{-k}X(z)$	That of $X(z)$, except $z = 0$ if $k > 0$ and $z = \infty$ if $k < 0$
Scaling in the z-domain	$a^n x(n)$	$X(a^{-1}z)$	$\lvert a\rvert r_2 < \lvert z\rvert < \lvert a\rvert r_1$
Time reversal	$x(-n)$	$X(z^{-1})$	$\frac{1}{r_1} < \lvert z\rvert < \frac{1}{r_2}$
Conjugation	$x^*(n)$	$X^*(z^*)$	ROC
Real part	$\text{Re}\{x(n)\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Includes ROC
Imaginary part	$\text{Im}\{x(n)\}$	$\frac{1}{2}j[X(z) - X^*(z^*)]$	Includes ROC
Differentiation in the z-domain	$nx(n)$	$-z\frac{dX(z)}{dz}$	$r_2 < \lvert z\rvert < r_1$
Convolution	$x_1(n) * x_2(n)$	$X_1(z)X_2(z)$	At least, the intersection of ROC_1 and ROC_2
Correlation	$r_{x_1x_2}(l) = x_1(l) * x_2(-l)$	$R_{x_1x_2}(z) = X_1(z)X_2(z^{-1})$	At least, the intersection of ROC of $X_1(z)$ and $X_2(z^{-1})$
Initial value theorem	If $x(n)$ causal	$x(0) = \lim_{z\to\infty} X(z)$	
Multiplication	$x_1(n)x_2(n)$	$\frac{1}{2\pi j}\oint_C X_1(v)X_2\left(\frac{z}{v}\right)v^{-1}\,dv$	At least, $r_{1l}r_{2l} < \lvert z\rvert < r_{1u}r_{2u}$
Parseval's relation	$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi j}\oint_C X_1(v)X_2^*(1/v^*)v^{-1}dv$		

Common z-Transform Paris

	Signal, $x(n)$	z-Transform, $X(z)$	ROC
1	$\delta(n)$	1	All z
2	$u(n)$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3	$a^n u(n)$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
4	$na^n u(n)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
5	$-a^n u(-n-1)$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
6	$-na^n u(-n-1)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
7	$(\cos\omega_0 n)u(n)$	$\frac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
8	$(\sin\omega_0 n)u(n)$	$\frac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
9	$(a^n\cos\omega_0 n)u(n)$	$\frac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
10	$(a^n\sin\omega_0 n)u(n)$	$\frac{az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$

Rational z-Transforms

- In the case of LTI discrete-time systems we are concerned with in this course, all pertinent z-transforms are rational functions of z^{-1}
- That is, they are ratios of two polynomials in z^{-1}

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

- The degree of the numerator polynomial $B(z)$ *is* M and the degree of the denominator polynomial $A(z)$ is N
- An alternate representation of a rational z-transform is as a ratio of two polynomials in z:

$$X(z) = \frac{b_0 z^{-M}}{a_0 z^{-N}} \frac{z^M + (b_1/b_0) z^{M-1} + \cdots + b_M / b_0}{z^N + (a_1/a_0) z^{N-1} + \cdots + a_N / a_0}$$

Rational *z*-Transform

- A rational *z*-transform can be alternately written in factored form as

$$X(z) = \frac{b_0}{a_0} z^{-M+N} \frac{(z-z_1)(z-z_2)\cdots(z-z_M)}{(z-p_1)(z-p_2)\cdots(z-p_N)}$$

$$= Gz^{-M+N} \frac{\prod_{k=1}^{M}(z-z_k)}{\prod_{k=1}^{N}(z-p_k)} \qquad G = \frac{b_0}{a_0}$$

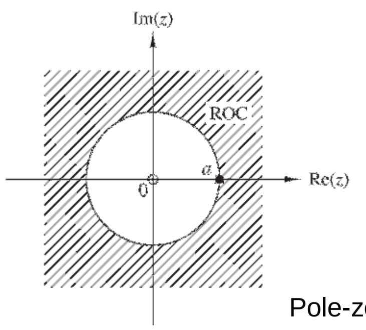
- At a root $z = z_k$ of the numerator polynomial $X(z_k) = 0$. These values of z are known as the **zeros** of $G(z)$
- At a root $z = p_k$ of the denominator polynomial $X(p_k) \to \infty$. These values of z are known as the **poles** of $G(z)$
- There are $|N - M|$ zeros (if $N > M$) or poles (if $N < M$) at z = 0

Rational *z*-Transform

- **Example** – the z-transform

$$X(z) = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \qquad \text{ROC: } |z| > a$$

has a zero at $z_1 = 0$ and a pole at $p_1 = a$



Pole-zero plot

Rational z-Transform

- **Example** – Determine the pole-zero plot for the signal

$$x(n)=\begin{cases}a^n, & 0\le n\le M-1 \\ 0, & \text{elsewhere}\end{cases} \qquad a>0$$

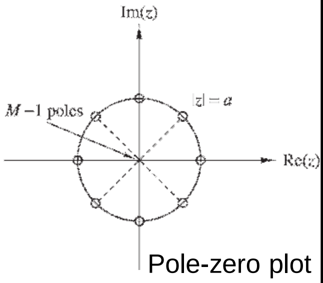
$$X(z)=\sum_{n=0}^{M-1}\left(az^{-1}\right)^n=\frac{1-\left(az^{-1}\right)^M}{1-az^{-1}}=\frac{z^M-a^M}{z^{M-1}(z-a)}$$

Since a > 0 $z^M = a^M$ has M roots at

$$z_k=ae^{j2\pi k/M} \qquad k=0,1,\ldots,M-1$$

The zero $z_0 = a$ cancels the pole at $z = a$. Thus

$$X(z)=\frac{(z-z_1)(z-z_2)\cdots(z-z_{M-1})}{z^{M-1}}$$



Pole-zero plot

Rational z-Transform

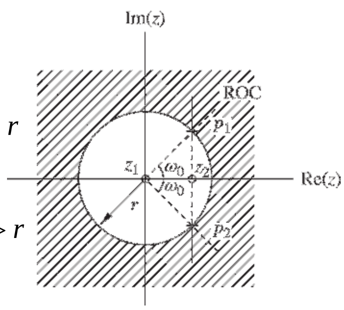
- **Example** – Determine the z-transform and the signal corresponds to the pole-zero plot
- There are 2 zeros (M = 2) at z_1 =0 and $z_2 = r\cos\omega_0$ and 2 poles (N = 2) at $p_1=re^{j\omega_0}$ and $p_2=re^{-j\omega_0}$

$$X(z)=G\frac{(z-z_1)(z-z_2)}{(z-p_1)(z-p_2)}$$

$$=G\frac{z(z-\cos\omega_0)}{\left(z-re^{j\omega_0}\right)\left(z-re^{-j\omega_0}\right)}, \quad \text{ROC: } |z|>r$$

