Generalization of the Pollaczek-Khinchin Formula for Throughput Analysis of Input-buffered Switches

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***Abstract*— Many switch architectures with buffers placed at input ports suffer from the head-of-line blocking (HOL) problem and thus can not achieve 100% throughput. For an input-buffered switch, the number of HOL packets is often characterized by the Lindley equation for a discrete-time queue, i.e., $q(t+1) = (q(t) - F)^+ + a(t)$, where $q(t)$ is the number of HOL packets at time t, $a(t)$ is the number of new HOL packets at time t, and F is the maximum number of HOL packets that can depart per unit of time. As the total number of HOL packets is bounded in a switch, it places an upper limit on the expected number of HOL packets. Thus, the maximum throughput is the utilization that makes the expected HOL packets equal to the upper limit. For the case with $F = 1$, the expected number of HOL packets can be found via the Pollaczek-Khinchin formula and the maximum throughput can be solved by a quadratic equation as reported in [5], [7], [13]. One of the main contributions of this paper is that we derive a generalized Pollaczek-Khinchin formula for the case $F > 1$. Such a formula is then used for finding the maximum throughput of several input-buffered switches. For the case $F >> 1$, numerical computation of the maximum throughput becomes difficult. For large F, we present several bounds and approximations for the throughput. Numerical studies and simulation results confirm that our approximation methods work well.**

***Index Terms*— discrete time queues, maximum throughput, input-buffered switches, Pollaczek-Khinchin formula**

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I. Introduction

As fiber optical technology becomes more and more mature, there is a demand for packet switches to match with the speed of fiber optics. Conventional packet switches are based on shared memory or shared medium architectures. It is well known that these architectures based on primarily output buffering suffer from scalability limitation. Input buffered switches are much more scalable than output buffered switches and have received a lot of attention recently. In order to keep up with the high-speed optical access lines, complicated computation and excessive information exchange among various modules in the switch must be minimized. Hence, the operation of the switch must be very simple and is mostly hardware based. As a result, congestion in the switch may arise because arrivals to a packet switch may occur simultaneously on different inputs destined for the same output. One of these contending packets for an output may be allowed to move forward, but the others must be queued for later transmission. As such, many switches proposed in the literature that are based on input buffers can not achieve 100% throughput [5], [7], [13], [3].

The main objective of this paper is to investigate whether the throughput of input-buffered switches can be found via a general analytic approach. It is well known that an output-buffer switch achieves 100% throughput while an input-buffered switch with First-In-First-Out (FIFO) queues does not. To gain the intuition behind this fact, we consider a discrete-time output-buffered switch with fixed size packets. Let F be the capacity of the output link, i.e., the maximum number of packets that can be transmitted per unit of time. Also, let $q(t)$ be the number of packets stored in the output buffer at time t and $a(t)$ be the number of packet arrivals at time t. Then, the switch is governed by the renowned Lindley equation:

$$q(t+1) = (q(t) - F)^+ + a(t), \tag{1}$$

where the operator $(x)^+$ denotes $\max(x, 0)$. Intuitively, as long as the average arrival rate, i.e., $\mathsf{E}[a(t)]$, is smaller than the output link capacity F, the queue will not blow up (see e.g., [1] for a more detailed description on stability of queues). As such, what comes in goes out.

On the other hand, for an input-buffered switch with FIFO queues, the quantity $q(t)$ represents the number of head-of-line (HOL) packets at the input buffers that are destined for a specific output. In general, there is a limit on the total number of HOL packets in an input-buffered switch (as a mean to limit the complexity of the switch). As such, in addition to the Lindley equation in (1), there is a constraint on the expected number of HOL packets for a specific output port in the steady state. Suppose that for some $s > 0$

$$\mathsf{E}[q(\infty)] \leq sF, \tag{2}$$

where $q(\infty)$ is the stead state random variable of $q(t)$. Then the maximum throughput of an input-buffered switch is simply the average arrival rate that achieves the limit in (2).

