Deterministic Traffic Specification via Projections under the Min-plus Algebra

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Abstract— **In this paper, we address the parameterization problem for traffic envelopes needed for deterministic traffic regulation and service guarantees. A parameterized function is a good "substitute" for a traffic envelope if (i) the substitute is not smaller than the envelope and (ii) no other functions not smaller than the envelope are smaller than the substitute. Analogous to the least square approximation problem in a vector space, we use *projections* under the $(\min, +)$-algebra to find a substitute for a traffic envelope. To facilitate the computation of operations under the $(\min, +)$-algebra, we develop the concept of *ordered orthogonal bases*. A substitute for a traffic envelope can be represented by a *coordinate vector* with respect to an ordered orthogonal basis. Operations under the $(\min, +)$-algebra, including pointwise minimum, convolution, subadditive closure, and pointwise maximum, can then be computed on the domain of coordinate vectors. A substitute and its coordinate vector forms a transform pair, called $\mathcal{C}$-transform in the paper. The $\mathcal{C}$-transform is related to the Legendre (or convex) transform and has many properties such as Parseval's formula.**

Keywords—**Traffic specification, projections, min-plus algebra**

I. Introduction

Recent advances in the filtering theory under the $(\min, +)$-algebra (see e.g., [5], [1], [3]) have shown that deterministic service guarantees can be achieved if traffic is properly constrained. The filtering theory is based on the traffic constraint functions (or sequences) in [6], [7]. For an arrival process A (with $A(t)$ being the number of arrivals by time t), it is f-upper constrained for some function f if

$$A(t) - A(s) \leq f(t-s), \quad s \leq t. \tag{1}$$

The tightest constraint function, known as the minimum envelope in [4], is

$$\hat{A}(t) = \sup_{s \geq 0} [A(t+s) - A(s)]. \tag{2}$$

To carry out numerical computation for service guarantees, the minimum envelope needs to be parameterized. For efficient parameterization, one has to find a good "substitute" for the minimum envelope that can be represented by a set of parameters. In view of (1), the criteria for a good "substitute" for the minimum envelope are (i) the substitute must not be smaller than the minimum envelope, and (ii) no other functions not smaller than the minimum envelope are smaller than the substitute. These are in the same spirit as the requirements for the traffic specification (TSpec) in [15]. In [17], an algorithm is developed to find a substitute of the form $f(t) = \min_{0 \leq i \leq n}[\sigma_i + \rho_i t]$. This type of functions are called Concave Piecewise Linear (CPL) functions in [14] and they can be implemented by a concatenation of (σ_i, ρ_i)-leaky buckets, $i = 1, \ldots, n$. In addition to the piecewise linear envelopes, nonlinear envelopes have been found useful in [12] to characterize the Bellcore local area network traffic [10]. All these motivate us to study the general parameterization problem for traffic envelopes.

Our approach for the general parameterization problem is analogous to the projection used in a vector space to find the least square approximation [11]. The projections under the $(\min, +)$-algebra were previously derived in [8] to solve the zero lateness scheduling problem for finite dimensional vectors. In this paper, we extend it to (infinite) sequences. We start from identifying a moduloid under the $(\min, +)$-algebra. A moduloid can be viewed as a vector space without inverse elements. A nonnegative and increasing sequence f is a vector in the moduloid and a real number $\alpha \in \{0\} \cup \mathcal{R}_+ \cup \{\infty\}$ is a scalar in the moduloid. A moduloid also has two operations: scalar multiplication and vector addition. Both operations follow the natural extensions of the scalar multiplication and the vector addition in a vector space. On the moduloid, we then define the scalar projection and the vector projection. The vector projection of a vector f onto a moduloid spanned by a set of vectors is shown to be the best substitute for f among all the vectors in that moduloid.

To facilitate the computation of operations under the $(\min, +)$-algebra, we develop the concept of ordered orthogonal bases. The vectors in the set of an ordered orthogonal basis are shown to be *linearly independent*. Moreover, the vector projection of a vector f onto a moduloid spanned by an ordered basis can be represented by a *coordinate vector*. For example, the set of vectors $f_i(t) = \rho_i t, i = 0, 1, \ldots, n$, with $\rho_0 > \rho_1 > \ldots > \rho_n \geq 0$ forms an ordered orthogonal basis. A CPL function of the form $f(t) = \min_{0 \leq i \leq n}[\sigma_i + \rho_i t]$ has the coordinate vector $(\sigma_0, \sigma_1, \ldots, \sigma_n)$ with respect to such an ordered orthogonal basis. Operations for nonnegative and increasing sequences under the $(\min, +)$-algebra, including pointwise minimum, convolution, subadditive closure, and pointwise maximum, can then computed on the domain of coordinate vectors. A vector and its coordinate vector forms a transform pair, called $\mathcal{C}$-transform in the paper. The $\mathcal{C}$-transform is related to the Legendre (or convex) transform [13] and has many properties such as Parseval's formula.

The paper is organized as follows: In Section II, we briefly review the $(\min, +)$-algebra. We define the scalar projection and the vector projection in Section III. The concept of ordered orthogonal bases is developed in Section IV. In Section V, we develop properties of the transform pairs of vectors and their

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corresponding coordinate vectors. We conclude the paper in Section VI. The proofs of various lemmas are presented in the appendices.

II. Review of the min-plus algebra

In this section, we briefly review the (min,+)-algebra (for more details of such an algebra, we refer to [2], [9], [5]). In such an algebra, one replaces the usual addition operator by the *min* operator and the usual multiplication operator by the addition operator. Following the notations in [5], we denote by $\oplus$ the $\min$ operator and $\otimes$ the $+$ operator. It is known that $(\mathcal{R} \cup \{\infty\}, \oplus, \otimes)$ is a commutative semifield with the zero element ∞ and the identity element 0 that possesses the following properties.

1. (Associativity) $\forall \alpha, \beta, \gamma \in \mathcal{R} \cup \{\infty\}$,

$$(\alpha \oplus \beta) \oplus \gamma = \alpha \oplus (\beta \oplus \gamma),$$
$$(\alpha \otimes \beta) \otimes \gamma = \alpha \otimes (\beta \otimes \gamma).$$

2. (Commutativity) $\forall \alpha, \beta \in \mathcal{R} \cup \{\infty\}$,

$$\alpha \oplus \beta = \beta \oplus \alpha,$$
$$\alpha \otimes \beta = \beta \otimes \alpha.$$

3. (Distributivity) $\forall \alpha, \beta, \gamma \in \mathcal{R} \cup \{\infty\}$,

$$(\alpha \oplus \beta) \otimes \gamma = (\alpha \otimes \gamma) \oplus (\beta \otimes \gamma).$$

Let $\mathcal{F}_{inc}$ (resp. $\mathcal{F}^0_{inc}$) be the set of increasing sequences with $f(0) \geq 0$ (resp. $f(0) = 0$). That is, a sequence $f \equiv \{f(t), t = 0, 1, 2, \ldots\} \in \mathcal{F}_{inc}$ (resp. $\mathcal{F}^0_{inc}$) satisfies $f(0) \geq 0$ (resp. $f(0) = 0$) and $f(s) \leq f(t)$ for all $s \leq t$. Two sequences are said to be equal, denoted by $f = g$, if $f(t) = g(t)$ for all t. Similarly, $f \leq g$ if $f(t) \leq g(t)$ for all t. Define the following two operations for f and g in $\mathcal{F}_{inc}$.