$$X(z)=G\frac{1-rz^{-1}\cos\omega_0}{1-2rz^{-1}\cos\omega_0+r^2z^{-2}}, \quad \text{ROC: } |z|>r$$

$$x(n)=G\left(r^n\cos\omega_0 n\right)u(n)$$

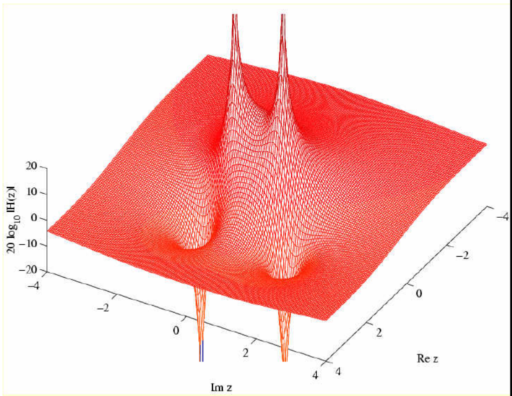


Pole-zero plot

Rational *z*-Transform

- A physical interpretation of the concepts of poles and zeros can be given by plotting the log-magnitude $20\log_{10}|X(z)|$ for

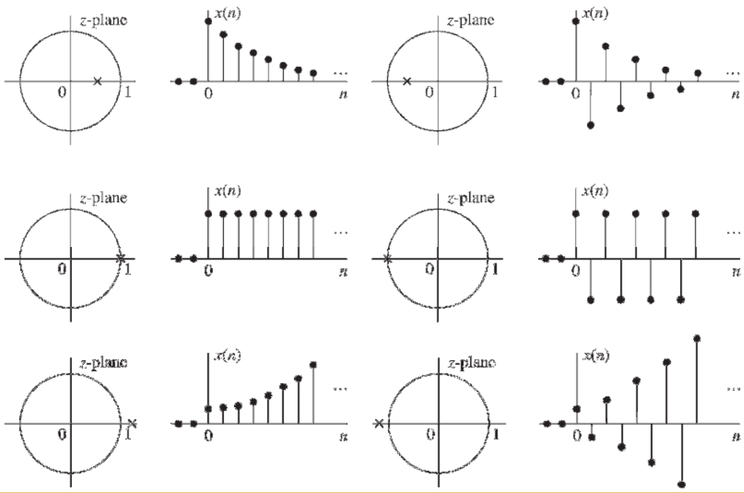
$$X(Z)=\frac{1-2.4z^{-1}+2.88z^{-2}}{1-0.8z^{-1}+0.64z^{-2}}$$



- The magnitude plot exhibits very large peaks around the poles of $X(z)$ ($z = 0.4 \pm j\,0.6928$)
- It also exhibits very narrow and deep wells around the location of the zeros ($z = 1.2 \pm j\,1.2$)

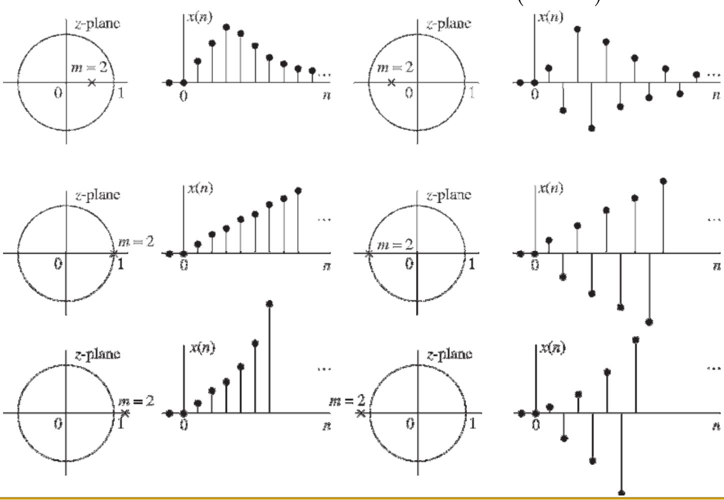
Behavior of a Single Real-Pole Causal Signal

$$x(n)=a^n u(n)\overset{z}{\longleftrightarrow}X(z)=\frac{1}{1-az^{-1}},\quad \text{ROC: } |z|>|a|$$



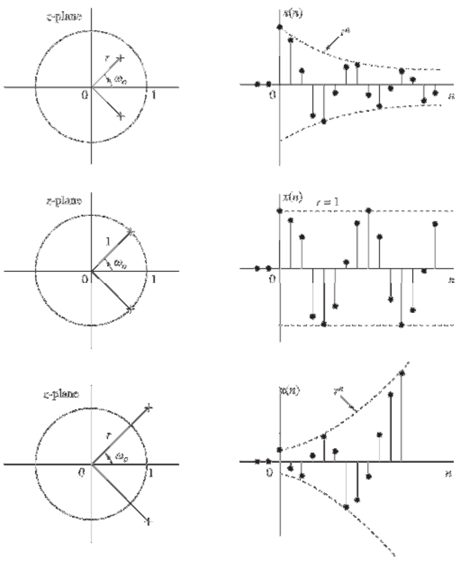
Behavior of a Double Real-Pole Causal Signal

$$x(n) = na^n u(n) \overset{z}{\longleftrightarrow} X(z) = \frac{1}{\left(1 - az^{-1}\right)^2}, \quad \text{ROC: } |z| > |a|$$



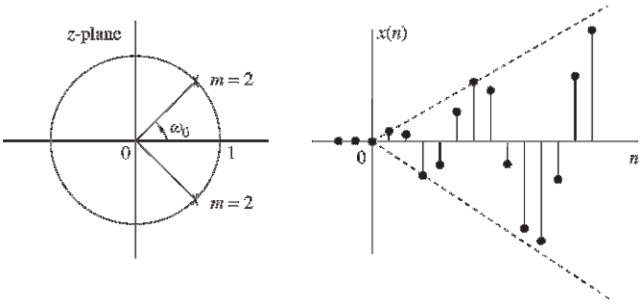
Behavior of a Causal Signal with a Pair of Complex-Conjugate Poles

$$x(n) = \left(r^n \cos \omega_0 n\right) u(n)$$



Behavior of a Causal Signal with a Double Pair of Complex-Conjugate Poles

$$x(n) = \left(n \cos \omega_0 n\right) u(n)$$



The System Function of an LTI System

- According to the convolution property, the I/O-relationship can be expressed as

$$Y(z) = H(z)X(z)$$

- The system function (impulse/unit-sample response) H(z) is

$$H(z) = \frac{Y(z)}{X(z)}$$

- If an LTI system is described by the following constant-coefficient difference equation

