Matrix Extensions of the Filtering Theory for Deterministic Traffic Regulation and Service Guarantees

Cheng-Shang Chang*
Dept. of Electrical Engineering
National Tsing Hua University
Hsinchu 30043
Taiwan, R.O.C.
Email: cschang@ee.nthu.edu.tw

(June 1997; revision: March 1998)

Abstract

In this paper, we extend the filtering theory in [6] for deterministic traffic regulation and service guarantees to the matrix setting. Such an extension enables us to model telecommunication networks as linear systems with multiple inputs and multiple outputs under the $(\min, +)$-algebra. Analogous to the scalar setting, there is an associated calculus in the matrix setting, including feedback, concatenation, "filter bank summation" and performance bounds. As an application of the calculus, we derive service guarantees for networks with nested window flow control. In particular, service guarantees for networks with tandem flow control can be solved explicitly by the Gauss elimination.

Keywords: traffic regulation, filtering, min-plus algebra, service guarantees

*This research is supported in part by the National Science Council, Taiwan, R.O.C., under Contracts NSC-86-2221-E007-036 and NSC 86-2115-M007-025.

1 Introduction

In the paper, Cruz [7] proposed the following deterministic traffic characterization for an increasing sequence $A \equiv \{A(t), t = 0, 1, 2, \ldots\}$ (with $A(0) = 0$). The sequence A is said to be f-upper constrained for some function f if

$$A(t) - A(s) \leq f(s), \quad \forall s \leq t.$$

Based on this traffic characterization, Cruz [7, 8] showed that deterministic service guarantees can be achieved in telecommunication networks. Since then, such a traffic characterization has been widely used in the field (see e.g., [1, 11, 12, 14]). In particular, Parekh and Gallager [12] used the characterization to analyze networks under the generalized processor sharing (GPS) scheme. They proposed the concept of service curves and were able to use that to compute end-to-end performance guarantees under the GPS scheme. In [9], Cruz made the first attempt to formalize the concept of service curves. His efforts lead to the currently accepted concept of service curves in [13, 6, 2, 4], i.e., a server that guarantees a service curve f if its output B satisfies $B(t) \geq B(s) + f(t-s)$ for some s.

Recently, a general filtering theory under the $(\min, +)$-algebra was developed in [6] for deterministic traffic regulation and service guarantees in telecommunication networks. The importance of the role of the $(\min, +)$-algebra in deterministic traffic regulation and service guarantees is also recognized by Agrawal and Rajan [2], Cruz and Okino [10] and Le Boudec [4]. In such an algebra, one replaces the usual addition by the min operator and the usual multiplication by the addition operator (see e.g., [3]). As in the classical linear system theory, the new filtering theory in [6] treats an arrival process A (or a departure process B) as a signal, and a network as a system (see Fig. 1). A signal $f \equiv \{f(t), t = 0, 1, 2, \ldots\}$ is a nonnegative and increasing sequence. For instance, $A(t)$, the cumulative number of arrivals by time t is nonnegative and increasing in time. A signal f is said to be not less than another signal g, denoted by $f \geq g$, if $f(t) \geq g(t)$ for all t. Two basic operations for signals under the $(\min, +)$-algebra are considered, the "addition" operation $\oplus$ and the "multiplication" operation $\star$.

(i) (min) the pointwise minimum of two signals:

$$(f \oplus g)(t) = \min[f(t), g(t)].$$

(ii) (convolution) the convolution of two signals:

$$(f \star g)(t) = \min_{0 \leq s \leq t}[f(s) + g(t-s)].$$

These two operations have the following algebraic properties and one may use them as the usual addition and multiplication.

1. Associativity: $(f \oplus g) \oplus h = f \oplus (g \oplus h)$, and $(f \star g) \star h = f \star (g \star h)$.

2. Commutativity: $f \oplus g = g \oplus f$ and $f \star g = g \star f$.

3. Distributivity: $(f \oplus g) \star h = (f \star h) \oplus (g \star h)$.

4. Zero element: $f \oplus \epsilon = f$, where ϵ is the "null"signal with $\epsilon(t) = \infty$ for all $t \geq 0$.

5. Absorbing zero element: $f \star \epsilon = \epsilon \star f = \epsilon$.

6. Identity element: $f \star \mathbf{e} = \mathbf{e} \star f = f$, where $\mathbf{e}$ is the "impulse" signal with $\mathbf{e}(0) = 0$ and $\mathbf{e}(t) = \infty$.

7. Idempotency of addition: $f \oplus f = f$.

8. Monotonicity: If $f \leq \tilde{f}$ and $g \leq \tilde{g}$, then $f \oplus g \leq \tilde{f} \oplus \tilde{g} \leq \tilde{f}$ and $f \star g \leq \tilde{f} \star \tilde{g}$. If $g(0) = 0$, then $f \star g \leq f$. If $f(0) = g(0) = 0$, then $f \oplus g \geq f \star g$.

There are two types of basic network elements: the maximal f-regulator (in Fig. 2) and the f-server (in Fig. 3). The maximal f-regulator with the input A yields the output $B = A \star f$ (when $f(0) = 0$ and f is subadditive, i.e., $f(s) + f(t) \geq f(s+t)$ for all $s, t \geq 0$), and the f-server for the input A guarantees the output $B \geq A \star f$. The maximal f-regulator has the following three important properties:

(TR 1) Traffic regulation: the output from the maximal f-regulator is f-upper constrained for any input.

(TR 2) Optimality: the maximal f-regulator is the best causal traffic regulator that one can implement in terms of maximizing the cumulative number of departures from the regulator at any moment in time.

(TR 3) Conformity: if the input to the maximal f-regulator is already f-upper constrained, then it is not affected by the regulator.

In particular, a (σ, ρ)-leaky bucket is the maximal f-regulator with $f(t) = \sigma + \rho t$ and $f(0) = 0$. Moreover, a concatenation of (σ_i, ρ_i)-leaky buckets is also the maximal f-regulator with $f(t) = \min_i[\sigma_i + \rho_i t]$ and $f(0) = 0$.

Schedulers, such as GPS, can be characterized by f-servers for certain f's. The representation of a server is not unique, and it may depend on the input. For instance, the minimum bandwidth guarantee property in a GPS server is *universal* as it holds for all inputs. However, representing a GPS server by the service curve derived by the all-greedy method in [12] is not *universal* as it depends on the inputs to the GPS server. Also, the maximal f-regulator is a universal f-server as it does traffic regulation for any input. Among all f-servers, the $\mathbf{O}_d$-server (with $\mathbf{O}_d(t) = 0$ for $0 \leq t \leq d$ and $\mathbf{O}_d(t) = \infty$ otherwise) is of particular interest. A server is the $\mathbf{O}_d$-server for an input if and only if the server guarantees maximum delay d for that input. For instance, a PGPS server (a packetized approximation for GPS in [12]) can be viewed as a concatenation of a GPS server and $\mathbf{O}_d$-server, where d is a bound for the time to transmit a packet. This is due to the fact that the departure time from a PGPS server is not later than that from the corresponding GPS server by the time to transmit a maximum size packet.

In terms of the linear system theory, both the maximal f-regulator and the f-server can be viewed as linear time invariant filters with the impulse response f (except the f-server is with inequality). Network elements can be joined by concatenation, "filter bank summation", and feedback to form a composite network element. The algebraic properties listed above were used to derive the impulse response of a composite network element in [6].

(i) Concatenation (see Fig. 4): a concatenation of an f_1-server for an input A and an f_2-server for the output from the f_1-server is an f-server for A, where $f = f_1 \star f_2$.

(ii) "filter bank summation" (see Fig. 5): the "filter bank summation" of an f_1-server for A and an f_2-server for A is an f-server for A, where $f = f_1 \oplus f_2$.

(iii) Feedback (see Fig. 6): the feedback of an f-server is an f^*-server for A if $f(0) > 0$ and $A(t) < \infty$ for all t, where f^*, called the subadditive closure in [6], can be computed recursively by the following equations:

$$\begin{aligned} &f^*(0) = 0, \\ &f^*(t) = \min[f(t), \min_{0<s<t}[f^*(s) + f^*(t-s)]], t > 0. \end{aligned}$$

Once the impulse response of a composite network element is obtained, the corresponding performance guarantees, such as queue length and delay, can be derived.

(iv) Performance bounds (see Fig. 7): if the input to an f_2-server is f_1-upper constrained, then the maximum queue length (resp. maximum delay) is bounded by the vertical (resp. horizontal) distance between f_1 and f_2.