Clearly, the first step of finding the maximum throughput involves solving the Lindley equation in (1). In this paper, we will assume that $\{a(t), t \geq 1\}$ is a sequence of independent Poisson random variables (with mean ρF). Such an assumption is quite common in the literature (see e.g., [5], [7], [13]) as the arrival process to a specific output port is a superposition of a large number of arrival processes at the inputs. For the special case $F = 1$, the solution of the Lindley equation can be derived from the famous Pollaczek-Khinchin formula for $M/GI/1$ queues (see e.g., [10]) and we have

$$\mathsf{E}[q(\infty)] = \frac{1}{2}\frac{2\rho - \rho^2}{1-\rho}, \tag{3}$$

where $\rho = \mathsf{E}[a(t)]$ is the average arrival rate. In [5], a single FIFO queue is used at each input port and that limits the number of HOL packets destined to a particular output to 1 (by the symmetry of all output ports). Thus, for the input-buffered switch with a single FIFO queue at each input, the maximum throughput is the solution of the fixed point equation

$$\frac{1}{2}\frac{2\rho - \rho^2}{1-\rho} = 1. \tag{4}$$

This leads to the well-known 58.6% throughput in [5]. Instead of using a single FIFO queue at each input, one can increase the throughput by using multiple FIFO queues at each input (see e.g., [7], [13]). Suppose that s FIFO queues are used at each input. This then limits the number of HOL packets to s. The maximum throughput is the solution of the fixed point equation

$$\frac{1}{2}\frac{2\rho - \rho^2}{1-\rho} = s. \tag{5}$$

This shows that the maximum throughput is $s + 1 - \sqrt{s^2 + 1}$ [7], [13].

For the general case $F > 1$, solving the Lindley equation involves the Wiener-Hopf spectral decomposition for $GI/GI/1$ queues, and $\mathsf{E}[q(\infty)]$ can be derived by the Spitzer identity (see e.g., [6]) as follows:

$$\mathsf{E}[q(\infty)] = \mathsf{E}[a(1)] + \sum_{t=1}^{\infty} \frac{1}{t} \mathsf{E}[(\sum_{s=1}^{t} a(s) - tF)^+]. \tag{6}$$

However, the Spitzer identity involves an infinite sum. As such, it is difficult to compute $\mathsf{E}[q(\infty)]$ based on (6).

One of the main contributions of this paper is that we derive a generalized Pollaczek-Khinchin formula that is much easier to compute than the Spitzer identity (see Section III for more details). Specifically, we show that

$$\mathsf{E}[q(\infty)] = \frac{1 - F(1-\rho)^2}{2(1-\rho)} + \sum_{j=1}^{F-1} \frac{1}{1-\alpha_j}, \tag{7}$$

where α_j is the unique root within the unit circle of the following equation:

$$z - e^{\mathrm{i}2\pi j/F} e^{\rho(z-1)} = 0, \tag{8}$$

for $j = 0, 1, \dots, F-1$. Note that $\mathrm{i} = \sqrt{-1}$ in (8). Based on (7), we then provide an algorithm for solving the fixed point problem with

$$\mathsf{E}[q(\infty)] = sF. \tag{9}$$

When F is large, i.e., $F >> 1$, solving the fixed point problem becomes difficult. To cope with the problem for large F, we have to resort to bounds and approximations in Section IV. These bounds and approximations are compared with exact analysis and simulations in Section V.

II. Motivation and Examples

To motivate our study, in this section we present two examples of input-buffered switches whose maximum throughput can be found by solving the fixed point problem in (1) and (9).

Example 1 (Input-buffered switch with line grouping) In Figure 1, we consider an input-buffered switch with M inputs and N outputs. Each input maintains a

FIFO queue. Each output has F parallel links. As such, there are FN output links and these FN output links are connected to the M inputs via an $M \times FN$ crossbar switch. A packet destined for a particular output can use any of the F links of that output. This is known as line grouping in [8].

To analyze the maximum throughput of this switch, we consider the well-known uniform i.i.d. traffic model as in [5], [2]. We assume that packets are the same size and time is slotted so that a packet can be transmitted within a time slot. Moreover, the arrival processes to the input ports satisfy the following conditions.

(A1) Arrivals at each input port are independent and identical Bernoulli processes.
(A2) All arrival processes have the same arrival rate ρ_a and every arrival process is independent of others.
(A3) The destination of every arrival at each input port is uniformly distributed over N outputs.
(A4) N is large.