(i) (min) the pointwise minimum of two sequences:

$$(f \oplus g)(t) = \min[f(t), g(t)].$$

(ii) (convolution) the convolution of two sequences under the $(\min, +)$-algebra:

$$(f \star g)(t) = \min_{0 \leq s \leq t}[f(s) + g(t-s)].$$

These two operations are closed in $\mathcal{F}_{inc}$. Let ϵ be the sequence with $\epsilon(t) = \infty$ for all t, and $\mathbf{e}$ be the sequence with $\mathbf{e}(0) = 0$ and $\mathbf{e}(t) = \infty$ for all $t > 0$. Then the following properties hold:

1. (Associativity) $\forall f, g, h \in \mathcal{F}_{inc}$,

$$(f \oplus g) \oplus h = f \oplus (g \oplus h),$$
$$(f \star g) \star h = f \star (g \star h).$$

2. (Commutativity) $\forall f, g \in \mathcal{F}_{inc}$,

$$f \oplus g = g \oplus f,$$
$$f \star g = g \star f.$$

3. (Distributivity) $\forall f, g, h \in \mathcal{F}_{inc}$,

$$(f \oplus g) \star h = (f \star h) \oplus (g \star h).$$

4. (Zero element) $\forall f \in \mathcal{F}_{inc}$,

$$f \oplus \epsilon = f.$$

5. (Absorbing zero element) $\forall f \in \mathcal{F}_{inc}$,

$$f \star \epsilon = \epsilon \star f = \epsilon.$$

6. (Identity element) $\forall f \in \mathcal{F}_{inc}$,

$$f \star \mathbf{e} = \mathbf{e} \star f = f.$$

7. (Idempotency of addition) $\forall f \in \mathcal{F}_{inc}$,

$$f \oplus f = f.$$

Thus, $(\mathcal{F}_{inc}, \oplus, \star)$ is a commutative dioid (see [2]) with the zero element ϵ and the identity element $\mathbf{e}$. In addition to the algebraic properties for a dioid, there is also monotonicity that plays an important role in the development in [5].

8. (Monotonicity) $\forall f, \tilde{f}, g, \tilde{g} \in \mathcal{F}_{inc}$, if $f \leq \tilde{f}$ and $g \leq \tilde{g}$, then

$$f \oplus g \leq \tilde{f} \oplus \tilde{g} \leq \tilde{f},$$
$$f \star g \leq \tilde{f} \star \tilde{g}.$$

If g is also in $\mathcal{F}^0_{inc}$, then $f \star g \leq f$. If both f and g are in $\mathcal{F}^0_{inc}$, then $f \oplus g \geq f \star g$.

As in [5], we define the subadditive closure as follows:

$$f^* = (\mathbf{e} \oplus f)^* = \lim_{n \to \infty} (\mathbf{e} \oplus f)^{(n)}$$
$$= \lim_{n \to \infty} (\mathbf{e} \oplus f \oplus f^{(2)} \oplus \ldots \oplus f^{(n)}), \qquad (3)$$

where $f^{(n)}$ is the self convolution of f for n times, i.e., $f^{(n)} = f^{(n-1)} \star f$, $n \geq 2$ and $f^{(1)} = f$. The subadditive closure is subadditive, i.e., $f^*(s) + f^*(t-s) \geq f^*(t)$ for all $s \leq t$.

In addition to the $\oplus$ and $\star$ operators for $\mathcal{F}_{inc}$, we also define the following pointwise maximum operation in $\mathcal{F}_{inc}$.

(iii) (max) the pointwise maximum of two sequences:

$$(f \vee g)(t) = \max[f(t), g(t)].$$

III. Projections under the Min-plus algebra

In this section, we define the scalar projection and the vector projection for sequences under the $(\min, +)$-algebra. We first identify a *moduloid* (see e.g., [2]) under the $(\min, +)$-algebra. A moduloid can be viewed as a *vector space* without inverse elements. Note that there are two operations in a vector space, scalar multiplication and vector addition. The vector addition under the $(\min, +)$-algebra can be defined by the pointwise minimum $\oplus$. Similarly, we define the scalar multiplication under the $(\min, +)$-algebra as follows.

Definition 1: Let $\bar{\mathcal{R}}_+ = \{0\} \cup \mathcal{R}_+ \cup \{\infty\}$. For any $\alpha \in \bar{\mathcal{R}}_+$ and $f \in \mathcal{F}_{inc}$, we define $\alpha \otimes f$ by

$$(\alpha \otimes f)(t) = \alpha \otimes f(t) = \alpha + f(t). \qquad (4)$$

As pointed out in [5], the scalar multiplication is a special case of convolution and

$$\alpha \otimes f = \mathbf{I}_\alpha \star f, \qquad (5)$$

where

$$\mathbf{I}_\alpha(t) = \begin{cases} \alpha & \text{if } t = 0 \\ \infty & \text{if } t > 0 \end{cases} . \quad (6)$$

It is easy to see that

$$\mathbf{I}_\alpha \star \mathbf{I}_\beta = \mathbf{I}_{\alpha \otimes \beta}. \quad (7)$$

and that

$$\mathbf{I}_\alpha \oplus \mathbf{I}_\beta = \mathbf{I}_{\alpha \oplus \beta}. \quad (8)$$

In view of this, one can easily verify from the properties in the dioid $(\mathcal{F}_{inc}, \oplus, \star)$ that the following properties hold:
1. $\oplus$ is associative, commutative;
2. $\alpha \otimes (f \oplus g) = (\alpha \otimes f) \oplus (\alpha \otimes g)$;
3. $(\alpha \oplus \beta) \otimes f = (\alpha \otimes f) \oplus (\beta \otimes f)$;
4. $\alpha \otimes (\beta \otimes f) = (\alpha \otimes \beta) \otimes f$;
5. $0 \otimes f = f$;
6. $\infty \otimes f = \infty$;

for all $\alpha, \beta \in \bar{\mathcal{R}}_+$ and $f, g \in \mathcal{F}_{inc}$.
Thus, $\mathcal{F}_{inc}$ with the pointwise minimum $\oplus$ as the vector addition and the scalar multiplication in Definition 1 is a moduloid over the semifield $(\bar{\mathcal{R}}_+, \oplus, \otimes)$. The moduloid also has the monotonicity that $\alpha \otimes f \geq \beta \otimes g$ provided that $\alpha \geq \beta$ and $f \geq g$.