The filtering theory in [6] only deals with systems with a single input and a single output (SISO). In this paper, we extend the filtering theory to the matrix setting so that systems with multiple inputs and multiple outputs (MIMO) can be analyzed. To do this, we consider matrices (or vectors) of signals, i.e., each element in a matrix is a signal. Following the natural matrix extension of the "addition" operation $\oplus$ and the "multiplication" operation $\star$, we define the "addition" operation and the "multiplication" operation for matrices of signals. For instance, the "addition" and "multiplication" of the 2×2 matrices F and G are as follows:

$$\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \oplus \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \begin{bmatrix} F_{11} \oplus G_{11} & F_{12} \oplus G_{12} \\ F_{21} \oplus G_{21} & F_{22} \oplus G_{22} \end{bmatrix},$$

and

$$\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \star \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \begin{bmatrix} (F_{11} \star G_{11}) \oplus (F_{12} \star G_{21}) & (F_{11} \star G_{12}) \oplus (F_{12} \star G_{22}) \\ (F_{21} \star G_{11}) \oplus (F_{22} \star G_{21}) & (F_{21} \star G_{12}) \oplus (F_{22} \star G_{22}) \end{bmatrix}.$$

Define the "null" matrix $\underline{\epsilon}$ to be the matrix that all its elements are the "null" signal ϵ, and the "identity" matrix $\underline{\mathbf{e}}$ to be the matrix that its diagonal elements are the "impulse" signal $\mathbf{e}$ and all its other elements are the "null" signal ϵ. For instance, the 2×2 "null" matrix and "identity" matrix are

$$\underline{\epsilon} = \begin{bmatrix} \epsilon & \epsilon \\ \epsilon & \epsilon \end{bmatrix} \quad \text{and} \quad \underline{\mathbf{e}} = \begin{bmatrix} \mathbf{e} & \epsilon \\ \epsilon & \mathbf{e} \end{bmatrix}.$$

Based on the matrix extension, most of the algebraic properties still hold except that the "multiplication" of two matrices are not commutative in general.

For traffic regulation, we extend Cruz's traffic regulation to the matrix setting. A $n \times 1$ matrix $A = (A_1, \ldots, A_n)^T$ is upper constrained by a $n \times n$ matrix F if $F_{ii}(0) = 0$, $i = 1, \ldots, n$, and

$$A_i(t) - A_j(s) \le F_{ij}(t-s), \quad \forall 0 \le s \le t.$$

In the matrix form, one has $A = F \star A$ if A is F-upper constrained. As in the scalar case, we consider the following two questions in this paper.

(i) Given an input matrix A, what is the minimum F such that A is F-upper constrained?

(ii) Given a constraint matrix F, how does one construct a regulator such that for any input matrix A, the output from the regulator is F-upper constrained?

The answer to the first question is the matrix extension of the minimum envelope process in [5], and it requires defining the "division" of two matrices. For the second question, we develop the matrix extension of the maximal F-regulator as a linear time invariant filter for an MIMO system with impulse response F^*, where F^* is the matrix extension of the subadditive closure in the scalar setting. We show that the three traffic regulation properties [TR 1-3] are still satisfied.

Analogous to the extension of the maximal F-traffic regulator, the F-server with input A and output B satisfies

$$B_i(t) - A_j(s) \ge F_{ij}(t-s), \quad \forall 0 \le s \le t.$$

In the matrix form, one has $B \ge F \star A$. Thus, the F-server can be viewed as an MIMO system that guarantees its output not less than the output from the linear time invariant filter with input A and impulse response F. As in the scalar case, there is a general MIMO filtering theory, including concatenation, "filter bank summation" and feedback. For systems with feedback, we propose the *primitive* condition as the "stability" condition (the existence of a unique solution). The primitive condition is a natural extension of the condition $f(0) > 0$ in the scalar setting.

For an application, we consider the nested window flow control problem in [2]. Such a problem can be formulated as an MIMO linear system with feedback, and its service guarantees can be derived by the filtering theory, provided that the primitive condition is satisfied. In particular, we found that the deadlock free condition in [2] is a necessary and sufficient condition for the primitive condition. A special case of the nested flow control problem is the tandem flow control problem (see Fig. 8), where the input of the node in the downstream is the output of the node in the up stream. For such a problem, service guarantees are derived by the Gauss elimination.

The paper is organized as follows: in Section 2, we provide the basic theory for the extension to the matrix setting. This includes the extension of the subadditive closure in [6], and matrix "division." In Section 3, we extend the results for traffic regulation to the matrix setting. In Section 4, we extend the result for F-servers to the matrix setting. In Section 5, we apply the MIMO filtering theory to the nested window control problems. The proofs of various theorems are presented in the Appendices.

2 Min-plus matrix algebra

As described in the introduction, we considered the following two operations for sequences (or signals, functions) indexed by $t = 0, 1, 2, \ldots$, under the $(\min, +)$-algebra in [6].

(i) (min) the pointwise minimum of two sequences:

$$(f \oplus g)(t) = \min[f(t), g(t)].$$

(ii) (convolution) the convolution of two sequences under the $(\min, +)$-algebra:

$$(f \star g)(t) = \min_{0 \le s \le t}[f(s) + g(t-s)].$$

Let $\mathcal{F}$ (resp. $\mathcal{F}_0$) be the set of increasing sequences with $f(0) \ge 0$ (resp. $f(0) = 0$). That is, a sequence $f \equiv \{f(t), t = 0, 1, 2, \ldots\} \in \mathcal{F}$ (resp. $\mathcal{F}_0$) satisfies $f(0) \ge 0$ (resp. $f(0) = 0$) and $f(s) \le f(t)$ for all $s \le t$. Clearly, these two operations are closed in $\mathcal{F}$. Moreover, $(\mathcal{F}, \oplus, \star)$ satisfies the algebraic properties described in the introduction and it is a commutative dioid (see [3]) with the zero element ϵ and the identity element $\mathbf{e}$, where ϵ is the sequence with $\epsilon(t) = \infty$ for all t, and $\mathbf{e}$ is the sequence with $\mathbf{e}(0) = 0$ and $\mathbf{e}(t) = \infty$ for all $t > 0$.

To extend the filtering theory in [6] to the matrix setting, we consider square $n \times n$ matrices with entries in $\mathcal{F}$. For a matrix $F \in \mathcal{F}^{n \times n}$, denote by F_{ij} its entry at the i^{th} row and the j^{th} column. For any two matrices F and G in $\mathcal{F}^{n \times n}$, we say $F = G$ (resp. $F \le G$) if $F_{ij}(t) = G_{ij}(t)$ (resp. $F_{ij}(t) \le G_{ij}(t)$) for all $i, j = 1, \ldots, n$ and $t = 0, 1, 2, \ldots$ The addition and multiplication of matrices are defined conventionally after the "addition"operator $\oplus$ and the "multiplication" operator $\star$ in $\mathcal{F}$. We still use $\oplus$ and $\star$ to denote the "matrix addition" operator and the "matrix multiplication" operator, i.e.,

$$(F \oplus G)_{ij} = F_{ij} \oplus G_{ij}, \tag{1}$$

$$(F \star G)_{ij} = [F_{i1} \star G_{1j}] \oplus [F_{i2} \star G_{2j}] \oplus \ldots \oplus [F_{in} \star G_{nj}]. \tag{2}$$

These are equivalent to

$$(F \oplus G)_{ij}(t) = \min[F_{ij}(t), G_{ij}(t)], \tag{3}$$

$$(F \star G)_{ij}(t) = \min_{1 \le k \le n} \min_{0 \le s \le t} [F_{ik}(s) + G_{kj}(t-s)], \tag{4}$$

for any $t \ge 0$. We note that for F and G in $\mathcal{F}^{n \times n}$, the matrices $F(t)$ and $G(t)$ are in $(R^+ \cup \{\infty\})^{n \times n}$. One may also write these in the following matrix forms

$$(F \oplus G)(t) = F(t) \oplus G(t) \tag{5}$$

$$(F \star G)(t) = [F(t) \otimes G(0)] \oplus [F(t-1) \otimes G(1)] \oplus \ldots \oplus [F(0) \otimes G(t)], \tag{6}$$

where $\oplus$ and $\otimes$ are the matrix "addition" and "multiplication" under the $(\min, +)$-algebra (see [6], Section 2.1), i.e.,

$$\begin{aligned}(F(t_1) \oplus G(t_2))_{ij} &= \min[F_{ij}(t_1), G_{ij}(t_2)] \\ (F(t_1) \otimes G(t_2))_{ij} &= \min_{1 \le k \le n} [F_{ik}(t_1) + G_{kj}(t_2)].\end{aligned}$$

One can easily verify that $(\mathcal{F}^{n \times n}, \oplus, \star)$ is still a dioid with the zero matrix $\underline{\epsilon}$ and the identity matrix $\underline{\mathbf{e}}$, where $\underline{\epsilon}$ has all its entries equal to ϵ, and $\underline{\mathbf{e}}$ has its diagonal entries equal to $\mathbf{e}$ and all other entries equal to ϵ. To be precise, we have the following properties:

1. (Associativity) $\forall F, G, H \in \mathcal{F}^{n \times n}$,

$$\begin{aligned}(F \oplus G) \oplus H &= F \oplus (G \oplus H) \\ (F \star G) \star H &= F \star (G \star H)\end{aligned}$$

2. (Commutativity) $\forall F, G \in \mathcal{F}^{n \times n}$,

$$F \oplus G = G \oplus F$$

3. (Distributivity) $\forall F, G, H \in \mathcal{F}^{n \times n}$,

$$\begin{aligned}(F \oplus G) \star H &= (F \star H) \oplus (G \star H) \\ H \star (F \oplus G) &= (H \star F) \oplus (H \star G)\end{aligned}$$

4. (Zero element) $\forall F \in \mathcal{F}^{n \times n}$,

$$F \oplus \underline{\epsilon} = F$$

5. (Absorbing zero element) $\forall F \in \mathcal{F}^{n \times n}$,

$$F \star \underline{\epsilon} = \underline{\epsilon} \star F = \underline{\epsilon}$$

6. (Identity element) $\forall F \in \mathcal{F}^{n \times n}$,

$$F \star \underline{\mathbf{e}} = \underline{\mathbf{e}} \star F = F$$

7. (Idempotency of addition) $\forall F \in \mathcal{F}^{n\times n}$,

$$F \oplus F = F$$

As in the usual matrix theory, we do not have the commutative property for $\star$ in $(\mathcal{F}^{n\times n}, \oplus, \star)$, i.e., $F \star G \neq G \star H$ in general.

Let $\mathcal{F}_0^{n\times n} = \{F \in \mathcal{F}^{n\times n} : F \oplus \underline{\mathbf{e}} = F\}$. That is, a matrix $F \in \mathcal{F}_0^{n\times n}$ if $F_{ii}(0) = 0$ for $i = 1, 2, \ldots, n$. As in the scalar case, we still have the following monotonicity

8. (Monotonicity)

$$\begin{aligned} &F \oplus G \leq \tilde{F} \oplus \tilde{G} \leq \tilde{F} \\ &F \star G \leq \tilde{F} \star \tilde{G}, \end{aligned}$$

for $F \leq \tilde{F}$ and $G \leq \tilde{G}$. If F (resp. G) is in $\mathcal{F}_0^{n\times n}$, then $F \star G \leq G$ (resp. $F \star G \leq F$). If both F and G are in $\mathcal{F}_0^{n\times n}$, then $F \oplus G \geq F \star G$.

Lemma 2.1 *Consider two sequences of matrices $\{F_m, m = 1, 2, \ldots\}$ and $\{G_m, m = 1, 2, \ldots\}$. If both sequences are decreasing in m, then*

$$(\lim_{m\to\infty} F_m) \star (\lim_{m\to\infty} G_m) = \lim_{m\to\infty} (F_m \star G_m).$$

The proof of Lemma 2.1 is in Appendix A. An important corollary of Lemma 2.1 is the distributivity holds for infinite "sums."

9. (Distributivity for infinite "sums") For any two sequences F_m and G_m in $\mathcal{F}^{n\times n}$,

$$\begin{aligned} &(F_1 \oplus F_2 \oplus \ldots \oplus F_m \oplus \ldots) \star (G_1 \oplus G_2 \oplus \ldots \oplus G_m \oplus \ldots) \\ &= (F_1 \star G_1) \oplus (F_1 \star G_2) \oplus (F_2 \star G_1) \oplus \ldots \oplus (F_m \star G_m) \oplus \ldots \end{aligned} \tag{7}$$

To see this, let $\tilde{F}_m = F_1 \oplus F_2 \oplus \ldots \oplus F_m$, $\tilde{G}_m = G_1 \oplus G_2 \oplus \ldots \oplus G_m$ and $\tilde{H}_m = (F_1 \star G_1) \oplus (F_1 \star G_2) \oplus (F_2 \star G_1) \oplus \ldots \oplus (F_m \star G_m)$. From the distributivity for finite "sums", one has $\tilde{F}_m \star \tilde{G}_m = \tilde{H}_m$. Also, from the monotonicity of $\oplus$, both $\tilde{F}_m$ and $\tilde{G}_m$ are decreasing in m. The result then follows from Lemma 2.1.

Denote by $F^{(m)}$ the matrix that is multiplied by F m times, i.e., $F^{(m)} = F^{(m-1)} \star F$, $m > 1$, and $F^{(1)} = F$. For any matrix $F \in \mathcal{F}^{n\times n}$, define the unitary operator (called the closure operation in this paper)

$$F^* = \lim_{m\to\infty} (F \oplus \underline{\mathbf{e}})^{(m)} = \lim_{m\to\infty} (\underline{\mathbf{e}} \oplus F \oplus F^{(2)} \oplus \ldots \oplus F^{(m)}), \tag{8}$$

As this sequence is decreasing, the limit always exists. The closure operation is the extension of the subadditive closure in the scalar case. In the following lemma, we derive properties for F^*. The proof is given in Appendix B.

Lemma 2.2 *Suppose that $F, G \in \mathcal{F}^{n\times n}$.*

(i) *(Monotonicity) If $F \le G$, then $F^* \le G^*$.*

(ii) *(Closure properties) $F^* = F^* \oplus \underline{\mathbf{e}} = F^* \star F^* = (F^*)^{(m)} = (F^*)^* \le F \oplus \underline{\mathbf{e}} \le F$.*

(iii) *(Maximum solution) F^* is the maximum solution of the equation $H = (F \star H) \oplus \underline{\mathbf{e}}$, i.e., for any H satisfying $H = (F \star H) \oplus \underline{\mathbf{e}}$, $H \le F^*$.*

In the extension to matrix operations, we note that F^* is no longer subadditive.

To extend the condition $f(0) > 0$ in the scalar setting for systems with feedback, we consider a special type of matrices, called primitive matrices below.

Definition 2.3 *(Primitive matrix) A matrix $F \in \mathcal{F}^{n\times n}$ is* primitive *if there is a finite m such that $F_{ij}^{(m)}(0) > 0$ for all i and j.*

In view of (6),

$$(F \star G)(0) = F(0) \otimes G(0). \tag{9}$$

where $\otimes$ is the matrix multiplication under the $(\min,+)$-algebra. Thus, F is primitive if and only if $F(0)$ is primitive under the $(\min,+)$-algebra (there is a finite m such that $(F(0))_{i,j}^{(m)} > 0$). This leads to the following condition for primitive matrices.

Lemma 2.4 *A matrix $F \in \mathcal{F}^{n\times n}$ is* primitive *if and only if given any cycle $k_1, \ldots, k_d, k_1$ for some $d \ge 1$, $\sum_{j=1}^{d-1} F_{k_j k_{j+1}}(0) + F_{k_d k_1}(0) > 0$.*

The condition in Lemma 2.4 is referred to the deadlock free condition in [2]. The proof of Lemma 2.4 is given in Appendix C.

The following theorem is the matrix counterpart of Theorem 2.4 in Chang [6] and its proof is given in Appendix D. Note that the condition $A(t) < \infty$ for all t in [6] is removed.

Theorem 2.5 *(Feedback)*

(i) *For the equation*

$$B = (F \star B) \oplus A, \tag{10}$$

$B = F^* \star A$ is the maximum solution.

(ii) *If F is* primitive*, then $B = F^* \star A$ is the unique solution.*

(iii) *Under the condition in (ii), if*

$$B \ge (F \star B) \oplus A,$$

then $B \ge F^* \star A$.

Remark 2.6 We note that the matrices defined in this section need not be square in order for the results in Theorem 2.5 to hold. We only need the condition that the two matrices can be multiplied, i.e., the number of rows in B is equal to the number of columns in F. In other words, one may view nonsquare matrices as square matrices with some elements being padded with the zero element ϵ.

In the scalar case, the subadditive closure f^* can be computed recursively from the following equations:

$$f^*(0) = 0,$$
$$f^*(t) = \min[f(t), \min_{0<s<t}[f^*(s) + f^*(t-s)]], t > 0.$$

Similarly, Theorem 2.5(i) can be used to derive a recursive algorithm for computing F^*. The proof of Lemma 2.7 is given in Appendix E.