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

- Taking the z-Transform at both sides

$$Y(z) = -\sum_{k=1}^{N} a_k Y(z) z^{-k} + \sum_{k=0}^{M} b_k X(z) z^{-k}$$

The System Function of an LTI System

$$Y(z)\left(1+\sum_{k=1}^{N} a_k z^{-k}\right) = X(z)\left(\sum_{k=0}^{M} b_k z^{-k}\right)$$

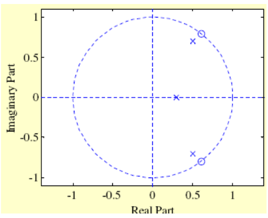
$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}}$$ **(pole-zero systems)**

- Therefore an LTI system described by a constant-coefficient difference equation has a rational system function
- Two important special forms of pole-zero systems:
  - If $a_k = 0$ for $1 \le k \le N$: $H(z) = \sum_{k=0}^{M} b_k z^{-k} = \frac{1}{z^M}\sum_{k=0}^{M} b_k z^{M-k}$ **(all-zero system)**
  - If $b_k = 0$ for $1 \le k \le M$: $H(z) = \frac{1}{1+\sum_{k=1}^{N} a_k z^{-k}} = \frac{b_0 z^N}{\sum_{k=0}^{N} a_k z^{N-k}}$ **(all-pole system)**

The System Function of an LTI System

- **Example** – A causal LTI IIR filter is described by a constant coefficient difference equation given by
- $y(n) = x(n-1) - 1.2\,x(n-2) + x(n-3) + 1.3\,y(n-1) - 1.04\,y(n-2) + 0.222\,y(n-3)$
- Its transfer function is therefore given by

$$H(z) = \frac{z^{-1} - 1.2z^{-2} + z^{-3}}{1 - 1.3z^{-1} + 1.04z^{-2} - 0.222z^{-3}}$$



- Alternate forms: $H(z) = \frac{z^2 - 1.2z + 1}{z^3 - 1.3z^2 + 1.04z - 0.222}$

$$= \frac{(z-0.6+j0.8)(z-0.6-j0.8)}{(z-0.3)(z-0.5+j0.7)(z-0.5-j0.7)}$$

- Note: Poles farthest from $z = 0$ have a magnitude $\sqrt{0.74}$
- ROC: $|z| > \sqrt{0.74}$

MATLAB Functions for Rational *z*-Transform

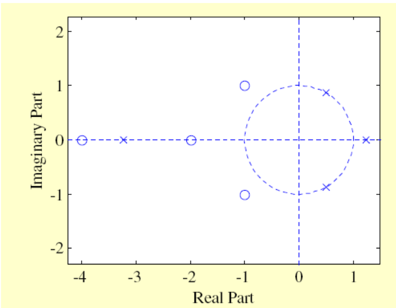
- In general, if the rational *z*-transform has *N* poles with *R* distinct magnitudes, then it has *R* + 1 ROCs, and *R* + 1 distinct sequences having the same *z*-transform
- Hence, a rational *z*-transform with a specified ROC has a unique sequence as its inverse *z*-transform
- MATLAB **[z,p,k] = tf2zp(num,den)** determines the zeros, poles, and the gain constant of a rational *z*-transform with the numerator coefficients specified by **num** and the denominator coefficients specified by **den**
- **[num,den] = zp2tf(z,p,k)** implements the reverse process
- The factored form of the *z*-transform can be obtained using **sos = zp2sos(z,p,k)**

MATLAB Functions for Rational *z*-Transform

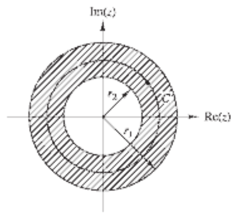
- The pole-zero plot is determined using the function **zplane**
- The *z*-transform can be either described in terms of its zeros and poles: **zplane(zeros,poles)** or, in terms of its numerator and denominator coefficients **zplane(num,den)**
- Example – The pole-zero plot of

$$X(z) = \frac{2z^4 + 16z^3 + 44z^2 + 56z + 32}{3z^4 + 3z^3 - 15z^2 + 18z - 12}$$

obtained using MATLAB



Inversion of the z-Transform



- The inverse z-transform of $X(z)$ is given by

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

- There are three methods for the evaluation of the inverse z-transform in practice:
  1. Direct evaluation by the contour integration
  2. Power series expansion
  3. Partial-fraction expansion and table lookup

Inverse z-Transform by Contour Integration

- **Cauchy's integral theorem**. Let $f(z)$ be a function of the complex variable z and C be a closed path in the z-plane. If the derivative $df(z)/dz$ exists on and inside the contour C and if $f(z)$ has no poles at $z = z_0$, then

$$\frac{1}{2\pi j} \oint_C \frac{f(z)}{z - z_0} dz = \begin{cases} f(z_0), & \text{if } z_0 \text{ is inside } C \\ 0, & \text{if } z_0 \text{ is outside } C \end{cases}$$

- More generally, if the (k+1)-order derivative of $f(z)$ exist and $f(z)$ has no poles at $z = z_0$, then

$$\frac{1}{2\pi j} \oint_C \frac{f(z)}{(z - z_0)^k} dz = \begin{cases} \left.\dfrac{1}{(k-1)!} \dfrac{d^{k-1} f(z)}{dz^{k-1}}\right|_{z=z_0}, & \text{if } z_0 \text{ is inside } C \\ 0, & \text{if } z_0 \text{ is outside } C \end{cases}$$

Inverse z-Transform by Contour Integration

- Suppose the integrand of the contour integral is a proper fraction *f*(*z*)/*g*(*z*), where *f*(*z*) has no poles inside the contour *C* and *g*(*z*) is a polynomial with distinct roots inside C. Then

$$\frac{1}{2\pi j}\oint_C \frac{f(z)}{g(z)}dz = \frac{1}{2\pi j}\oint_C \left[\sum_{i=1}^{n}\frac{A_i}{z-z_i}\right]dz$$

$$= \sum_{i=1}^{n}\frac{1}{2\pi j}\oint_C \frac{A_i}{z-z_i}dz = \sum_{i=1}^{n}A_i \qquad A_i = (z-z_i)\frac{f(z)}{g(z)}\bigg|_{z=z_i}$$

- In the case of the inverse *z*-transform, we have

$$x(n) = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz$$

$$= \sum_{\text{all poles } \{z_i\} \text{ inside } C}\left[\text{residue of } X(z)z^{n-1} \text{ at } z = z_i\right]$$

$$= \sum_i (z-z_i)X(z)z^{n-1}\Big|_{z=z_i}$$

Inverse z-Transform by Contour Integration

- **Example** – Evaluate the inverse *z*-transform of

$$X(z) = \frac{1}{1-az^{-1}}, \qquad |z| > |a|$$

Using the complex inversion integral

- Sol: $$x(n) = \frac{1}{2\pi j}\oint_C \frac{z^{n-1}}{1-az^{-1}}dz = \frac{1}{2\pi j}\oint_C \frac{z^n}{z-a}dz$$