Following the notations in a vector space, we call $\alpha \in \bar{\mathcal{R}}_+$ a *scalar* and $f \in \mathcal{F}_{inc}$ a *vector*. As in a vector space, we say a vector f can be written as a *linear combination* of a set of vectors $\{f_0, f_1, \ldots, f_n\}$ if there exist scalars α_i, $i = 0, 1, \ldots, n$ such that

$$f = (\alpha_0 \otimes f_0) \oplus (\alpha_1 \otimes f_1) \oplus \ldots \oplus (\alpha_n \otimes f_n). \quad (9)$$

To ease the presentation, in this paper we use $\bigoplus_{i=0}^n \alpha_i \otimes f_i$ to denote the right hand side of (9). The set of all linear combinations of $\{f_0, f_1, \ldots, f_n\}$ is called the *span* of $\{f_0, f_1, \ldots, f_n\}$. The span of $\{f_0, f_1, \ldots, f_n\}$ is denoted by Span$(f_0, f_1, \ldots, f_n)$. Clearly, Span$(f_0, f_1, \ldots, f_n)$ is also a moduloid.

Now we define the scalar projection and the vector projection for sequences under the $(\min, +)$-algebra.

Definition 2: For $f, g \in \mathcal{F}_{inc}$, define the scalar projection from a vector f onto a vector g as follows:

$$\langle f, g \rangle = \sup_{t \geq 0} [f(t) - g(t)]^+, \quad (10)$$

where $x^+ = \max[0, x]$. In this paper, we use the convention that $\infty - \infty = 0$. Define the vector projection from f to g by $\langle f, g \rangle \otimes g$.

Note that the scalar projection is a mapping from $\mathcal{F}_{inc} \times \mathcal{F}_{inc}$ to $\bar{\mathcal{R}}_+$, while the vector projection is a mapping from $\mathcal{F}_{inc} \times \mathcal{F}_{inc}$ to $\mathcal{F}_{inc}$. Intuitively, one may view the projection operator under the $(\min, +)$-algebra as the counterpart of the inner product in a vector space. However, as shown in the following lemma, some of their properties might be different from those for the inner product in a vector space.

Lemma 3: For all $f, f_1, f_2, g, g_1, g_2, h \in \mathcal{F}_{inc}$ and $\alpha, \beta \in \bar{\mathcal{R}}_+$, we have the following properties for scalar projection.
(i) $\langle \alpha \otimes f, g \rangle = \alpha \otimes \langle f, g \rangle = \langle f, g \rangle + \alpha$.
(ii) $\langle f_1 \oplus f_2, g \rangle \leq \langle f_1, g \rangle \oplus \langle f_2, g \rangle$.
(iii) $\langle f, \alpha \otimes g \rangle = [\langle f, g \rangle - \alpha]^+$.
(iv) $\langle f, g_1 \oplus g_2 \rangle = \max[\langle f, g_1 \rangle, \langle f, g_2 \rangle]$.
(v) $\langle f_1 \vee f_2, g \rangle = \max[\langle f_1, g \rangle, \langle f_2, g \rangle]$.
(vi) (Triangle inequality) $\langle f, g \rangle \leq \langle f, h \rangle + \langle h, g \rangle$.
(vii) (Monotonicity) $\langle f_1, g_1 \rangle \geq \langle f_2, g_2 \rangle$ if $f_1 \geq f_2$ and $g_1 \leq g_2$.
(viii) $\langle f, g \rangle = 0$ if and only if $f \leq g$.

Proof. We only prove (iii). The rest of the results are obvious. Note that

$$\begin{aligned} \langle f, \alpha \otimes g \rangle &= [\sup_{t \geq 0}[f(t) - g(t)] - \alpha]^+ \\ &\leq [\sup_{t \geq 0}[f(t) - g(t)]^+ - \alpha]^+ = [\langle f, g \rangle - \alpha]^+. \end{aligned}$$

On the other hand, note from $(x - \alpha)^+ \geq x^+ - \alpha$ for $\alpha \geq 0$ that

$$\begin{aligned} \langle f, \alpha \otimes g \rangle &= \sup_{t \geq 0}[f(t) - g(t) - \alpha]^+ \\ &\geq \sup_{t \geq 0}[f(t) - g(t)]^+ - \alpha = \langle f, g \rangle - \alpha. \end{aligned}$$

As the scalar projection is nonnegative,

$$\langle f, \alpha \otimes g \rangle \geq [\langle f, g \rangle - \alpha]^+.$$

■

In the following lemma, we show that the vector projection from a vector f onto a vector g is the least vector in Span(g) that is not smaller than f.

Lemma 4: Let $p = \langle f, g \rangle \otimes g$ be the vector projection from f to g. Then for any vector $h \in$ Span(g) and $h \geq f$, one has $h \geq p \geq f$.

Proof. We first show that $p \geq f$. In view of Lemma 3(viii), it suffices to show that $\langle f, p \rangle = 0$. Note from Lemma 3(iii) that

$$\langle f, p \rangle = \langle f, \langle f, g \rangle \otimes g \rangle = [\langle f, g \rangle - \langle f, g \rangle]^+ = 0.$$

We next show that $p \leq h$. Since $h \in$ Span(g), $h = \alpha \otimes g$ for some $\alpha \in \bar{\mathcal{R}}_+$. In view of Lemma 3(i), one has

$$\langle h, g \rangle = \langle \alpha \otimes g, g \rangle = \alpha.$$

As it is assumed that $h \geq f$, it follows from the monotonicity of the scalar projection in Lemma 3(vii) that

$$\alpha = \langle h, g \rangle \geq \langle f, g \rangle.$$

Thus, we have from the monotonicity of the scalar multiplication that

$$h = \alpha \otimes g \geq \langle f, g \rangle \otimes g = p.$$

■

Definition 5: Consider a set of vectors $\{f_0, f_1 \ldots, f_n\}$. Let $p_i = \langle f, f_i \rangle \otimes f_i$ be the vector projection from f onto f_i, $i = 0, 1, \ldots, n$, and

$$p = p_0 \oplus p_1 \oplus \ldots \oplus p_n. \quad (11)$$

The vector p in (11) is called the vector projection of f onto Span$(f_0, f_1, \ldots, f_n)$.

We note that without taking x^+ in (10), our definition for vector projection onto the span of a set of vectors is equivalent to that in [8], pp. 169, when considering finite sequences.

In the following theorem, we extend the result in Lemma 4 to the span of a set of vectors.

Theorem 6: Consider a set of vectors $\{f_0, f_1 \ldots, f_n\}$. Let p be the vector projection from f onto Span$(f_0, f_1, \ldots, f_n)$. Then for any vector $h \in$ Span$(f_0, f_1, \ldots, f_n)$ and $h \geq f$, one has $h \geq p \geq f$, i.e., p is the least vector in Span$(f_0, f_1, \ldots, f_n)$ that is not smaller than f.