Lemma 2.7 *Let $F^\circ = F \oplus \underline{\mathbf{e}}$. Then*

$$F^*(0) = (F^\circ(0))^{(n-1)} = \underline{\mathbf{e}}(0) \oplus F(0) \oplus (F(0))^{(2)} \oplus \ldots \oplus (F(0))^{(n-1)}, \tag{11}$$

and for $t > 0$

$$F^*(t) = F^*(0) \otimes \Big[[F^\circ(1) \otimes F^*(t-1)] \oplus [F^\circ(2) \otimes F^*(t-2)] \oplus \ldots \oplus [F^\circ(t) \otimes F^*(0)]\Big]. \tag{12}$$

In the scalar case, one has $f^*(0) = 0$ and Lemma 2.7 leads to another recursive algorithm for computing f^* as follows:

$$f^*(t) = \min_{0 \le s < t}[f^*(s) + f(t-s)].$$

Beside the matrix "addition" operator $\oplus$ and the matrix "multiplication" operator $\star$, we consider the following matrix operator $\div$ in $\mathcal{F}^{n\times n}$

$$(F \div G)_{ij}(t) = \sup_{s\ge 0} \max_{1\le k\le n}[F_{ik}(t+s) - G_{jk}(s)]^+, \tag{13}$$

where $x^+ = \max(0, x)$. Here we follow the convention that $\infty - \infty = 0$.

Lemma 2.8 *(Sup-division property) For F, G and H in $\mathcal{F}^{n\times n}$, $F \div G \le H$ if and only if $F \le H \star G$.*

As the statement holds only for one side of the inequality, the operator $\div$ is called the "sup-division" operator in this paper. In the following lemma, we derive properties of the "sup-division" operator. The proofs of Lemma 2.8 and Lemma 2.9 are given in Appendix F. We note that some results in Lemma 2.9 for the scalar setting were reported in [2, 4].

Lemma 2.9 *Suppose that $F, \tilde{F}, G, \tilde{G}$ and H are in $\mathcal{F}^{n\times n}$.*

(i) *(Monotonicity) $F \div G \le \tilde{F} \div \tilde{G}$ if $F \le \tilde{F}$ and $G \ge \tilde{G}$. If $G \in \mathcal{F}_0^{n\times n}$, then $F \div G \ge F$.*

(ii) *$F \div (H \star G) = (F \div G) \div H$.*

(iii) *$F \div F$ is the minimum solution of the equation $F = H \star F$, i.e., for any H satisfying $F = H \star F$, $H \ge F \div F$.*

(iv) *$F \div F = (F \div F) \star (F \div F) = (F \div F)^*$*

(v) *$F \div F = F$ if and only if $F = F^*$.*

(vi) *$(F \div F) \div (F \div F) = F \div F$.*

3 Traffic regulation

Consider a matrix $A \in \mathcal{F}^{n\times 1}$. Let A_i, $i = 1, 2, \dots, n$, be the i^{th} element in A. One may view $A_i(t)$ as the cumulative arrival of a source by time t. The following traffic characterization is an extension of Cruz's [7] traffic characterization to the matrix setting.

Definition 3.1 *A matrix $A \in \mathcal{F}^{n\times 1}$ is F-upper constrained for some matrix $F \in \mathcal{F}_0^{n\times n}$ if for all $i, j = 1, \dots, n$ and $t = 0, 1, \dots,$*

$$A_i(t) - A_j(s) \le F_{ij}(t-s), \quad \forall 0 \le s \le t. \tag{14}$$

Such an extension enables us to model the correlation among traffic sources. For example, in the case of multicasting, one has $A_1 = A_2 = \ldots = A_n$. If A_1 is f-upper constrained for some $f \in \mathcal{F}_0$, then A is F-upper constrained with $F_{ij} = f$ for all $1 \le i, j \le n$.

This characterization has the following equivalent statements. The proof of Lemma 3.2 is analogous to that in Chang [6], Lemma 3.2, and thus omitted.

Lemma 3.2 *Suppose that $A \in \mathcal{F}^{n\times 1}$ and $F \in \mathcal{F}_0^{n\times n}$. The following four statements are equivalent: (i) A is F-upper constrained, (ii) $A = F \star A$, (iii) $A = F^* \star A$, and (iv) A is F^*-upper constrained.*

As in the scalar case, we provide answers to the following two questions.

(Q1) Given an input matrix $A \in \mathcal{F}^{n\times 1}$, what is the minimum $F \in \mathcal{F}_0^{n\times n}$ such that A is F-upper constrained?

(Q2) Given a constraint matrix $F \in \mathcal{F}_0^{n\times n}$, how does one construct a regulator such that for any input matrix $A \in \mathcal{F}^{n\times 1}$, the output from the regulator is F-upper constrained?

We answer the first question in Theorem 3.3, which is the matrix extension of the minimum envelope process in [5].

Theorem 3.3 *Let* $\hat{\mathbf{e}}$ *be the matrix in* $\mathcal{F}^{1\times n}$ *with all its elements being* $\mathbf{e}$*. For any input matrix* $A \in \mathcal{F}^{n\times 1}$*, let* $\hat{A} = A\hat{\mathbf{e}}$ *(*$\hat{A}$ *is a matrix in* $\mathcal{F}^{n\times n}$ *with identical columns). Then* $\hat{A} \div \hat{A}$ *is the minimum* $F \in \mathcal{F}_0^{n\times n}$ *such that* A *is* F*-upper constrained.*

As the matrix $\hat{A}$ has identical columns, we have from (13) that

$$(\hat{A} \div \hat{A})_{ij}(t) = \sup_{s\geq 0}[A_i(t+s) - A_j(s)]^+.$$

Proof. Suppose that A is F-upper constrained. From Lemma 3.2(ii), we have $A = F \star A$. Since the matrix $\hat{A}$ has identical columns, this holds if and only if $\hat{A} = F \star \hat{A}$. As a direct consequence of Lemma 2.9(iii), the minimum solution is $\hat{A} \div \hat{A}$. ■

Corollary 3.4 *For any matrix* $A \in \mathcal{F}^{n\times 1}$*, let* $\hat{A} = A\hat{\mathbf{e}}$ *as in Theorem 3.3. Then* A *is* F*-upper constrained for some* $F \in \mathcal{F}_0^{n\times n}$ *if and only if* $\hat{A} \div \hat{A} \leq F$*.*

Proof. We have shown the *only if* part in Theorem 3.3. To see the *if* part, suppose that $\hat{A} \div \hat{A} \leq F$. From the sup-division property in Lemma 2.8, it follows that $\hat{A} \leq F \star \hat{A}$. This is equivalent to $A \leq F \star A$. As F is in $\mathcal{F}_0^{n\times n}$, we have the monotonicity of $\star$ that $F \star A \leq A$. Thus, we indeed have $A = F \star A$ and A is F-upper constrained (Lemma 3.2(ii)). ■

We answer the second question in the following theorem. The proof of Theorem 3.5 is omitted as it follows a similar argument in Chang [6], Theorem 3.3.

Theorem 3.5 *Suppose that* $A \in \mathcal{F}^{n\times 1}$ *and* $F \in \mathcal{F}_0^{n\times n}$*. Let* $B = F^* \star A$*.*

(i) *(Traffic regulation)* B *is* F^**-upper constrained and thus* F*-constrained.*

(ii) *(Optimality) For any* F*-upper constrained* $\tilde{B} \in \mathcal{F}^{n\times 1}$ *with* $\tilde{B} \leq A$*, one has* $\tilde{B} \leq B$*.*

(iii) *(Conformity)* A *is* F*-upper constrained if and only if* $B = A$*.*

The construction $B = F^* \star A$ *is called the* maximal F-regulator *(for the input* A*) in the paper.*

As in the scalar case, the condition $\tilde{B} \leq A$ corresponds to one of the causal conditions in [1] as the number of departures cannot be larger than the number of arrivals. Theorem 3.5(i) shows that for any input A, the maximal F-regulator generates an F-upper constrained output B. Theorem 3.5(ii) shows that it is the best construction that one can implement if one would like to maximize the number of departures by time t. Finally, Theorem 3.5(iii) shows that if

A is already F-upper constrained, then it will pass through the maximal F-regulator without any change.

In the following lemma, we show how one can transform the traffic characterization from the matrix setting to the scalar setting.

Lemma 3.6 *(Multiplexing) Suppose that $A \in \mathcal{F}^{n\times 1}$ is F-upper constrained for some $F \in \mathcal{F}_0^{n\times n}$. Then $\sum_{i=1}^n A_i$ is f-upper constrained, where $f(t) = \min_\pi \sum_{i=1}^n F_{i\pi(i)}(t)$ and π is a permutation of $\{1, 2, \ldots, n\}$.*

Proof. Note that for any permutation π of $\{1, 2, \ldots, n\}$

$$\sum_{i=1}^n A_i(t) - \sum_{i=1}^n A_i(s) = \sum_{i=1}^n A_i(t) - \sum_{i=1}^n A_{\pi(i)}(s)$$
$$\leq \sum_{i=1}^n A_i(t) - A_{\pi(i)}(s) \leq \sum_{i=1}^n F_{i\pi(i)}(t-s),$$

as we assume that A is F-upper constrained. ■

4 Service guarantees

In the section we extend the concept of service curves developed by Cruz [9] and Sariowan [13] to the matrix setting.