(1) If $n \ge 0$, *f*(*z*) has only zeros and hence no poles inside *C*. The only pole inside *C* is *z* = *a*. Hence $x(n) = f(z_0) = a^n, \quad n \ge 0$

(2) If $n < 0$, *f*(*z*) has an nth-order poles at z = 0 which is also inside *C*. If *n* = 1, we have

$$x(-1) = \frac{1}{2\pi j}\oint_C \frac{1}{z(z-a)}dz = \frac{1}{z-a}\bigg|_{z=0} + \frac{1}{z}\bigg|_{z=a} = 0$$

If *n* = 2, we have

$$x(-2) = \frac{1}{2\pi j}\oint_C \frac{1}{z^2(z-a)}dz = \frac{d}{dz}\left(\frac{1}{z-a}\right)\bigg|_{z=0} + \frac{1}{z^2}\bigg|_{z=a} = 0$$

x(n) = 0 for n < 0

$$x(n) = a^n u(n)$$

Inverse z-Transform by Power Series Expansion

- Expand $X(z)$ into a power series as

$$X(z)=\sum_{n=-\infty}^{\infty} c_n z^{-n}$$

which converges in the given ROC. Then, by the uniqueness of the z-transform, $x(n) = c_n$ for all n.

- **Example**: Determine the inverse z-transform of

$$X(z)=\frac{1}{1-1.5z^{-1}+0.5z^{-2}}$$

If (a) ROC: $|z| > 1$, (b) ROC: $|z| < 0.5$

Sol: (a) $$X(z)=\frac{1}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}=1+\frac{3}{2}z^{-1}+\frac{7}{4}z^{-2}+\frac{15}{8}z^{-3}+\frac{31}{16}z^{-4}+\cdots$$

$$X(n)=\left\{\underline{1},\frac{3}{2},\frac{7}{4},\frac{15}{8},\frac{31}{16},\cdots\right\}$$

Inverse z-Transform by Partial-Fraction Expansion

- A rational $X(z)$ can be expressed as

$$X(z)=\frac{B(z)}{B(z)}=\frac{\sum_{i=0}^{M} b_i z^{-i}}{\sum_{i=0}^{N} a_i z^{-i}}=\frac{b_0+b_1z^{-1}+\cdots+b_M z^{-M}}{1+a_1z^{-1}+\cdots+a_N z^{-N}}$$

- $X(z)$ is called improper if $M \geq N$. It can be re-expressed as

$$X(z)=\sum_{\ell=0}^{M-N} c_\ell z^{-\ell}+\frac{B_1(z)}{A(z)}=c_0+c_1z^{-1}+\cdots+c_{M-N}z^{M-N}+\frac{B_1(z)}{A(z)}$$

where the degree of $B_1(z)$ is less than N

- The rational function $B_1(z)/A(z)$ is called a **proper fraction**
- **Example** – Consider $$X(z)=\frac{2+0.8z^{-1}+0.5z^{-2}+0.3z^{-3}}{1+0.8z^{-1}+0.2z^{-2}}$$

By long division we arrive at

$$X(z)=-3.5+1.5z^{-1}+\frac{5.5+2.1z^{-1}}{1+0.8z^{-1}+0.2z^{-2}}$$

Inverse z-Transform by Partial-Fraction Expansion

- **Simple Poles**: In most practical cases, the rational z-transform of interest $X(z)$ is a proper fraction with simple poles
- Let the poles of $X(z)$ be at $z = p_k$ $1 \le k \le N$
- A partial-fraction expansion of $X(z)$ is then of the form

$$X(z) = \sum_{\ell=1}^{N} \left(\frac{A_\ell}{1 - p_\ell z^{-1}} \right) = \frac{A_1}{1 - p_1 z^{-1}} + \frac{A_2}{1 - p_2 z^{-1}} + \cdots \frac{A_N}{1 - p_N z^{-1}}$$

- The constants in the partial-fraction expansion are called the **residues** and are given by

$$A_\ell = \left(1 - p_\ell z^{-1}\right) X(z) \Big|_{z = p_\ell}$$

Inverse z-Transform by Partial-Fraction Expansion

- $x(n) = Z^{-1}\{X(z)\}$ can be obtained by inverting each term as:

$$Z^{-1}\left\{ \frac{1}{1 - p_k z^{-1}} \right\} = \begin{cases} (p_k)^n u(n), & \text{if ROC: } |z| > |p_k| \text{ (causal sequence)} \\ -(p_k)^n u(-n-1) & \text{if ROC: } |z| < |p_k| \text{ (anticausal sequence)} \end{cases}$$

- If $x(n)$ is causal, the ROC is $|z| > p_{max}$ where $p_{max} = \max\{|p_1|, |p_2|, \ldots, |p_N|\}$, and $x(n)$ becomes

$$x(n) = \left(A_1 p_1^n + A_2 p_2^n + \cdots + A_N p_N^n \right) u(n)$$

- The above approach with a slight modification can also be used to determine the inverse of a rational z-transform of a non-causal sequence

Inverse z-Transform by Partial-Fraction Expansion

- **Example** – Determine the inverse *z*-transform of

$$X(z)=\frac{1}{1-1.5z^{-1}+0.5z^{-2}}=\frac{1}{\left(1-z^{-1}\right)\left(1-0.5z^{-1}\right)}$$

If (a) ROC: |*z*| > 1, (b) ROC: |*z*| < 0.5, and (c) 0.5 < |*z*| < 1

- Ans:

$$X(z)=\frac{A_1}{1-z^{-1}}+\frac{A_2}{1-0.5z^{-1}}$$

$$A_1=\left(1-z^{-1}\right)H(z)\big|_{z=1}=2,\quad A_2=\left(1-0.5z^{-1}\right)H(z)\big|_{z=0.5}=-1$$

(a) ROC: |*z*| > 1 $\quad x(n)=\left(2-0.5^n\right)u(n)$

(b) ROC: |*z*| < 0.5 $\quad x(n)=\left(-2+0.5^n\right)u(-n-1)$

(b) ROC: 0.5 < |*z*| < 1 $\quad x(n)=-2u(-n-1)-\left(0.5\right)^n u(n)$

Inverse z-Transform by Partial-Fraction Expansion

- **Multiple Poles**: If *X*(*z*) has multiple poles, the partial-fraction expansion is of slightly different form
- Let the pole at *z* = *v* be of multiplicity *L* and the remaining *N* − *L* poles be simple and at *z* = λ, 1 ≤ *l* ≤ *N* − *L*
- Then the partial-fraction expansion of *G*(*z*) is of the form

$$G(z)=\sum_{\ell=0}^{M-N}\eta_\ell z^{-\ell}+\sum_{\ell=1}^{N-L}\frac{\rho_\ell}{1-\lambda_\ell z^{-1}}+\sum_{i=1}^{L}\frac{\gamma_i}{\left(1-vz^{-1}\right)^i}$$

where the constants are computed using

$$\gamma_i=\frac{1}{(L-i)!(-v)^{L-i}}\frac{d^{L-i}}{d(z^{-1})^{L-i}}\left[\left(1-vz^{-1}\right)^L G(z)\right]_{z=v},\qquad 1\le i\le L$$