Proof. We first show that $p \geq f$. From Lemma 4, we have $p_i \geq f$. It then follows from the monotonicity and idempotency of $\oplus$ that

$$p = p_0 \oplus p_1 \oplus \ldots \oplus p_n \geq f \oplus f \oplus \ldots \oplus f = f.$$

We next show that $p \leq h$. Since $h \in$ Span$(f_0, f_1, \ldots, f_n)$, $h = \bigoplus_{i=0}^{n} \alpha_i \otimes f_i$ for some $\alpha_i \in \bar{\mathcal{R}}_+$, $i = 0, 1, \ldots, n$. As it is assumed that $h \geq f$, it follows from Lemma 3(i), the monotonicity of $\oplus$ and the monotonicity of the scalar projection in Lemma 3(vii) that

$$\alpha_i = \langle \alpha_i \otimes f_i, f_i \rangle \geq \langle \bigoplus_{j=0}^{n} \alpha_j \otimes f_j, f_i \rangle = \langle h, f_i \rangle \geq \langle f, f_i \rangle.$$

Thus, we have from the monotonicity of the scalar multiplication and the vector addition that

$$h = \bigoplus_{i=0}^{n} \alpha_i \otimes f_i \geq \bigoplus_{i=0}^{n} \langle f, f_i \rangle \otimes f_i = p.$$

■

Corollary 7: (i) If f is in Span$(f_0, f_1, \ldots, f_n)$, then the vector projection p of f onto Span$(f_0, f_1, \ldots, f_n)$ is itself, i.e.,

$$f = p = \bigoplus_{i=0}^{n} \langle f, f_i \rangle \otimes f_i.$$

(ii) For any $f \in \mathcal{F}_{inc}$ with the vector projection p of f onto Span$(f_0, f_1, \ldots, f_n)$, one has $\langle p, f_i \rangle = \langle f, f_i \rangle$ for all i.

Proof. (i) It is a direct consequence from Theorem 6 by choosing $h = f$.

(ii) Since $p = \bigoplus_{i=0}^{n} \langle f, f_i \rangle \otimes f_i$, it follows from the monotonicity of $\oplus$ and Theorem 6 that

$$\langle f, f_i \rangle \otimes f_i \geq p \geq f.$$

Using the monotonicity of the projection in Lemma 3(vii) yields

$$\langle \langle f, f_i \rangle \otimes f_i, f_i \rangle \geq \langle p, f_i \rangle \geq \langle f, f_i \rangle.$$

Note from Lemma 3(i) and (viii) that

$$\langle \langle f, f_i \rangle \otimes f_i, f_i \rangle = \langle f, f_i \rangle \otimes \langle f_i, f_i \rangle = \langle f, f_i \rangle.$$

Thus, $\langle p, f_i \rangle = \langle f, f_i \rangle$. ■

IV. Ordered orthogonal bases under the min-plus algebra

In a vector space, a basis is a set of linearly independent vectors that spans the vector space. For a vector space with an inner product (an inner product space), a set of orthogonal vectors are linearly independent and thus can be used as a basis for that vector space. Parallel to these concepts in a vector space, we develop the concept of ordered orthogonal bases under the $(\min, +)$-algebra. As there does not exist inverse elements on the moduloid, there are several levels of *linear independence* discussed in [16]. In the following, we adopt the definition of linear independence in [2]. This corresponds to IP2 in [16].

Definition 8: A set of vectors $\{f_0, f_1, \ldots, f_n\}$ are said to be *linearly dependent* if scalars $\alpha_0, \alpha_1, \ldots, \alpha_n$ not all ∞, and a subset $I \in \{0, 1, \ldots, n\}$ exist such that

$$\bigoplus_{k \in I} \alpha_k \otimes f_k = \bigoplus_{k \notin I} \alpha_k \otimes f_k. \tag{12}$$

The set of vectors $\{f_0, f_1, \ldots, f_n\}$ are said to be *linearly independent* if they are not linearly dependent.

Lemma 9: Consider a set of vectors $\{f_0, f_1, \ldots, f_n\}$. If

$$\langle f_i, f_j \rangle = \begin{cases} 0 & \text{if } i > j \\ \infty & \text{if } i < j \end{cases}, \tag{13}$$

then these vectors are linearly independent.

The proof of Lemma 9 is given in Appendix A. As discussed in the previous section, if we view the scalar projection under the $(\min, +)$-algebra as the counterpart of an inner product in a vector space, the condition in (13) may be viewed as the counterpart of orthogonality in an inner product space. This leads us to define the concept of ordered orthogonal bases in Definition 10.

Definition 10: Consider a set of vectors $S \in \mathcal{F}_{inc}$. If a set of *linearly independent* vectors $\{f_0, f_1, \ldots f_n\}$ that spans S, then the set of vectors $\{f_0, f_1, \ldots f_n\}$ is called a *basis* of S. A basis is called an *ordered orthogonal basis* in this paper if (13) holds.

In view of Lemma 3(viii) and (10), a set of vectors $\{f_0, f_1, \ldots f_n\}$ is an ordered orthogonal basis if and only if for $i = 0, 1, \ldots, n-1$ and all t

$$f_i(t) \geq f_{i+1}(t),$$
$$\sup_{t \geq 0} [f_i(t) - f_{i+1}(t)] = \infty.$$

Example 11: **(Standard basis)** *Consider the set of vectors* $\{\mathbf{O}_d, d = 0, 1, 2, \ldots\}$*, where*

$$\mathbf{O}_d(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq d \\ \infty & \text{otherwise} \end{cases}. \tag{14}$$

It is easy to see that

$$\langle \mathbf{O}_{d_1}, \mathbf{O}_{d_2} \rangle = \begin{cases} 0 & \text{if } d_1 > d_2 \\ \infty & \text{if } d_1 < d_2 \end{cases}. \tag{15}$$

Thus, the set of vectors are linearly independent from Lemma 9. Moreover, for any $f \in \mathcal{F}_{inc}$,

$$f = \bigoplus_{d=0}^{\infty} f(d) \otimes \mathbf{O}_d = \bigoplus_{d=0}^{\infty} \langle f, \mathbf{O}_d \rangle \otimes \mathbf{O}_d. \tag{16}$$

This shows that the set of vectors $\{\mathbf{O}_d, d = 0,1,2,\ldots\}$ is an ordered orthogonal basis for $\mathcal{F}_{inc}$. Such a basis is called the standard basis in the paper. As $\langle f, \mathbf{O}_d\rangle = f(d)$ for any $f \in \mathcal{F}_{inc}$, one can easily verify that

$$\langle f \oplus g, \mathbf{O}_d\rangle = \langle f, \mathbf{O}_d\rangle \oplus \langle g, \mathbf{O}_d\rangle. \tag{17}$$

This is an important property as in general we only have inequalities in Lemma 3(ii).

Example 12: Consider the set of vectors $f_i(t) = \rho_i t^{\alpha_i}, i = 0,1,\ldots,n$, with $\rho_0 > \rho_1 > \ldots > \rho_n \geq 0$ and $\alpha_0 \geq \alpha_1 \geq \ldots \geq \alpha_n > 0$. It is easy to see that

$$\langle f_i, f_j\rangle = \begin{cases} 0 & \text{if } i > j \\ \infty & \text{if } i < j \end{cases}. \tag{18}$$

Thus, the set of vectors $\{f_i(t) = \rho_i t^{\alpha_i}, i = 0,1,\ldots,n\}$ is an ordered orthogonal basis for Span$(f_0, f_1, \ldots, f_n)$. From Corollary 7(i), for any vector f in Span$(f_0, f_1, \ldots, f_n)$,

$$f = \bigoplus_{i=0}^{n} \langle f, f_i\rangle \otimes f_i.$$

In particular, when $\alpha_i = 1$ for all i, we note that any f in Span$(f_0, f_1, \ldots, f_n)$ can be written as $f(t) = \min_{0\leq i\leq n}[\rho_i t + \sigma_i]$, where

$$\sigma_i = \langle f, f_i\rangle = \sup_{t\geq 0}[f(t) - \rho_i t]^+, \quad i = 0,1,\ldots,n.$$

Thus, Span$(f_0, f_1, \ldots, f_n)$ corresponds to the family of CPL sequences in [14].