Definition 4.1 *(F-server) A server is called a F-server ($F \in \mathcal{F}^{n\times n}$) for an input matrix $A \in \mathcal{F}^{n\times 1}$ if its output matrix satisfies $B \geq F \star A$. If the inequality is satisfied for all input matrices, then we say the F-server is* universal*. If the inequality is an equality, we say the F-server is* exact*.*

Clearly, the maximal F-regulator is a *universal* and *exact* F^*-server. Analogous to the scalar case, one has the following properties for F-servers. The proofs of Theorems 4.2-4.4 are omitted as they are analogous to those in Chang [6], Theorems 3.5, 3.7 and 5.5.

Theorem 4.2 *(Concatenation (cf. Fig. 4)) A concatenation of an F_1-server for an input matrix A and an F_2-server for the output from the F_1-server is an F-server for A, where $F = F_2 \star F_1$.*

Theorem 4.3 *(Filter bank summation (cf. Fig. 5)) Consider an input matrix A. Let B_1 (resp. B_2) be the output from an F_1-server (resp. F_2-server) for A. The output from the "filter bank summation", denoted by B, is $B_1 \oplus B_2$. Then the "filter bank summation" of an F_1-server for A and an F_2-server for A is an F-server for A, where $F = F_1 \oplus F_2$.*

Theorem 4.4 *(Feedback (cf. Fig. 6)) Consider an input matrix* $A \in \mathcal{F}^{n\times 1}$ *and an* F*-server for* B*, where* $B = A \oplus A_1$*, and* A_1 *is the output from the* F*-server. If* F *is primitive, then the feedback system is an* F^**-server for* A*.*

In the following theorems, we derive performance bounds for F-servers.

Theorem 4.5 *Consider an* F_2*-server for* A*. Let* B *be the output. Also, let* $q_i = \sup_{s\geq 0}[A_i(s) - B_i(s)]^+$ *be the maximum queue length at the* i^{th} *server. Suppose that* A *is* F_1*-upper constrained.*

(i) *(Queue length)* $q_i \leq (F_1^* \div F_2)_{ii}(0)$*.*

(ii) *(Output burstiness) If* $B \leq A$*, then* B *is* $(F_1^* \div F_2)^*$*-upper constrained.*

Proof. (i) As in the proof of Theorem 3.3, let $\hat{A} = A\hat{\mathbf{e}}$ and $\hat{B} = B\hat{\mathbf{e}}$, where $\hat{\mathbf{e}}$ is the matrix in $\mathcal{F}^{1\times n}$ with all its elements being $\mathbf{e}$. In view of the definition of the sup-division in (13), $q_i = (\hat{A} \div \hat{B})_{ii}(0)$. Thus, it suffices to show that $\hat{A} \div \hat{B} \leq F_1^* \div F_2$. Since we assume that B is the output from the F_2-server for A,

$$\hat{B} \geq F_2 \star \hat{A}. \tag{15}$$

From the monotonicity of $\div$ and Lemma 2.9(ii), it follows that

$$\hat{A} \div \hat{B} \leq \hat{A} \div (F_2 \star \hat{A}) = (\hat{A} \div \hat{A}) \div F_2.$$

In Theorem 3.3, we know that $\hat{A} \div \hat{A}$ is the minimum $F \in \mathcal{F}_0^{n\times n}$ such that A is F-upper constrained. As we assume that A is F_1-upper constrained (and thus F_1^*-upper constrained), we have $\hat{A} \div \hat{A} \leq F_1^*$. Using the monotonicity of $\div$ yields

$$\hat{A} \div \hat{B} \leq F_1^* \div F_2. \tag{16}$$

(ii) As we assume that $B \leq A$ (and thus $\hat{B} \leq \hat{A}$), it follows from the monotonicity of $\div$ and (16) that $\hat{B} \div \hat{B} \leq F_1^* \div F_2$. Applying Lemma 2.9(iv) and the monotonicity of the closure operation yield

$$\hat{B} \div \hat{B} = (\hat{B} \div \hat{B})^* \leq (F_1^* \div F_2)^*.$$

As a direct result of Corollary 3.4, B is $(F_1^* \div F_2)^*$-upper constrained. ■

Theorem 4.6 *(Virtual delay) Consider an* F_2*-server for* A*. Let* B *be the output. Let* $d_i(t) = \inf\{d \geq 0 : B_i(t+d) \geq A_i(t)\}$ *be the virtual delay of the last customer from the* i^{th} *source that arrives at time* t*. Suppose that* A *is* F_1*-upper constrained. If* $F_{1,ij}^*(0) \leq F_{2,ij}(0)$ *for all* $j \neq i$*, then* $d_i(t) \leq \inf\{d \geq 0 : F_{1,ij}^*(s) \leq F_{2,ij}(s+d), s = 0, 1, \ldots, t, j = 1, 2, \ldots, n\}$*.*

We note that the bound for delay is more complicated than that for queue length in Theorem 4.5. It also requires the condition that $F^*_{1,ij}(0) \le F_{2,ij}(0)$ for all $j \neq i$.

Proof. Let $d^* = \inf\{d \ge 0 : F^*_{1,ij}(s) \le F_{2,ij}(s+d), s = 0,1,\dots,t, j = 1,2,\dots,n\}$. It suffices to show that for all $d \ge d^*$, $A_i(t) \le B_i(t+d)$. Note from (4) and $B \ge F_2 \star A$ that

$$A_i(t) - B_i(t+d) \le \max_{1\le j\le n} \max_{0\le s\le t+d} [A_i(t) - A_j(s) - F_{2,ij}(t+d-s)].$$

Now we consider the following three cases:

Case 1. $0 \le s \le t$:

In this case, for all $d \ge d^*$

$$A_i(t) - A_j(s) - F_{2,ij}(t+d-s) \le F^*_{1,ij}(t-s) - F_{2,ij}(t-s+d) \le 0,$$

as we assume that A is F_1-upper constrained.

Case 2. $t+1 \le s \le t+d$ and $j = i$:

In this case, $A_i(t) - A_i(s) \le 0$ as $A_i(t)$ is increasing in t. Also, $F_{2,ii}(t+d-s) \ge F_{2,ii}(0) \ge 0$. Thus, $A_i(t) - A_i(s) - F_{2,ii}(t+d-s) \le 0$.

Case 3. $t+1 \le s \le t+d$ and $j \neq i$:

Since we assume that A is F-upper constrained, $A_i(t) - A_j(s) \le A_i(t) - A_j(t) \le F^*_{1,ij}(0)$. Also, $F_{2,ij}(t+d-s) \ge F_{2,ij}(0)$. Thus,

$$A_i(t) - A_j(s) - F_{2,ij}(t+d-s) \le F^*_{1,ij}(0) - F_{2,ij}(0) \le 0,$$

as we assume that $F^*_{1,ij}(0) - F_{2,ij}(0) \le 0$ for all $j \neq i$. ■

5 Nested window flow control

In this section, we apply our results to the nested window control problem in [2]. Let $A \in \mathcal{F}^{n\times 1}$ be the input matrix to a network (with n nodes) and $B \in \mathcal{F}^{n\times 1}$ be the output matrix. Suppose that the network enforces a nested window flow control for the input A with the window size matrix W (W_{ij} is the window size between node i and node j). For the nested window flow control system, the effective input matrix to the network, denoted by $\tilde{A}$, satisfies (cf. (5.1) in [2])

$$\tilde{A}_i(t) = \min\Big[A_i(t), \min_{1\le j\le n}[W_{ij} + B_j(t)]\Big], \;\; i = 1,\dots,n. \tag{17}$$

Note that $\min_{1\le j\le n}[W_{ij} + B_j(t)] = (\tilde{W} \star B)_i(t)$, where $\tilde{W}$ is the matrix with $\tilde{W}_{ij}(t) = \infty$ for $t > 0$ and $\tilde{W}_{ij}(0) = W_{ij}$. One may rewrite (17) as follows:

$$\tilde{A} = A \oplus (\tilde{W} \star B). \tag{18}$$

Also, we assume that the network is an F-server for the effective input matrix $\tilde{A}$, i.e.,

$$B \geq F \star \tilde{A}. \tag{19}$$

In conjunction with (19),

$$B \geq F \star \tilde{A} = F \star (A \oplus (\tilde{W} \star B)) = ((F \star \tilde{W}) \star B) \oplus (F \star A),$$

where we apply the commutativity of $\oplus$, the distributive property and the associativity of $\star$. If the matrix $F \star \tilde{W}$ is primitive, then we have from Theorem 2.5(iii) that

$$B \geq (F \star \tilde{W})^* \star (F \star A),$$

and the nested window flow control system is a $(F \star \tilde{W})^* \star F$-server.