- The residues ρ_l are calculated as before

Decomposition of Rational *z*-Transforms

- Suppose we have a rational *z*-transform $X(z)$ expressed as

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}} = b_0 \frac{\prod_{k=1}^{M}\left(1-z_k z^{-1}\right)}{\prod_{k=1}^{N}\left(1-p_k z^{-1}\right)}$$

- If $M \geq N$ (i.e., $X(z)$ is improper), we convert $X(z)$ to a sum of a polynomial and a proper function

$$X(z) = \sum_{k=0}^{M-N} c_k z^{-k} + X_{\text{pr}}(z)$$

- If the roots are distinct, it can be expanded in partial fractions as

$$X(z) = A_1 \frac{1}{\left(1-p_1 z^{-1}\right)} + A_2 \frac{1}{\left(1-p_2 z^{-1}\right)} + \cdots + A_N \frac{1}{\left(1-p_N z^{-1}\right)}$$

Decomposition of Rational *z*-Transforms

- If there some complex conjugate pairs of poles, we can group and combine the pairs as

$$\frac{A}{1-pz^{-1}} + \frac{A^*}{1-p^*z^{-1}} = \frac{\left(A+A^*\right)+\left(-Ap^*-A^*p\right)z^{-1}}{1+(-p-p^*)z^{-1}+\left(pp^*\right)z^{-2}} = \frac{b_0+b_1z^{-1}}{1+a_1z^{-1}+a_2z^{-2}}$$

- The partial fraction expansion becomes

$$X(z) = \sum_{k=0}^{M-N} c_k z^{-k} + \sum_{k=1}^{K_1} \frac{b_k}{1+a_k z^{-1}} + \sum_{k=1}^{K_2} \frac{b_{0k}+b_{1k}z^{-1}}{1+a_{1k}z^{-1}+a_{2k}z^{-2}}$$

where $N = K_1 + 2K_2$

Decomposition of Rational *z*-Transforms

- An alternative form is obtained by expressing X(z) as a product of simple terms. To avoid complex coefficients in the decomposition, the complex-conjugate poles and zeros should be combined as

$$\frac{\left(1-z_k z^{-1}\right)\left(1-z_k^* z^{-1}\right)}{\left(1-p_k z^{-1}\right)\left(1-p_k^* z^{-1}\right)}=\frac{1+b_{1k}z^{-1}+b_{2k}z^{-2}}{1+a_{1k}z^{-1}+a_{2k}z^{-2}}$$

- Assuming for simplicity that $M = N$, $X(z)$ can be expressed as

$$X(z)=b_0\cdot\prod_{k=1}^{K_1}\frac{1+b_k z^{-1}}{1+a_k z^{-1}}\cdot\prod_{k=1}^{K_2}\frac{1+b_{1k}z^{-1}+b_{2k}z^{-2}}{1+a_{1k}z^{-1}+a_{2k}z^{-2}}$$

where $N = K_1 + 2K_2$

LTI Discrete-Time Systems in the Transform Domain

- An LTI discrete-time system is completely characterized in the time-domain by its impulse response sequence $\{h(n)\}$
- Thus, the transform-domain representation of a discrete-time signal can also be equally applied to the transform-domain representation of an LTI discrete-time system
- Besides providing additional insight into the behavior of LTI systems, it is easier to design and implement these systems in the transform-domain for certain applications
- We consider now the use of the DTFT and the *z*-transform in developing the transform-domain representations of an LTI system

LTI Discrete-Time Systems in the Transform Domain

- Consider LTI discrete-time systems characterized by linear constant coefficient difference equations of the form

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

- Applying the *z*-transform to both sides of the difference equation and making use of the linearity and the time-invariance properties we arrive at

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

- A more convenient form of the *z*-domain representation of the difference equation is given by

$$\left(\sum_{k=0}^{N} a_k z^{-k}\right) Y(z) = \left(\sum_{k=0}^{M} b_k z^{-k}\right) X(z)$$

Frequency Response from Transfer Function

- If the ROC of the transfer function $H(z)$ includes the unit circle, then the frequency response $H(e^{j\omega})$ of the LTI digital filter can be obtained simply as follows:

$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}}$$

- For a real coefficient transfer function $H(z)$ it can be shown that $\left|H\left(e^{j\omega}\right)\right|^2 = H\left(e^{j\omega}\right)H^*\left(e^{j\omega}\right) = H\left(e^{j\omega}\right)H\left(e^{-j\omega}\right) = H(z)H\left(z^{-1}\right)\big|_{z=e^{j\omega}}$
- For a stable rational transfer function in the form

$$H(z) = \frac{b_0}{a_0} z^{N-M} \frac{\prod_{k=1}^{M}(z - z_k)}{\prod_{k=1}^{N}(z - p_k)}$$

- the factored form of the frequency response is given

$$H\left(e^{j\omega}\right) = \frac{b_0}{a_0} e^{j\omega(N-M)} \frac{\prod_{k=1}^{M}\left(e^{j\omega} - z_k\right)}{\prod_{k=1}^{N}\left(e^{j\omega} - p_k\right)}$$

Geometric Interpretation of Frequency Response Computation

- It is convenient to visualize the contributions of the **zero factor** $(z - z_k)$ and the **pole factor** $(z - p_k)$ from the factored form of the frequency response
- The magnitude function is given by

$$\left|H\left(e^{j\omega}\right)\right| = \left|\frac{b_0}{a_0}\right| \left|e^{j\omega(N-M)}\right| \frac{\prod_{k=1}^{M}\left|e^{j\omega} - z_k\right|}{\prod_{k=1}^{N}\left|e^{j\omega} - p_k\right|}$$

which reduces to

$$\left|H\left(e^{j\omega}\right)\right| = \left|\frac{b_0}{a_0}\right| \frac{\prod_{k=1}^{M}\left|e^{j\omega} - z_k\right|}{\prod_{k=1}^{N}\left|e^{j\omega} - p_k\right|}$$

- The phase response for a rational transfer function is of the form

$$\arg H\left(e^{j\omega}\right) = \arg\left(b_0 / a_0\right) + \omega\left(N - M\right) + \sum_{k=1}^{M} \arg\left(e^{j\omega} - z_k\right) - \sum_{k=1}^{N} \arg\left(e^{j\omega} - p_k\right)$$