V. $\mathcal{C}$-TRANSFORM UNDER THE MIN-PLUS ALGEBRA

In the previous section, we developed the concept of ordered orthogonal bases. It is natural to consider the coordinate vectors with respect to ordered orthogonal bases. In this section, we show that a vector and its coordinate vector forms a transform pair, called $\mathcal{C}$-transform in Definition 13. Operations on vectors, such as pointwise minimum, convolution, subadditive closure, and pointwise maximum, can be derived by operations on coordinate vectors.

Definition 13: Consider a set of vectors $S_1 \in \mathcal{F}_{inc}$ with an ordered orthogonal basis $\{f_0, f_1, \ldots, f_n\}$. Let $S_2 =$ Span$(\mathbf{O}_0, \mathbf{O}_1, \ldots, \mathbf{O}_n)$. Define a mapping $\mathcal{C} : \mathcal{F}_{inc} \mapsto S_2$ as follows:

$$\mathcal{C}(f) = \bigoplus_{i=0}^{n} \langle f, f_i\rangle \otimes \mathbf{O}_i. \tag{19}$$

We call $\mathcal{C}(f)$ the $\mathcal{C}$-transform of f. Also, define the "inverse" mapping $\mathcal{C}^{-1} : \mathcal{F}_{inc} \mapsto S_1$ as follows:

$$\mathcal{C}^{-1}(f) = \bigoplus_{i=0}^{n} \langle f, \mathbf{O}_i\rangle \otimes f_i. \tag{20}$$

We call $\mathcal{C}^{-1}(f)$ the inverse $\mathcal{C}$-transform of f.

As Span$(\mathbf{O}_0, \mathbf{O}_1, \ldots, \mathbf{O}_n)$ is the standard basis, one has $\mathcal{C}(f)(m) = \langle f, f_m\rangle$, $m = 0,1,\ldots,n$. Thus, the $\mathcal{C}$-transform of f may be viewed the *coordinate vector* of f with respect to the ordered orthogonal basis $\{f_0, f_1, \ldots, f_n\}$. In particular, if we consider the ordered orthogonal basis $\{f_m(t) = (n-m)t, m = 0,1,\ldots,n\}$. Then

$$\mathcal{C}(f)(m) = \sup_{t\geq 0}[f(t) - (n-m)t]^+, \quad m = 0,1,\ldots,n,$$

and

$$\begin{aligned} \mathcal{C}^{-1}(g)(t) &= \min_{0\leq m\leq n}[g(m) + (n-m)t] \\ &= \min_{0\leq m\leq n}[g(n-m) + mt], \quad t \geq 0. \end{aligned}$$

Thus,

$$\mathcal{C}^{-1}(\mathcal{C}(f))(t) = \min_{0\leq m\leq n}[\sup_{\tau\geq 0}[f(\tau) - m\tau]^+ + mt].$$

When extending m from the bounded set $\{0,1,\ldots,n\}$ to all the nonnegative real numbers, this particular $\mathcal{C}$-transform pair reduces to the (one-sided) Legendre transform pair [13].

In the following, we derive some properties for the transform pair. The proofs of these properties are given in Appendix B.

Lemma 14: Consider the $\mathcal{C}$-transform and its inverse transform in Definition 13.

(i) $\mathcal{C}$-transform is a sublinear transformation, i.e.,

$$\begin{aligned} \mathcal{C}(\alpha \otimes f) &= \alpha \otimes \mathcal{C}(f), \\ \mathcal{C}(f \oplus g) &\leq \mathcal{C}(f) \oplus \mathcal{C}(g). \end{aligned}$$

(ii) $\mathcal{C}^{-1}$-transform is a linear transformation, i.e.,

$$\begin{aligned} \mathcal{C}^{-1}(\alpha \otimes f) &= \alpha \otimes \mathcal{C}^{-1}(f), \\ \mathcal{C}^{-1}(f \oplus g) &= \mathcal{C}^{-1}(f) \oplus \mathcal{C}^{-1}(g). \end{aligned}$$

(iii) Both $\mathcal{C}$-transform and $\mathcal{C}^{-1}$-transform are isotone, i.e., for $f \leq g$

$$\begin{aligned} \mathcal{C}(f) &\leq \mathcal{C}(g), \\ \mathcal{C}^{-1}(f) &\leq \mathcal{C}^{-1}(g). \end{aligned}$$

(iv) For any $f \in \mathcal{F}_{inc}$, $\langle \mathcal{C}(f), \mathbf{O}_i\rangle = \langle f, f_i\rangle$ for all i.
(v) For any $f \in \mathcal{F}_{inc}$, $\langle \mathcal{C}^{-1}(f), f_i\rangle \leq \langle f, \mathbf{O}_i\rangle$ for all i.
(vi) (Transformation of bases) $\mathcal{C}(f_i) = \mathbf{O}_i$ and $\mathcal{C}^{-1}(\mathbf{O}_i) = f_i$ for all i.
(vii) For any $f \in \mathcal{F}_{inc}$, $\mathcal{C}^{-1}(\mathcal{C}(f))$ is the vector projection of f onto S_1. Thus,

$$\mathcal{C}^{-1}(\mathcal{C}(f)) \geq f, \tag{21}$$

with equality holding if $f \in S_1$.
(viii) If $f \in S_2$, then $\mathcal{C}(\mathcal{C}^{-1}(f)) \leq f$.
(ix) For any $f \in \mathcal{F}_{inc}$, $\mathcal{C}\Big(\mathcal{C}^{-1}(\mathcal{C}(f))\Big) = \mathcal{C}(f)$. That is, a vector f and its vector projection onto S_1 have the same coordinate vector.

With the properties derived in Lemma 14, we show that the operations for vectors, including pointwise minimum, convolution, subadditive closure, and pointwise maximum, may be computed by their coordinate vectors if they have the same ordered orthogonal basis that satisfies certain properties. The proofs of these results are given in Appendix C.

Lemma 15: Consider the $\mathcal{C}$-transform and its inverse transform in Definition 13.