A sufficient condition for $F \star \tilde{W}$ to be primitive is that $\tilde{W}(0) = W$ is primitive under the $(\min, +)$-algebra. From Lemma 2.4, the necessary and sufficient condition for W to be primitive is the deadlock free condition in [2], i.e., given *any* cycle of network elements $k_1, \ldots, k_d, k_1$, for some $d \geq 1$, one has $\sum_{j=1}^{d-1} W_{k_j k_{j+1}} + W_{k_d k_1} > 0$.

A special case of the nested window flow control problem is the tandem window flow control problem in [2, 10], in which $0 < W_{i-1,i} < \infty$ for $i = 2, \ldots, n$ and $W_{ij} = \infty$ otherwise (see Fig. 8 for $n = 3$). In other words, window flow control is only enforced between node $i-1$ and i, $i = 2, \ldots, n$. Also, assume there is only one external input A_1. As the network is in tandem, the only input to the i^{th} node is the output from the $i - 1^{th}$ node (the external input for the first node). Thus, $A_i = B_{i-1}$ for all $i = 2, \ldots, n$ and $F_{ij} = \epsilon$ for $j \neq i$, $i = 1, \ldots, n$. Recall that ϵ is the zero element with $\epsilon(t) = \infty$ for all t. In this case, the matrix equation in (18) is equivalent to the following equations:

$$\tilde{A}_1 = A_1 \oplus (\tilde{W}_{1,2} \star B_2), \tag{20}$$

$$\tilde{A}_i = B_{i-1} \oplus (\tilde{W}_{i,i+1} \star B_{i+1}), \quad i = 2, \ldots, n-1 \tag{21}$$

$$\tilde{A}_n = B_{n-1}. \tag{22}$$

Also, the F-server is equivalent to the following equations:

$$B_i \geq F_{ii} \star \tilde{A}_i, \quad i = 1, \ldots, n. \tag{23}$$

In conjunction with (20),(21) and (22), one has

$$B_1 \geq (F_{11} \star A_1) \oplus (F_{11} \star \tilde{W}_{1,2} \star B_2), \tag{24}$$

$$B_i \geq (F_{ii} \star B_{i-1}) \oplus (F_{ii} \star \tilde{W}_{i,i+1} \star B_{i+1}), \quad i = 2, \ldots, n-1, \tag{25}$$

$$B_n \geq F_{nn} \star B_{n-1}, \tag{26}$$

where we apply the distributivity.

The equations in (24), (25) and (26) can be solved by the Gauss elimination. For example, take $n=3$ in Fig. 8. Replacing B_3 in (25) for $i=2$ yields

$$B_2 \geq (F_{2,2} \star B_1) \oplus (F_{2,2} \star \tilde{W}_{2,3} \star F_{3,3} \star B_2). \tag{27}$$

As $\tilde{W}_{2,3}(0) = W_{2,3} > 0$, it follows from Theorem 2.5(iii) that

$$B_2 \geq (F_{2,2} \star \tilde{W}_{2,3} \star F_{3,3})^* \star F_{2,2} \star B_1 = F_{2,2} \star (F_{2,2} \star \tilde{W}_{2,3} \star F_{3,3})^* \star B_1, \tag{28}$$

where we apply the commutativity of $\star$ in the scalar case. Replacing (28) in (24) and using Theorem 2.5(iii) yield

$$B_1 \geq (F_{1,1} \star \tilde{W}_{1,2} \star F_{2,2} \star (F_{2,2} \star \tilde{W}_{2,3} \star F_{3,3})^*)^* \star F_{1,1} \star A_1. \tag{29}$$

Now apply back substitution. One then has

$$B_2 \geq (F_{1,1} \star \tilde{W}_{1,2} \star F_{2,2} \star (F_{2,2} \star \tilde{W}_{2,3} \star F_{3,3})^*)^* \star (F_{2,2} \star \tilde{W}_{2,3} \star F_{3,3})^* \star F_{2,2} \star F_{1,1} \star A_1. \tag{30}$$

Since $\star$ is commutative for the scalar case, one can easily verify from the distributivity for infinite "sums" that $(f \star g^*)^* \star g^* = f^* \star g^*$. Thus,

$$B_2 \geq (F_{2,2} \star \tilde{W}_{2,3} \star F_{3,3})^* \star (F_{1,1} \star \tilde{W}_{1,2} \star F_{2,2})^* \star F_{2,2} \star F_{1,1} \star A_1. \tag{31}$$

From (26), it then follows

$$B_3 \geq (F_{2,2} \star \tilde{W}_{2,3} \star F_{3,3})^* \star (F_{1,1} \star \tilde{W}_{1,2} \star F_{2,2})^* \star F_{3,3} \star F_{2,2} \star F_{1,1} \star A_1 \tag{32}$$

In general, one can show that the tandem window flow control is a $(\prod_{i=1}^n F_{i,i}) \star \prod_{i=2}^n (F_{i-1,i-1} \star \tilde{W}_{i-1,i} \star F_{i,i})^*$-server.

A Appendix A

In this section, we prove Lemma 2.1.

Let $H_m = F_m \star G_m$. Note from the monotonicity of $\star$, H_m is also decreasing in m. Thus, the limits of F_m, G_m and H_m exist. As we assume that both F_m and G_m are decreasing in m, it follows from the monotonicity of $\star$ that for all m

$$(\lim_{m\to\infty} F_m) \star (\lim_{m\to\infty} G_m) \leq F_m \star G_m = H_m.$$

Letting $m \to \infty$ yields

$$(\lim_{m\to\infty} F_m) \star (\lim_{m\to\infty} G_m) \leq \lim_{m\to\infty} H_m. \tag{33}$$

To prove the other inequality, note from (4) that

$$H_{m,ij}(t) = \min_{1\leq k\leq n} \min_{0\leq s\leq t} [F_{m,ik}(s) + G_{m,kj}(t-s)]. \tag{34}$$

This implies that for all $s = 0, 1, \ldots, t$ and $k = 1, \ldots, n$,

$$H_{m,ij}(t) \leq F_{m,ik}(s) + G_{m,kj}(t-s).$$

Letting $m \to \infty$ yields

$$\begin{aligned}\lim_{m\to\infty} H_{m,ij}(t) &\leq \lim_{m\to\infty}[F_{m,ik}(s) + G_{m,kj}(t-s)] \\ &= \lim_{m\to\infty} F_{m,ik}(s) + \lim_{m\to\infty} G_{m,kj}(t-s).\end{aligned}$$

As this holds for all $s = 0, 1, \ldots, t$ and $k = 1, \ldots, n$, one has

$$\lim_{m\to\infty} H_{m,ij}(t) \leq \min_{1\leq k\leq n} \min_{0\leq s\leq t}[\lim_{m\to\infty} F_{m,ik}(s) + \lim_{m\to\infty} G_{m,kj}(t-s)].$$

In matrix form, we have (cf. (34))

$$\lim_{m\to\infty} H_m \leq (\lim_{m\to\infty} F_m) \star (\lim_{m\to\infty} G_m) \tag{35}$$

B Appendix B

In this section, we prove Lemma 2.2.

(i) This follows from the monotonicity of $\oplus$.

(ii) From the idempotency of $\oplus$ and the definition of the closure operation in (8), one has $F^* = F^* \oplus \underline{\mathbf{e}}$. That $F^* \star F^* = F^*$ follows from the distributivity (for the infinite "sums") and the idempotency of $\oplus$. By induction, one has $(F^*)^{(m)} = F^*$. From the definition of the closure operation, we then have

$$(F^*)^* = \lim_{n\to\infty} \underline{\mathbf{e}} \oplus F^* \oplus (F^*)^{(2)} \oplus \ldots \oplus (F^*)^{(n)} = \lim_{n\to\infty} \underline{\mathbf{e}} \oplus F^* = F^*.$$

The inequalities follow trivially from the monotonicity of $\oplus$.

(iii) From the distributivity for the infinite "sums", one can easily verify that F^* is a solution of the equation $H = (F \star H) \oplus \underline{\mathbf{e}}$. Now we show it is the maximum solution. By iterating the equation, we have

$$H = (F^{(m+1)} \star H) \oplus (F^{(m)} \oplus \ldots \oplus F \oplus \underline{\mathbf{e}}).$$

It then follows from the monotonicity of $\oplus$ that $H \leq F^{(m)} \oplus \ldots \oplus F \oplus \underline{\mathbf{e}}$ for all m. Letting $m \to \infty$ completes the argument.

C Appendix C

In this section, we prove Lemma 2.4.