Geometric Interpretation of Frequency Response Computation

- The magnitude-squared function of a real-coefficient transfer function can be computed using

$$\left|H\left(e^{j\omega}\right)\right|^2 = \left|\frac{b_0}{a_0}\right|^2 \frac{\prod_{k=1}^{M}\left(e^{j\omega} - z_k\right)\left(e^{-j\omega} - z_k^*\right)}{\prod_{k=1}^{N}\left(e^{j\omega} - p_k\right)\left(e^{-j\omega} - p_k^*\right)}$$

- The factored form of the frequency response

$$H\left(e^{j\omega}\right) = \frac{b_0}{a_0} e^{j\omega(N-M)} \frac{\prod_{k=1}^{M}\left(e^{j\omega} - z_k\right)}{\prod_{k=1}^{N}\left(e^{j\omega} - p_k\right)}$$

is convenient to develop a geometric interpretation of the frequency response computation from the pole-zero plot as ω varies from 0 to 2π on the unit circle
- The geometric interpretation can be used to obtain a sketch of the response as a function of the frequency

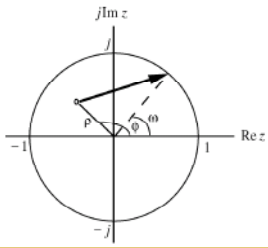
Geometric Interpretation of Frequency Response Computation

- A typical factor in the factored form of the frequency response is given by

$$(e^{j\omega} - \rho e^{j\phi})$$

where $\rho e^{j\phi}$ is a zero (pole) if it is zero (pole) factor

- As shown below in the z-plane the factor $(e^{j\omega} - \rho e^{j\phi})$ represents a vector starting at the point $z = \rho e^{j\phi}$ and ending on the unit circle at $z = e^{j\omega}$



- As ω is varied from 0 to 2π, the tip of the vector moves counter-clockwise from the point $z = 1$ tracing the unit circle and back to the point $z = 1$

Geometric Interpretation of Frequency Response Computation

- As indicated by

$$\left|H\left(e^{j\omega}\right)\right| = \left|\frac{b_0}{a_0}\right| \frac{\prod_{k=1}^{M}\left|e^{j\omega} - z_k\right|}{\prod_{k=1}^{N}\left|e^{j\omega} - p_k\right|}$$

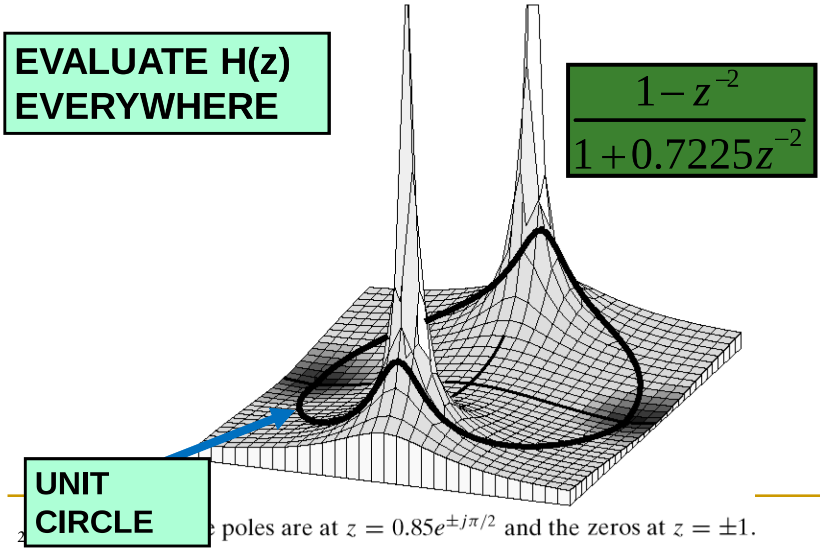
the magnitude response $|H(e^{j\omega})|$at a specific value of ω is given by the product of the magnitudes of all zero vectors divided by the product of the magnitudes of all pole vectors

- Likewise, from

$$\arg H\left(e^{j\omega}\right) = \arg\left(b_0 / a_0\right) + \omega\left(N - M\right) + \sum_{k=1}^{M}\arg\left(e^{j\omega} - z_k\right) - \sum_{k=1}^{N}\arg\left(e^{j\omega} - p_k\right)$$

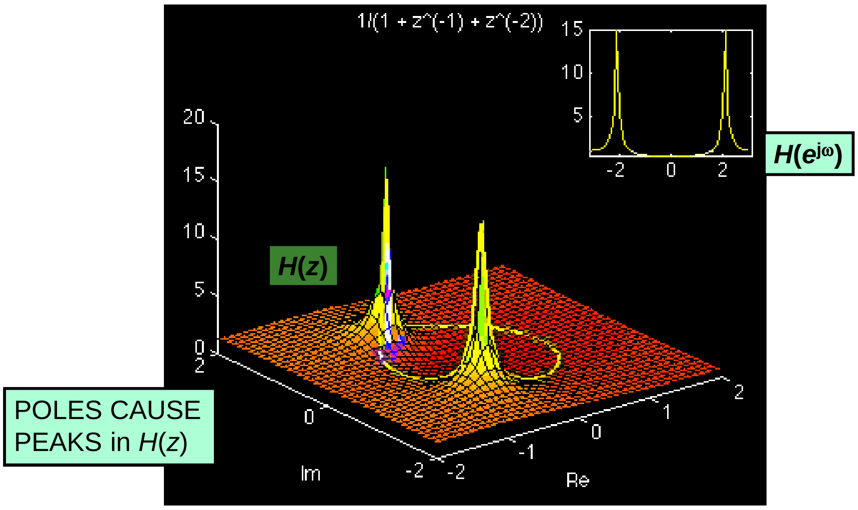
we observe that the phase response at a specific value of ω is obtained by adding the phase of the term b_0/a_0 and the linear-phase term $\omega(N - M)$ to the sum of the angles of the zero vectors minus the angles of the pole vectors

3-D View of Transfer Function & Frequency Response



… poles are at $z = 0.85e^{\pm j\pi/2}$ and the zeros at $z = \pm 1$.