(i) For any $f, g \in S_1$,

$$f \oplus g = \mathcal{C}^{-1}(\mathcal{C}(f) \oplus \mathcal{C}(g)). \tag{22}$$

(ii) If f_i's are *subadditive*, i.e., $f_i(s) + f_i(t-s) \geq f_i(t)$, $s \leq t$, and $f_i(0) = 0$ for all i, then for any $f, g \in S_1$,

$$\begin{aligned} f \star g &= (\langle g, f_0 \rangle \otimes f) \oplus (\langle f, f_0 \rangle \otimes g) \\ &= \mathcal{C}^{-1}\Big((\langle \mathcal{C}(g), \mathbf{O}_0 \rangle \otimes \mathcal{C}(f)) \oplus (\langle \mathcal{C}(f), \mathbf{O}_0 \rangle \otimes \mathcal{C}(g)) \Big). \end{aligned}$$

(iii) Under the conditions in (ii), if $f_0 = \mathbf{O}_0$, then for any $f \in S_1$,

$$f^* = f_0 \oplus f = \mathcal{C}^{-1}(\mathbf{O}_0 \oplus \mathcal{C}(f)). \tag{23}$$

(iv) For any $f, g \in \mathcal{F}_{inc}$,

$$\mathcal{C}^{-1}(\mathcal{C}(f \vee g)) = \mathcal{C}^{-1}(\mathcal{C}(f) \vee \mathcal{C}(g)). \tag{24}$$

In view of Lemma 15(i), to compute the pointwise minimum of two vectors f and g in S_1, one can compute the pointwise minimum of their coordinate vectors and then take the inverse transform. As the number of vectors in the basis of S_1 is usually finite, this reduces the computation complexity from the pointwise minimum of two infinite sequences to the pointwise minimum of two finite sequences. Lemma 15(ii) implies that the computation of the convolution of two vectors f and g in S_1 can be easily computed on the domain of coordinate vectors if the ordered orthogonal basis $\{f_0, f_1, \ldots, f_n\}$ consists of a set of subadditive sequences (with $f_i(0) = 0$). Note that $\langle h, \mathbf{O}_0 \rangle = h(0)$ for any $h \in \mathcal{F}_{inc}$. Thus, for $m = 0, 1 \ldots, n$,

$$\begin{aligned} &\Big((\langle \mathcal{C}(g), \mathbf{O}_0 \rangle \otimes \mathcal{C}(f)) \oplus (\langle \mathcal{C}(f), \mathbf{O}_0 \rangle \otimes \mathcal{C}(g)) \Big)(m) \\ &= \min[\mathcal{C}(g)(0) + \mathcal{C}(f)(m), \mathcal{C}(f)(0) + \mathcal{C}(g)(m)]. \end{aligned}$$

The complexity of computing the convolution of vectors in S_1 is almost the same as that of computing the pointwise minimum. To compute the subadditive closure for a vector in S_1, in addition to the conditions in Lemma 15(ii), we also require the basis to contain the identity element $\mathbf{e}$ in Lemma 15(iii). Note that $\mathbf{O}_0$ is in fact the identity element $\mathbf{e}$ and that

$$\Big(\mathbf{O}_0 \oplus \mathcal{C}(f)\Big)(m) = \Big(\mathbf{e} \oplus \mathcal{C}(f)\Big)(m) = \begin{cases} 0 & \text{if } m = 0 \\ \mathcal{C}(f)(m) & \text{otherwise} \end{cases}.$$

As the pointwise maximum of two vectors in S_1 may not be in S_1, one needs to find its projection onto S_1 as a substitute. Lemma 15(iv) implies that this can be done by taking the pointwise maximum of the coordinate vectors of f and g.

Lemma 16 below shows that the projection of a vector f onto a vector g may be computed (or bounded) by the projection of the coordinate vector of f onto the coordinate vector of g. This is analogous to Parseval's formula for orthonormal bases in a vector space (see e.g., [11]). The proof of Lemma 16 is given in Appendix D.

Lemma 16: Consider the $\mathcal{C}$-transform and its inverse transform in Definition 13.

(i) For any $f \in \mathcal{F}_{inc}$ and $g \in S_1$,

$$\langle f, g \rangle = \langle \mathcal{C}(f), \mathcal{C}(g) \rangle. \tag{25}$$

(ii) For any $f \in \mathcal{F}_{inc}$ and $g \in S_2$,

$$\langle f, g \rangle \geq \langle \mathcal{C}^{-1}(f), \mathcal{C}^{-1}(g) \rangle. \tag{26}$$

VI. Conclusions

In this paper, we used the vector projection under the $(\min, +)$-algebra for the parameterization problem of a traffic envelope. We showed that the vector projection is the best substitute in the sense that it is the least vector among the span of a set of vectors that are not smaller than the traffic envelope.

We also developed the concept of ordered orthogonal bases. The vector projection of a traffic envelope can then be represented by a coordinate vector with respect to an ordered orthogonal basis. We showed that operations under the $(\min, +)$-algebra, including pointwise minimum, convolution, subadditive closure, and pointwise maximum, could be computed on the domain of coordinate vectors. These are based on the properties derived for the $\mathcal{C}$-transform pair of the vector projection and its coordinate vector. Finally, we note that our approach is also applicable to functions (continuous-time setting), though we only discussed sequences (discrete-time setting) in this paper.

Appendix

I. Appendix A

We prove Lemma 9 by contradiction. Suppose these n vectors are linearly dependent. From Definition 8, there exists scalars $\alpha_0, \alpha_1, \ldots, \alpha_n$ not all ∞, and a subset $I \in \{0, 1, \ldots, n\}$ exists such that

$$\bigoplus_{k \in I} \alpha_k \otimes f_k = \bigoplus_{k \notin I} \alpha_k \otimes f_k. \tag{27}$$

By removing scalars that are equal to ∞, the above statement is equivalent to the existence of two disjoint subset I_1 and I_2 such that

$$\bigoplus_{k \in I_1} \alpha_k \otimes f_k = \bigoplus_{k \in I_2} \alpha_k \otimes f_k, \tag{28}$$

and all the scalars $\alpha_k \in I_1 \cup I_2$ ($I_1 \cup I_2$ is nonempty) are not equal to ∞. Let i_1 be the largest index in $I_1 \cup I_2$. Without loss of generality, assume that $i_1 \in I_1$. From the monotonicity of $\oplus$, we have

$$\alpha_{i_1} \otimes f_{i_1} \geq \bigoplus_{k \in I_1} \alpha_k \otimes f_k = \bigoplus_{k \in I_2} \alpha_k \otimes f_k \geq \bigoplus_{k \in I_2} f_k, \tag{29}$$

as $\alpha_k \geq 0$. Let i_2 be the largest index in I_2. As I_1 and I_2 are disjoint, $i_2 < i_1$. Since we assume that $\langle f_i, f_j \rangle = 0$ for all $j < i$, we have that $f_j \geq f_{i_2}$ for all $j < i_2$ and thus $\bigoplus_{k \in I_2} f_k = f_{i_2}$. Using the monotonicity of projections in Lemma 3(vii) and (29), we have

$$\langle \alpha_{i_1} \otimes f_{i_1}, f_{i_1} \rangle \geq \langle f_{i_2}, f_{i_1} \rangle = \infty, \tag{30}$$

as we assume that $\langle f_j, f_i \rangle = \infty$ for all $j < i$. Note from Lemma 3(i) that the left hand side of (30) is simply $\alpha_{i,1}$ as $\langle f_{i_1}, f_{i_1} \rangle = 0$. This shows that $\alpha_{i,1} \geq \infty$ and we reach a contradiction.

II. Appendix B

In this section, we prove Lemma 14.

(i) This follows directly from Lemma 3(i) and (ii).