To find out the maximum throughput, we may assume that there are already infinite many packets queued at the input buffers. As such, there are exactly M HOL packets. Since the traffic is uniform, it suffices to look at a particular output due to symmetry. Let $q(t)$ be the number of HOL packets destined for output port 1 at time t and $a(t)$ be the number of packets that become HOL packets and choose output port 1 as their destination at time t. Since there are F parallel links for output 1, there are at most F HOL packets that can be transmitted to output 1. Thus, we have the following Lindley equation

$$q(t+1) = (q(t) - F)^+ + a(t). \tag{10}$$

Moreover, as argued in [5], the quantity $a(t)$ is a (random) sum of independent Bernoulli random variables and it converges to a Poisson distribution as $N \to \infty$. As there are exactly M HOL packets at any time, we then have from symmetry that

$$\mathsf{E}[q(\infty)] = \frac{M}{N}. \tag{11}$$

Thus, the maximum throughput of an input-buffered switch with line grouping can be found by solving the fixed point problem in (1) and (9) (with $s = M/FN$).

Example 2 (Frame based mailbox switch) In [3], a two-stage input buffered switch called the mailbox switch was proposed. This switch architecture has very simple operation and incurs very little computation and communication overheads. In this example, we present

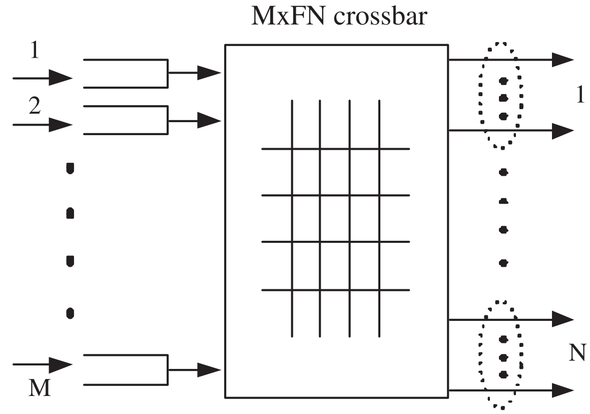

Fig. 1. An input-buffered switch with line grouping

a frame based mailbox switch. Time is partitioned into frames. Each frame consists of exactly F time slots. As in the mailbox switch, the frame based mailbox switch has two $N \times N$ crossbar switch fabrics (see Figure 2) and buffers between the two crossbar switch fabrics. The buffers between the two switch fabrics are called mailboxes. There are N mailboxes, indexed from 1 to N. Each mailbox contains N bins (indexed from 1 to N), and each bin contains F cells (indexed from 1 to F). Each cell can store exactly one packet. Cells in the i^{th} bin of a mailbox are used for storing packets that are destined for the i^{th} output port of the second switch. In addition to these, a First In First Out (FIFO) queue is added in front of each input port of the first stage.

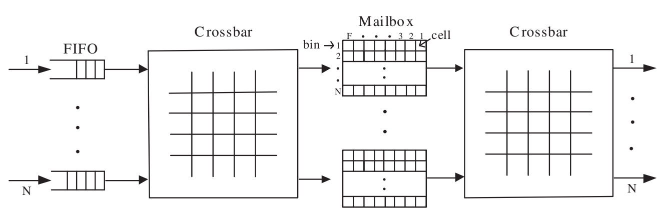

Fig. 2. The switch architecture

The connection patterns of the two switching fabrics are set up in the same way as those of the original mailbox switches, except that the connection patterns in the original mailbox switches are changed in every time slot and the connection patterns in the frame based mailbox switches are changed in every frame. Specifically, During the t^{th} frame, input port i is connected to output port j if

$$j = h(i,t) = \Big((t-i) \bmod N\Big) + 1. \tag{12}$$

A nice property of the connection patterns generated by (12) is that such connection patterns are *symmetric*, i.e.,

$$i = h(h(i,t),t).$$

In words, input port i and output port j are connected if and only if input port j and output port i are connected. As such, we call a switch fabric that implements the connection patterns in (12) a *symmetric Time Division Multiplexing (TDM) switch*. As mentioned in [3], this symmetric property of the connection patterns implies that if input port i of the first stage is connected to mailbox j, the second stage connects mailbox j to output port i of the second stage. If we build the input i of the first switch and output port i of the second switch on the same line card, the symmetric connection patterns provide a bi-directional communication link between a line card and a mailbox. This helps to solve the out of sequence problem in the original mailbox in [3]. In the framed based mailbox switch, there is no need for resequencing. Such a bi-directional communication link is only used for back pressure, i.e., feeding back the information to an input port when a packet is blocked due to a full bin.