Let $S_{ij}(m)$ be the set of paths that start from i to j in m steps. A typical path in $S_{ij}(m)$ is $(i_0, i_1, \ldots, i_{m-1}, i_m)$ with $i_0 = i$ and $i_m = j$. Then

$$(F(0))_{ij}^{(m)} = \min_{(i_0,i_1,\ldots,i_{m-1},i_m)\in S_{ij}(m)} [\sum_{k=1}^{m} (F(0))_{i_{k-1},i_k}].$$

One may view $\sum_{k=1}^{m}(F(0))_{i_{k-1},i_k}$ as the weight associated with the path $(i_0, i_1, \ldots, i_{m-1}, i_m)$. Thus, the matrix $F(0)$ is primitive if and only if there exists a finite m such that every path in $\cup_{i,j}S_{i,j}(m)$ has a positive weight.

Now we prove the *if* part. Note that for $m \geq n$, every path in $S_{i,j}(m)$ contains at least one cycle (as we are throwing $m+1$ balls into n urns). Thus, every path in $S_{i,j}(m)$ has a positive weight as the cycle contained in the path is assumed to have a positive weight.

For the *only if* part, note that if every path in $\cup_{i,j}S_{i,j}(m)$ has a positive weight, then every path in $\cup_{i,j}S_{i,j}(m+1)$ also has a positive weight as it contains a path in $\cup_{i,j}S_{i,j}(m)$. This implies that if $(F(0))^{(m)} > 0$, then $(F(0))^{\ell} > 0$ for all $\ell \geq m$. Thus, we only need to consider cycles with length not greater than m. For such a cycle, we may consider joining k identical cycles for some k large enough to form a path in $\cup_{i,j}S_{i,j}(\ell)$ with $\ell \geq m$. Thus, k times of the weight of such a cycle is not less than the weight of a certain path in $\cup_{i,j}S_{i,j}(\ell)$. Since every path in $\cup_{i,j}S_{i,j}(m)$ has a positive weight, the weight of such a cycle is also positive.

D Appendix D

In this section, we prove Theorem 2.5.

(i) The proof is analogous to that in Chang [6], Theorem 2.4(i) and thus omitted. .

(ii) We first show that $F^* \star A$ is the unique solution under the condition that $A_{ij}(t) < \infty$ for all i, j, t. This condition will be removed later on. Since we assume that F is primitive, there is a finite m such that $F_{ij}^{(m)}(0) > 0$. Let $\delta = \min_{i,j} F_{ij}^{(m)}(0) > 0$. Iterating the equation in (10) yields

$$B = (F^{(km)} \star B) \oplus \Big((F^{(km-1)} \star A) \oplus \ldots \oplus (F \star A) \oplus A\Big). \tag{36}$$

It then follows from (9) that

$$(F^{(km)} \star B)(0) = (F(0))^{(km)} \otimes B(0) \geq (F(0))^{(km)} = ((F(0))^{(m)})^{(k)}.$$

Thus, $(F^{(km)} \star B)_{ij}(0) \geq k\delta$. From the assumption that $A_{ij}(t) < \infty$ for all i, j, t, it follows that for any fixed t there is a finite k such that $A_{i,j}(t) < k\delta$ for all i, j and thus $F^{(km)}(t) \star B \geq A(t)$. In view of (36), for any fixed t there is a finite k such that

$$\begin{aligned} B(t) &= \Big((F^{(km-1)} \star A) \oplus \ldots \oplus (F \star A) \oplus A\Big)(t) \\ &= \Big((F^{(km-1)} \oplus \ldots \oplus F \oplus \underline{\mathbf{e}}) \star A\Big)(t) \geq (F^* \star A)(t). \end{aligned}$$

This shows that $B \geq F^* \star A$. In conjunction with (i), $F^* \star A$ is the unique solution.

Now we remove the condition that $A_{ij}(t) < \infty$ for all i, j, t. We do this by truncation. Let T_c be the matrix that $T_{c,ij}(t) = c$ for all i, j, t. From the monotonicity of $\oplus$,

$$B = (F \star B) \oplus A \geq ((F \star B) \oplus A) \oplus T_c = (F \star B) \oplus (A \oplus T_c).$$

Note that $(A \oplus T_c)_{ij}(t) \le c$ for all i, j, t. It then follows from the bounded case above that

$$B \ge F^* \star (A \oplus T_c).$$

Applying the distributive property yields

$$B \ge (F^* \star A) \oplus (F^* \star T_c).$$

Observe that $(F^* \star T_c)(t) \ge (F^* \star T_c)(0) \ge T_c(0) = c = T_c(t)$. Thus, $F^* \star T_c \ge T_c$ and we have from the monotonicity of $\oplus$ that

$$B \ge (F^* \star A) \oplus T_c.$$

This is equivalent to $B_{ij}(t) \ge \min[(F^* \star A)_{ij}(t), c]$ for all i, j, t. As this holds for all $c > 0$, letting $c \to \infty$ yields $B_{ij}(t) \ge (F^* \star A)_{ij}(t)$ for all i, j, t. Thus, $B \ge F^* \star A$.

(iii) Follow the same argument in (ii).

E Appendix E

In this section, we prove Lemma 2.7.

As $F^* = (F^\circ)^*$, we have from Lemma 2.2(iii) that F^* is the maximum solution of the equation $H = (F^\circ \star H) \oplus \underline{\mathbf{e}}$. In view of (6) and $(\underline{\mathbf{e}}(t))_{ij} = \infty$ for all $t > 0$, $1 \le i, j \le n$, F^* is the maximum solution of the equations

$$H(0) = F^\circ(0) \otimes H(0) \oplus \underline{\mathbf{e}}(0), \tag{37}$$

and for $t > 0$

$$\begin{aligned} H(t) &= (F^\circ \star H)(t) \\ &= [F^\circ(0) \otimes H(t)] \oplus \Big[[F^\circ(1) \otimes H(t-1)] \oplus [F^\circ(2) \otimes H(t-2)] \oplus \ldots \oplus [F^\circ(t) \otimes H(0)]\Big] \end{aligned} \tag{38}$$

Observe that $F^\circ(0) \oplus \underline{\mathbf{e}}(0) = F^\circ(0)$ since $F^\circ \oplus \underline{\mathbf{e}} = F^\circ$. Thus, by iterating (37),

$$F^*(0) = \lim_{m \to \infty} (F^\circ(0))^{(m)}. \tag{39}$$

Note that (39) can also be obtained directly from (8) and (6). To show (11), it suffices to show that $(F^\circ(0))^{(m)} = (F^\circ(0))^{(n-1)}$ for $m \ge n$. Note that

$$\begin{aligned} (F^\circ(0))_{ij}^{(m+1)} &= \min_{1 \le k \le n} [(F^\circ(0))_{ik}^{(m)} + (F^\circ(0))_{kj}] \\ &\ge (F^\circ(0))_{ij}^{(m)} + (F^\circ(0))_{jj}. \end{aligned}$$

As $(F^\circ(0))_{jj} = 0$ for all j, $(F^\circ(0))^{(m)}$ is decreasing in m. Thus, it suffices to show that $(F^\circ(0))^{(m)} \le (F^\circ(0))^{(m-1)}$ for all $m \ge n$.

As in the proof of Lemma 2.4, let $S_{ij}(m)$ be the set of paths that start from i to j in m steps. A typical path in $S_{ij}(m)$ is $(i_0, i_1, \ldots, i_{m-1}, i_m)$ with $i_0 = i$ and $i_m = j$. Then

$$(F^\circ(0))_{ij}^{(m)} = \min_{(i_0,i_1,\ldots,i_{m-1},i_m)\in S_{ij}(m)} [\sum_{k=1}^{m} (F^\circ(0))_{i_{k-1},i_k}]$$

and $\sum_{k=1}^{m}(F^\circ(0))_{i_{k-1},i_k}$ may be viewed as the weight associated with the path $(i_0, i_1, \ldots, i_{m-1}, i_m)$.

To show that $(F^\circ(0))_{ij}^{(m)} \leq (F^\circ(0))_{ij}^{(m-1)}$ for $m \geq n$, it suffices to argue that for every path in $S_{ij}(m)$, there is a path in $S_{ij}(m-1)$ with less or equal weight. Note that for every path in $S_{ij}(m)$ with $m \geq n$, there exists at least a cycle (as we are throwing $m+1$ balls into n urns), i.e, there exists $0 \leq m_1 < m_2 \leq m$ such that $i_{m_1} = i_{m_2}$. Thus,

$$\begin{aligned}
\sum_{k=1}^{m}(F^\circ(0))_{i_{k-1},i_k} &\geq \sum_{k=1}^{m_1}(F^\circ(0))_{i_{k-1},i_k} + \sum_{k=m_2+1}^{m}(F^\circ(0))_{i_{k-1},i_k} \\
&= \sum_{k=1}^{m_1}(F^\circ(0))_{i_{k-1},i_k} + \sum_{k=m_2+1}^{m}(F^\circ(0))_{i_{k-1},i_k} + \sum_{k=m+1}^{m+m_2-m_1-1}(F^\circ(0))_{j,j}, \qquad (40)
\end{aligned}$$

as $(F^\circ(0))_{jj} = 0$ for all j. Note that (40) is the weight of the path

$$(i_0, \ldots, i_{m_1}, i_{m_2+1}, \ldots, i_{m-1}, i_m, i_{m+1}, \ldots, i_{m+m_2-m_1-1})$$

with $i_m = i_{m+1} = \ldots = i_{m+m_2-m_1-1} = j$. This path is in $S_{ij}(m-1)$. This completes the argument for (11)

Now by induction on t, we have from (38) that $F^*(t)$ is the maximum solution of the equation

$$H(t) = [F^\circ(0) \otimes H(t)] \oplus \Big[[F^\circ(1) \otimes F^*(t-1)] \oplus [F^\circ(2) \otimes F^*(t-2)] \oplus \ldots \oplus [F^\circ(t) \otimes F^*(0)]. \quad (41)$$

Once again, iterating (41) yields (12).