(ii) This is a direct consequence of Lemma 3(i) and (17).

(iii) That both transforms are isotone follows from the monotonicity of scalar projection in Lemma 3(vii).

(iv) Note from (17), Lemma 3(i), and (15) that

$$\begin{aligned}\langle \mathcal{C}(f), \mathbf{O}_i\rangle &= \langle \bigoplus_{j=0}^{n} \langle f, f_j\rangle \otimes \mathbf{O}_j, \mathbf{O}_i\rangle \\ &= \bigoplus_{j=0}^{n} \langle\langle f, f_j\rangle \otimes \mathbf{O}_j, \mathbf{O}_i\rangle \\ &= \bigoplus_{j=0}^{n} \langle f, f_j\rangle \otimes \langle \mathbf{O}_j, \mathbf{O}_i\rangle \\ &= \bigoplus_{j=i}^{n} \langle f, f_j\rangle.\end{aligned}$$

From the triangle inequality in Lemma 3(vi) and (13), it follows that for $j > i$

$$\langle f, f_i\rangle \le \langle f, f_j\rangle + \langle f_j, f_i\rangle = \langle f, f_j\rangle.$$

Thus, $\langle \mathcal{C}(f), \mathbf{O}_i\rangle = \langle f, f_i\rangle$.

(v) Note from Lemma 3(vii), (i), and (13) that

$$\begin{aligned}\langle \mathcal{C}^{-1}(f), f_i\rangle &= \langle \bigoplus_{j=0}^{n} \langle f, \mathbf{O}_j\rangle \otimes f_j, f_i\rangle \\ &\le \langle\langle f, \mathbf{O}_i\rangle \otimes f_i, f_i\rangle \\ &= \langle f, \mathbf{O}_i\rangle \otimes \langle f_i, f_i\rangle = \langle f, \mathbf{O}_i\rangle.\end{aligned}$$

(vi) In view of (13) that

$$\mathcal{C}(f_i) = \bigoplus_{j=0}^{n} \langle f_i, f_j\rangle \otimes \mathbf{O}_j = \bigoplus_{j=0}^{i} \mathbf{O}_j = \mathbf{O}_i.$$

Also, it follows from (15) that

$$\mathcal{C}^{-1}(\mathbf{O}_i) = \bigoplus_{j=0}^{n} \langle \mathbf{O}_i, \mathbf{O}_j\rangle \otimes f_j = \bigoplus_{j=0}^{i} f_j = f_i,$$

as $f_i \le f_j$ for all $j < i$.

(vii) As we have shown that $\mathcal{C}^{-1}$ is a linear transformation in (ii), it then follows from (vi) that

$$\begin{aligned}\mathcal{C}^{-1}(\mathcal{C}(f)) &= \mathcal{C}^{-1}(\bigoplus_{i=0}^{n} \langle f, f_i\rangle \otimes \mathbf{O}_i) \\ &= \bigoplus_{i=0}^{n} \langle f, f_i\rangle \otimes \mathcal{C}^{-1}(\mathbf{O}_i) = \bigoplus_{i=0}^{n} \langle f, f_i\rangle \otimes f_i,\end{aligned}$$

which is exactly the vector projection of f onto S_1. The rest of the results in (vi) then follow from Theorem 6 and Corollary 7(i).

(viii) Since $\mathcal{C}$ is a sublinear transformation in (i), we have from (vi) that

$$\begin{aligned}\mathcal{C}(\mathcal{C}^{-1}(f)) &= \mathcal{C}(\bigoplus_{i=0}^{n} \langle f, \mathbf{O}_i\rangle \otimes f_i) \\ &\le \bigoplus_{i=0}^{n} \langle f, \mathbf{O}_i\rangle \otimes \mathcal{C}(f_i) = \bigoplus_{i=0}^{n} \langle f, \mathbf{O}_i\rangle \otimes \mathbf{O}_i.\end{aligned}$$

As we assume that $f \in S_2$, we have from Corollary 7(i) that $f = \bigoplus_{i=0}^{n} \langle f, \mathbf{O}_i\rangle \otimes \mathbf{O}_i$. Thus, $\mathcal{C}(\mathcal{C}^{-1}(f)) \le f$.

(ix) As $\mathcal{C}$-transform is isotone in (iii), we have from (vii) that

$$\mathcal{C}\Big(\mathcal{C}^{-1}(\mathcal{C}(f))\Big) \ge \mathcal{C}(f).$$

Since $\mathcal{C}(f) \in S_2$, we also have from (viii) that

$$\mathcal{C}\Big(\mathcal{C}^{-1}(\mathcal{C}(f))\Big) \le \mathcal{C}(f).$$

III. Appendix C

In this section, we prove Lemma 15.

(i) As $\mathcal{C}^{-1}$-transform is linear, we have from Lemma 14(vii) that for $f, g \in S_1$,

$$\mathcal{C}^{-1}(\mathcal{C}(f) \oplus \mathcal{C}(g)) = \mathcal{C}^{-1}(\mathcal{C}(f)) \oplus \mathcal{C}^{-1}(\mathcal{C}(g)) = f \oplus g.$$

(ii) As we assume that both f and g are in S_1,

$$\begin{aligned}f &= \bigoplus_{i=0}^{n} \langle f, f_i\rangle \otimes f_i, \\ g &= \bigoplus_{i=0}^{n} \langle g, f_i\rangle \otimes f_i.\end{aligned}$$

Let $\alpha_i = \langle f, f_i\rangle$ and $\beta_i = \langle g, f_i\rangle$. Since $\langle f_i, f_j\rangle = 0$ for $i > j$ in (13), $f_i \le f_j$ for $i > j$. We have from the monotonicity in Lemma 3(vii) that $\alpha_0 \le \alpha_1 \le \ldots \le \alpha_n$ and $\beta_0 \le \beta_1 \le \ldots \le \beta_n$.

It then follows from (5), distributivity and idempotency of the dioid $(\mathcal{F}_{inc}, \oplus, \star)$ that

$$\begin{aligned}f \star g &= \Big(\bigoplus_{i=0}^{n} \mathbf{I}_{\alpha_i} \star f_i\Big) \star \Big(\bigoplus_{i=0}^{n} \mathbf{I}_{\beta_i} \star f_i\Big) \\ &= \bigoplus_{i=0}^{n} \bigoplus_{j=0}^{n} (\mathbf{I}_{\alpha_i} \star \mathbf{I}_{\beta_j} \star f_i \star f_j) \\ &= \bigoplus_{i=0}^{n} \bigoplus_{j=0}^{i} (\mathbf{I}_{\alpha_i} \star \mathbf{I}_{\beta_j} \star f_i \star f_j) \\ &\quad \oplus \bigoplus_{i=0}^{n} \bigoplus_{j=0}^{i} (\mathbf{I}_{\alpha_j} \star \mathbf{I}_{\beta_i} \star f_j \star f_i).\end{aligned}$$

As f_i is subadditive and $f_i(0) = 0$, one has for $i \ge j$

$$(f_i \star f_j)(t) = \min_{0 \le s \le t} [f_i(s) + f_j(t-s)] = f_i(t). \qquad (31)$$