Flying Thru *z*-Plane



Response of Systems with Rational System function

- Assume that the input signal x(n) has a rational *z*-transform

$$X(z) = \frac{N(z)}{Q(z)}$$

- For a initially relaxed system, the output becomes

$$Y(z) = H(z)X(z) = \frac{B(z)N(z)}{A(z)Q(z)}$$

$$= \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}} + \sum_{k=1}^{L} \frac{Q_k}{1 - q_k z^{-1}}$$

- The output $y(n)$ can be subdivided into two parts. The first part is a function of $\{p_k\}$, called the **natural response** of the system. The second is a function of $\{q_k\}$, called the **forced response**

$$y(n) = \underbrace{\sum_{k=1}^{N} A_k (p_k)^n u(n)}_{\text{the natral response } y_{\text{nr}}(n)} + \underbrace{\sum_{k=1}^{L} Q_k (q_k)^n u(n)}_{\text{the forced response } y_{\text{fr}}(n)}$$

Transient & Steady-State Responses

- The zero-state response of a system to a given input can be separate into two components: the **natural response** and the **forced response**

$$y_{\text{nr}}(n) = \sum_{k=1}^{N} A_k (p_k)^n u(n)$$

where $\{p_k\}$, $k = 1, 2, \ldots, N$ are the poles of the system.

- If $|p_k| < 1$ for all k, then $y_{\text{nr}}(n)$ decays to zero as n approaches infinity. In such a case, we refer to the natural response of the system as the **transient response**
- The forced response of the system has the form

$$y_{\text{fr}}(n) = \sum_{k=1}^{L} Q_k (q_k)^n u(n)$$

- where $\{p_k\}$, $k = 1, 2, \ldots, N$ are the poles in the forcing function

Transient & Steady-State Responses

- When the causal input signal is a sinusoid, the forced response is also a sinusoid. In such case the forced output is called **steady-state response** of the system
- Example – Determine the transient and steady-state response of the following system (assuming initially rest)

$$y(n) = 0.5y(n-1) + x(n)$$

where the input is $x(n) = 10\cos(\pi n/4)u(n)$

- Sol

$$H(z) = \frac{1}{1-0.5z^{-1}} \qquad X(z) = \frac{10\left(1-\left(1/\sqrt{2}\right)z^{-1}\right)}{1-\sqrt{2}z^{-1}+z^{-2}}$$

$$Y(z) = \frac{10\left(1-\left(1/\sqrt{2}\right)z^{-1}\right)}{\left(1-0.5z^{-1}\right)\left(1-e^{j\pi/4}z^{-1}\right)\left(1-e^{-j\pi/4}z^{-1}\right)} = \boxed{\frac{6.3}{\left(1-0.5z^{-1}\right)}} + \boxed{\frac{6.78e^{-j28.7^\circ}}{\left(1-e^{j\pi/4}z^{-1}\right)} + \frac{6.78e^{j28.7^\circ}}{\left(1-e^{-j\pi/4}z^{-1}\right)}}$$

$$y_{\text{nr}}(n) = 6.3(0.5)^n u(n) \qquad y_{\text{fr}}(n) = 13.56\cos\left(\pi n/4 - 28.7^\circ\right)u(n)$$

Causality and Stability

- As defined previously, a causal LTI system is one whose unit sample response $h(n)$ satisfies the condition

$$h(n) = 0, \quad n < 0$$

- The ROC of a causal sequence is the exterior of a circle
- A linear LTI system is causal **if and only if** the ROC of the system function is the **exterior of a circle** of radius $r < \infty$, including the point $z = \infty$
- Recall that a necessary and sufficient condition for a LTI system to be BIBO stable is

$$S = \sum_{n=-\infty}^{\infty} \left|h(n)\right| < \infty$$

- In turn, this condition implies that $H(z)$ must contain the **unit circle** within its ROC

Causality and Stability

- Since

$$|H(z)| = \left|\sum_{n=-\infty}^{\infty} h(n)z^{-n}\right| \le \sum_{n=-\infty}^{\infty} \left|h(n)z^{-n}\right| = \sum_{n=-\infty}^{\infty} |h(n)||z^{-n}|$$

- When evaluated on the unit circle (i.e., $|z| = 1$)

$$|H(z)| \le \sum_{n=-\infty}^{\infty} |h(n)|$$

- Hence, if the system is BIBO stable, the unit circle is contained in the ROC of $H(z)$, vice versa
- Therefore, a LTI system is BIBO stable **if and only if** the ROC of the system function includes the unit circle
- Note that causality and stability are different and that one does not imply the other
- A causal LTI system is BIBO stable if and only if all the poles of $H(z)$ are **inside the unit circle** (why?)

Causality and Stability

- Example – A linear LTI system is characterized by the system function

$$H(z) = \frac{3-4z^{-1}}{1-3.5z^{-1}+1.5z^{-2}} = \frac{1}{1-0.5z^{-1}} + \frac{2}{1-3z^{-1}}$$

Specify the ROC of $H(z)$ and determine $h(n)$ for the following conditions:

(a) The system is stable.

(b) The system is causal.

(c) The system is anticausal.

Pole-Zero Cancelation

- Pole-zero cancelation: When a z-transform has pole that is the same location as a zero, the pole is canceled by the zero and, consequently, the term containing that pole in the inverse z-transform vanishes
- Pole-zero cancelations can occur either in the system function $H(z)$ itself (a **reduced-order** system) or in the product of $H(z)X(z)$
- Properly selecting the position of the zeros of the input signal is possible to suppress one or more modes of the system function, or vice versa
- When the zero is located very near the pole but not exactly at the same location, the term in the response has very small amplitude. The nonexact pole-zero cancellation can occur in practice due to insufficient numerical precision

Pole-Zero Cancelation

- **Example** – Determine the response of the system

$$y(n)=\frac{1}{2}y(n-1)+x(n)-3x(n-1)$$

to the input signal $x(n)=\delta(n)-\frac{1}{3}\delta(n-1)$

Sol:

The system function is

$$H(z)=\frac{1}{1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}}=\frac{1}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)}$$

- And the z-transform of the input is

$$X(z)=1-\frac{1}{3}z^{-1}$$

$$Y(z)=H(z)X(z)=\frac{1}{1-\frac{1}{2}z^{-1}} \qquad y(n)=\left(\frac{1}{2}\right)^n u(n)$$

Stability Condition in Terms of the Pole Locations

- In addition, for a stable and causal digital filter for which $h[n]$ is a right-sided sequence, the ROC will include the unit circle and entire z-plane including the point $z = \infty$
- An FIR digital filter with bounded impulse response is always stable
- On the other hand, an IIR filter may be unstable if not designed properly
- In addition, an originally stable IIR filter characterized by infinite precision coefficients may become unstable when coefficients get quantized due to implementation

Stability of Second-Order Systems

- Consider a causal two-pole system described by the second-order difference equation,

$$y(n) = -a_1 y(n-1) - a_2 y(n-2) + b_0 x(n)$$

- The system function is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0}{1 + a_1 z^{-1} + a_2 z^{-2}} = \frac{b_0 z^2}{z^2 + a_1 z^1 + a_2}$$

- This system has two zeros at the origin and poles at

$$p_1, p_2 = -\frac{a_1}{2} \pm \sqrt{\frac{a_1^2 - 4a_2}{4}}$$

- The system is BIBO stable is the poles lie inside the unit circle, that is, if $|p_1| < 1$ and $|p_2| < 1$
- Thus, if the ROC includes the unit circle $|z| = 1$, then the system is stable, and vice versa