F Appendix F

In this section, we prove Lemma 2.8 and Lemma 2.9.

Proof. (Lemma 2.8) We first show the *if* part. Note that $F \leq H \star G$ implies

$$F_{ik}(t+s) \leq \min_{1\leq j\leq n} \min_{0\leq\tau\leq t+s} [H_{ij}(t+s-\tau) + G_{jk}(\tau)],$$

for all $1 \leq i, k \leq n$ and $t, s \geq 0$. Thus, taking $\tau = s$ yields

$$F_{ik}(t+s) \leq H_{ij}(t) + G_{jk}(s),$$

for all $1 \leq i, j, k \leq n$ and $t, s \geq 0$. This in turns implies

$$\max_{1\leq k\leq n} \sup_{s\geq 0} [F_{ik}(t+s) - G_{jk}(s)] \leq H_{ij}(t),$$

for all $t \geq 0$ and $1 \leq i,j \leq n$. As $H \in \mathcal{F}^{n\times n}$, $H_{ij}(t) \geq 0$. Thus,

$$\max_{1\leq k\leq n} \sup_{s\geq 0}[F_{ik}(t+s) - G_{jk}(s)]^+ \leq H_{ij}(t),$$

for all $t \geq 0$ and $1 \leq i,j \leq n$. In matrix form, we then have $F \div G \leq H$.

To see the *only if* part, note that $F \div G \leq H$ implies

$$F_{ik}(t+s) - G_{jk}(s) \leq H_{ij}(t),$$

for all $1 \leq i,j,k \leq n$ and $t,s \geq 0$. Letting $\tau = t+s$ yields

$$F_{ik}(\tau) \leq \min_{1\leq j\leq n} \min_{0\leq s\leq \tau}[H_{ij}(\tau - s) + G_{jk}(s)],$$

for all $\tau \geq 0$. Thus, $F \leq H \star G$. ■

Proof. (Lemma 2.9) (i) That (i) holds follows trivially from the definition in (13).

(ii) Let $F_1 = F \div (H\star G)$. From the sup-division property in Lemma 2.8 and the associativity of $\star$, it follows that

$$F \leq F_1 \star (H \star G) = (F_1 \star H) \star G.$$

Thus, we have from the sup-division property that $F \div G \leq F_1 \star H$. Applying the sup-division property once more yields

$$(F \div G) \div H \leq F_1 = F \div (H \star G).$$

To show the other inequality, let $F_2 = (F \div G) \div H$. From the sup-division property, it follows that $F \div G \leq F_2 \star H$. Applying the sup-division property once more yields $F \leq (F_2 \star H) \star G$. From the associativity of $\star$, we have $F \leq F_2 \star (H \star G)$. Once again, using the sup-division property to obtain

$$F \div (H \star G) \leq F_2 = (F \div G) \div H.$$

(iii) From the sup-division property, it follows immediately that $F \div F \leq H$ for any H satisfying $F = H \star F$. Thus, it remains to show that $F \div F$ is indeed a solution. In view of the equality $F \div F = F \div F$, one has from the sup-division property that $F \leq (F \div F) \star F$.

On the other hand, as $(F \div F)_{ii}(0) = 0$ for all i, $F \div F$ is in $\mathcal{F}_0^{n\times n}$. Thus, it follows from the monotonicity of $\star$ that $F \geq (F \div F) \star F$.

(iv) As $F \div F$ is a solution of $F = (F \div F) \star F$, iterating the equation yields that

$$F = ((F \div F) \star (F \div F)) \star F.$$

Thus, $(F \div F) \star (F \div F)$ is also a solution. In (iii), we have shown that $F \div F$ is the minimum solution. Thus,

$$(F \div F) \star (F \div F) \geq F \div F. \tag{42}$$

On the other hand, as $F \div F$ is in $\mathcal{F}_0^{n\times n}$, it follows from the monotonicity of $\star$ that $(F \div F) \star (F \div F) \leq F \div F$. In conjunction with (42),

$$(F \div F) \star (F \div F) = F \div F.$$

Since $F \div F$ is in $\mathcal{F}_0^{n\times n}$,

$$(F \div F)^* = \lim_{m\to\infty} (F \div F)^{(m)} = F \div F.$$

(v) If $F \div F = F$, then it follows from (iv) that

$$F^* = (F \div F)^* = F \div F = F.$$

Now suppose $F = F^*$. We first show that $(F \div F) \div F = F \div F$ under such a condition. From (ii) and Lemma 2.2(ii), it follows that

$$\begin{aligned}&(F \div F) \div F = F \div (F \star F) = F \div (F^* \star F^*)\\&= F \div F^* = F \div F.\end{aligned}$$

Applying the sup-division property yields

$$F \div F \leq (F \div F) \star F.$$

As a result of (iii), i.e., $(F \div F) \star F = F$, we have $F \div F \leq F$. For the other inequality, we observe from the monotonicity of $\div$ that

$$F \div F = F \div F^* \geq F,$$

as $F^* \in \mathcal{F}_0^{n\times n}$.

(vi) That (vi) holds follows directly from (iv) and (v). ■

References

[1] V. Anantharam and T. Konstantopoulos. "Optimality and interchangeability of leaky buckets in tandem," Univ. of Texas at Austin, Tech. Report SCC-95-01, 1995. Initial version also in: Proc. 32nd Allerton Conference, pp. 235-244, Monticello, IL, October 1994.

[2] R. Agrawal and R. Rajan, "Performance bounds for guaranteed and adaptive services," *IBM RC 20649*, 1996. Also in *Proc. 34th Allerton Conf. on Comm., Cont. & Comp.*, Monticello, IL, Oct. 1996.

[3] F. Baccelli, G. Cohen, G. Oslder, and J. Quadrat, *Synchronization and Linearity: An Algebra for Discrete Event Systems,* Wiley, 1992.

[4] J.Y. Le Boudec, "Network calculus made easy," *preprint.*

[5] C.S. Chang, "Stability, queue length and delay of deterministic and stochastic queueing networks," *IEEE Trans. Automat. Contr.*, Vol. 39, pp. 913-931, 1994.

[6] C.S. Chang, "On deterministic traffic regulation and service guarantees: a systematic approach by filtering," to appear in *IEEE Trans. Inform. Theory*, May, 1998. Abbreviated version "A filtering theory for deterministic traffic regulation," in *IEEE INFOCOM'97*, Kobe, Japan.

[7] R.L. Cruz, "A calculus for network delay, Part I: Network elements in isolation," *IEEE Trans. Inform. Theory*, Vol. 37, pp. 114-131, 1991.

[8] R.L. Cruz, "A calculus for network delay, Part II: Network analysis," *IEEE Trans. Inform. Theory*, Vol. 37, pp. 132-141, 1991.

[9] R.L. Cruz, "Quality of service guarantees in virtual circuit switched networks," *IEEE J. Selected Areas in Communications*, Vol. 13, pp. 1048-1056, 1995.

[10] R.L. Cruz and C.M. Okino, "Service guarantees for window flow control," *Proc. 34th Allerton Conf. on Comm., Cont. & Comp.*, Monticello, IL, Oct. 1996.

[11] T. Konstantopoulos and V. Anantharam. "Optimal flow control schemes that regulate the burstiness of traffic," *IEEE/ACM Trans. Networking*, Vol. 3, pp. 423-432, 1995.

[12] A.K. Parekh and R.G. Gallager, "A generalized processor sharing approach to flow control in integrated service networks: the single-node case," *IEEE/ACM Trans. Networking*, Vol. 1, pp. 344-357, 1993.

[13] H. Sariowan, "A Service-Curve Approach to Performance Guarantees in Integrated-Service Networks," Ph.D. Dissertation, University of California, San Diego, 1996.

[14] H. Zhang, "Service displines for guaranteed performance service in packet-switching networks," *Proceedings of the IEEE*, Vol. 83, pp. 1374-1396, 1995.