Thus,

$$\begin{aligned}f \star g &= \bigoplus_{i=0}^{n} \bigoplus_{j=0}^{i} (\mathbf{I}_{\alpha_i} \star \mathbf{I}_{\beta_j} \star f_i) \oplus \bigoplus_{i=0}^{n} \bigoplus_{j=0}^{i} (\mathbf{I}_{\alpha_j} \star \mathbf{I}_{\beta_i} \star f_i) \\ &= \bigoplus_{i=0}^{n} \Big(\bigoplus_{j=0}^{i} (\mathbf{I}_{\alpha_i} \star \mathbf{I}_{\beta_j}) \oplus (\mathbf{I}_{\alpha_j} \star \mathbf{I}_{\beta_i}) \star f_i\Big).\end{aligned}$$

Using (7), (8), and (5) yields

$$
\begin{aligned}
f \star g &= \bigoplus_{i=0}^{n} \Big(\bigoplus_{j=0}^{i} (\mathbf{I}_{\alpha_i \otimes \beta_j}) \oplus (\mathbf{I}_{\alpha_j \otimes \beta_i}) \star f_i \Big) \\
&= \bigoplus_{i=0}^{n} \Big(\bigoplus_{j=0}^{i} \mathbf{I}_{(\alpha_i \otimes \beta_j) \oplus (\alpha_j \otimes \beta_i)} \star f_i \Big) \\
&= \bigoplus_{i=0}^{n} \Big(\bigoplus_{j=0}^{i} (\alpha_i \otimes \beta_j) \oplus (\alpha_j \otimes \beta_i) \Big) \otimes f_i.
\end{aligned}
$$

As both α_i and β_i are increasing in i,

$$
f \star g = \bigoplus_{i=0}^{n} \Big((\alpha_i \otimes \beta_0) \oplus (\alpha_0 \otimes \beta_i) \Big) \otimes f_i.
$$

Using distributivity, we have

$$
\begin{aligned}
f \star g &= \Big(\bigoplus_{i=0}^{n} (\alpha_i \otimes \beta_0) \otimes f_i \Big) \oplus \Big(\bigoplus_{i=0}^{n} (\alpha_0 \otimes \beta_i) \otimes f_i \Big) \\
&= \Big(\beta_0 \otimes (\bigoplus_{i=0}^{n} \alpha_i \otimes f_i) \Big) \oplus \Big(\alpha_0 \otimes (\bigoplus_{i=0}^{n} \beta_i \otimes f_i) \Big) \\
&= (\beta_0 \otimes f) \oplus (\alpha_0 \otimes g) \\
&= (\langle g, f_0 \rangle \otimes f) \oplus (\langle f, f_0 \rangle \otimes g).
\end{aligned}
$$

From Lemma 14(iv) and (22), we have

$$
\begin{aligned}
f \star g &= (\langle \mathcal{C}(g), \mathbf{O}_0 \rangle \otimes f) \oplus (\langle \mathcal{C}(f), \mathbf{O}_0 \rangle \otimes g) \\
&= \mathcal{C}^{-1} \Big((\langle \mathcal{C}(g), \mathbf{O}_0 \rangle \otimes \mathcal{C}(f)) \oplus (\langle \mathcal{C}(f), \mathbf{O}_0 \rangle \otimes \mathcal{C}(g) \Big).
\end{aligned}
$$

(iii) From (ii), we have $f^{(2)} = \langle f, f_0 \rangle \otimes f \geq 0 \otimes f = f$ for any $f \in S_1$. By induction, $f^{(n)} \geq f$. From (3), it then follows that

$$
f^* = \lim_{n \to \infty} (\mathbf{e} \oplus f \oplus f^{(2)} \oplus \ldots \oplus f^{(n)}) = \mathbf{e} \oplus f.
$$

Note that $\mathbf{O}_0 = \mathbf{e}$. Under the assumption that $f_0 = \mathbf{O}_0$, $f^* = f_0 \oplus f$ is still in S_1. That $f^* = \mathcal{C}^{-1}(\mathbf{O}_0 \oplus \mathcal{C}(f))$ follows from (i) and $\mathcal{C}(f_0) = \mathbf{O}_0$ in Lemma 14(vi).

(iv) From Lemma 14(vii), we know that $\mathcal{C}^{-1}(\mathcal{C}(f \vee g))$ is the vector projection of $f \vee g$ onto S_1. Thus, we have from Corollary 7(i) that

$$
\mathcal{C}^{-1}(\mathcal{C}(f \vee g)) = \bigoplus_{i=0}^{n} \langle f \vee g, f_i \rangle \otimes f_i.
$$

Applying Lemma 3(ii) and Lemma 14(iv) yields

$$
\begin{aligned}
\mathcal{C}^{-1}(\mathcal{C}(f \vee g)) &= \bigoplus_{i=0}^{n} \max[\langle f, f_i \rangle, \langle g, f_i \rangle] \otimes f_i \\
&= \bigoplus_{i=0}^{n} \max[\langle \mathcal{C}(f), \mathbf{O}_i \rangle, \langle \mathcal{C}(g), \mathbf{O}_i \rangle] \otimes f_i \\
&= \bigoplus_{i=0}^{n} \langle \mathcal{C}(f) \vee \mathcal{C}(g), \mathbf{O}_i \rangle \otimes f_i \\
&= \mathcal{C}^{-1}(\mathcal{C}(f) \vee \mathcal{C}(g)),
\end{aligned}
$$

where we use the definition of $\mathcal{C}^{-1}$-transform in the last identity.

IV. Appendix D

In this section, we prove Lemma 16.

(i) As we assume that g is in S_1,

$$
g = \bigoplus_{i=0}^{n} \langle g, f_i \rangle \otimes f_i.
$$

Applying Lemma 3(iv) and (iii) yields

$$
\begin{aligned}
\langle f, g \rangle &= \langle f, \bigoplus_{i=0}^{n} \langle g, f_i \rangle \otimes f_i \rangle \\
&= \max_{0 \le i \le n} [\langle f, \langle g, f_i \rangle \otimes f_i \rangle] = \max_{0 \le i \le n} [\langle f, f_i \rangle - \langle g, f_i \rangle]^+.
\end{aligned}
$$

Since $\langle f, f_i \rangle = \langle \mathcal{C}(f), \mathbf{O}_i \rangle$ in Lemma 14(iv), we have from Lemma 3(iii) and (iv) that

$$
\begin{aligned}
\langle f, g \rangle &= \max_{0 \le i \le n} [\langle \mathcal{C}(f), \mathbf{O}_i \rangle - \langle g, f_i \rangle]^+ \\
&= \max_{0 \le i \le n} [\langle \mathcal{C}(f), \langle g, f_i \rangle \otimes \mathbf{O}_i \rangle] \\
&= \langle \mathcal{C}(f), \bigoplus_{i=0}^{n} \langle g, f_i \rangle \otimes \mathbf{O}_i \rangle = \langle \mathcal{C}(f), \mathcal{C}(g) \rangle.
\end{aligned}
$$

(ii) The proof for (ii) is similar except one replaces Lemma 14(iv) by Lemma 14(v).

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