Stability of Second-Order Systems

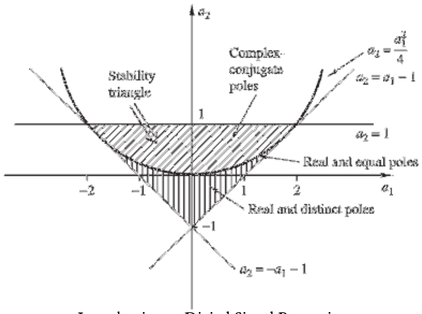
- The coefficients are related to the roots by

$$a_1 = -(p_1 + p_2), \qquad a_2 = p_1 p_2$$

- For stability, the following conditions must be satisfied

$$|a_2| = |p_1 p_2| < 1, \qquad |a_1| < 1 + a_2$$

- Therefore a two-pole system is stable if and only if the two coefficients satisfy the two conditions.



Stability of Second-Order Systems

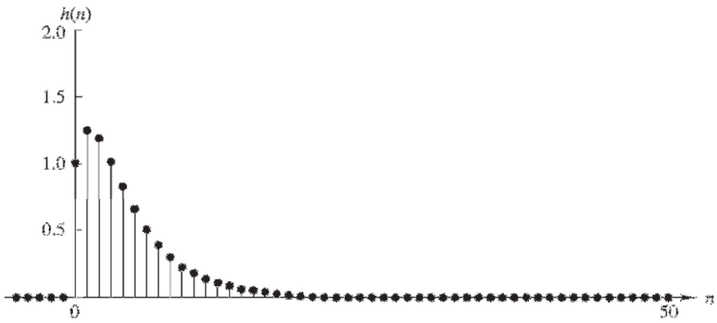


Figure 3.5.2 Plot of $h(n)$ given by (3.5.16) with $p_1 = 0.5$, $p_2 = 0.75$; $h(n) = [1/(p_1 - p_2)](p_1^{n+1} - p_2^{n+1})u(n)$.

Stability of Second-Order Systems

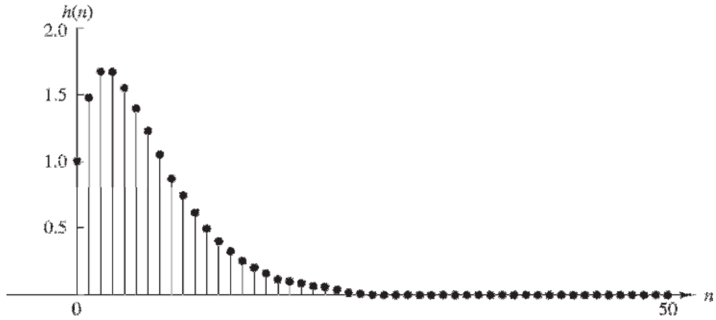


Figure 3.5.3 Plot of $h(n)$ given by (3.5.18) with $p = \frac{3}{4}$; $h(n) = (n+1)p^n u(n)$.

Stability of Second-Order Systems

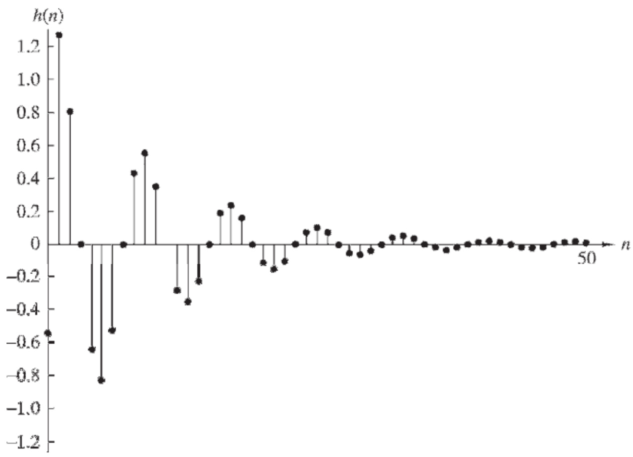


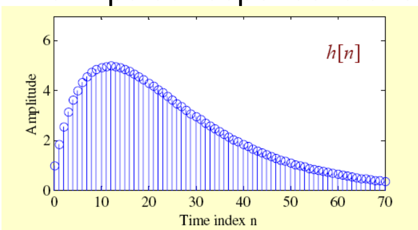
Figure 3.5.4 Plot of $h(n)$ given by (3.5.22) with $b_0 = 1$, $\omega_0 = \pi/4$, $r = 0.9$; $h(n) = [b_0 r^n/(\sin\omega_0)]\sin[(n+1)\omega_0]u(n)$.

Stability of Second-Order Systems

- **Example** – Consider the causal LTI IIR transfer function:

$$H(z) = \frac{1}{1 - 1.845z^{-1} + 0.850586z^{-2}}$$

- The plot of the impulse response is shown below



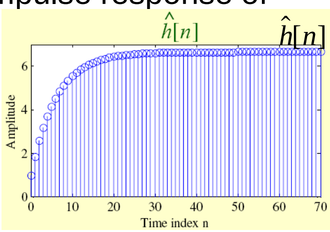
- As can be seen from the above plot, the impulse response coefficient *h*(*n*) decays rapidly to zero value as *n* increases
- The absolute summability condition of *h*(*n*) is satisfied, $\Rightarrow$ *H*(*z*) is a stable transfer function

Stability of Second-Order Systems

- Now, consider the case when the transfer function coef. are rounded to values with 2 digits after the decimal point:

$$\hat{H}(z) = \frac{1}{1 - 1.85z^{-1} + 0.85z^{-2}}$$

- A plot of the impulse response of is shown below



- In this case, the impulse response coefficient $\hat{h}(n)$ increases rapidly to a constant value as *n* increases
- Hence, $\hat{h}(n)$ is an unstable transfer function

Stability Condition in Terms of the Pole Locations

- **Example**: The factored form of

$$H(z) = \frac{1}{1 - 0.845z^{-1} + 0.850586z^{-2}}$$

is

$$H(z) = \frac{1}{(1 - 0.902z^{-1})(1 - 0.943z^{-1})}$$

which has a real pole at z = 0.902 and a pole at z = 0.943

- Since both poles are inside the unit circle $H(z)$ is BIBO stable
- **Example**: The factored form of

$$\hat{H}(z) = \frac{1}{1 - 1.85z^{-1} + 0.85z^{-2}}$$

is

$$\hat{H}(z) = \frac{1}{(1 - z^{-1})(1 - 0.85z^{-1})}$$

which has a pole at z = 1 and the other inside the unit circle

- Since one pole is not inside the unit circle, $H(z)$ is not BIBO stable