We now describe how a frame based mailbox switch works. The connection patterns of the two switch fabrics are changed at the beginning of every frame. Every input port of the first stage maintains a counter. These counters are reset to one in the beginning of every frame. Denote by $c_i(\tau, t)$ the value of the counter of the i^{th} input port in the t^{th} time slot of the τ^{th} frame. In every time slot in a frame, the following operations are repeated for each input/ouput port.

(i) **Retrieving mails**: in the t^{th} time slot of the τ^{th} frame, the i^{th} output port of the second switch is connected to the $h(i, \tau)^{th}$ mailbox. The packet in the first cell of the i^{th} bin is transmitted to the i^{th} output port. Packets in cells $2, 3, \ldots, F$ of the i^{th} bin are moved forward to cells 1,2,..., $F - 1$.

(iiA) **Sending mails**: Assume that the destination of the $c_i(\tau, t)^{th}$ packet is j. If the j^{th} bin is completely full, this information is fed back to the input port by the second switch fabric. The $c_i(\tau, t)^{th}$ packet is blocked at the input port and the counter is incremented. If the j^{th} bin is not full, the $c_i(\tau, t)^{th}$ packet is placed in the first empty cell in the j^{th} bin, and all the packets after the $c_i(\tau, t)^{th}$ packet are moved one place forward in the input buffer.

To analyze the maximum throughput of this switch, we also consider the well-known uniform i.i.d. traffic model as in [5], [2]. Assume that the arrival processes to the input ports of the first switch satisfy the following conditions.

(A1) Arrivals at each input port are independent and identical Bernoulli processes.

(A2) All arrival processes have the same arrival rate ρ_a and every arrival process is independent of others.

(A3) The destination of every arrival at each input port is uniformly distributed over N outputs.

(A4) N is large.

As the traffic is uniform, we only need to consider a particular output port of the second switch, say, the first output. During the t^{th} frame, it is connected to the $h(1, t)^{th}$ mailbox, and this mailbox is also connected to the first input port of the first switch. Since the frame size is F, at most F packets can be transmitted from the first bin of the $h(1, t)^{th}$ mailbox to the first output port of the second switch. If the first bin of the $h(1, t)^{th}$ mailbox is occupied at the beginning of the t^{th} frame, then all the packets are retrieved by the first output port and the first bin becomes empty. In any event, we know that the first bin of the $h(1, t)^{th}$ mailbox is empty at the beginning of the t^{th} frame.

Let $Y_i(t)$ be the number of packets in the first F positions from the head of the queue at the i^{th} input port that are destined for the first output port of the second switch in frame t. Let

$$q(t) = \sum_{i=1}^{N} Y_i(t + i - 1). \tag{13}$$

As the $h(1, t)^{th}$ mailbox is connected to the first input port during the t^{th} frame, it will be connected to the i^{th} input port in frame $t+i-1$. Thus, $q(t)$ is the total number of HOL packets that can be placed in the first bin of the $h(1, t)^{th}$ mailbox from frame t to frame $t + N - 1$. If $q(t) \geq F$, then there are exactly F HOL packets that will be placed in the first bin of the $h(1, t)^{th}$ mailbox as the bin is empty at the beginning of the t^{th} frame. Other blocked HOL packets remain the HOL packets and they can be placed in the first bin of the $h(1, t+1)^{th}$ mailbox from frame $t + 1$ to frame $t + N$. The quantity $a(t)$ is the number of packets that becomes the HOL packets and can be placed in the first bin of the $h(1, t + 1)^{th}$ mailbox from frame $t+1$ to frame $t+N$. The maximum throughput is the solution of the fixed point equation (9) with $s = 1$.

The advantage of using the frame based mailbox switch is its simplicity. It no longer requires feeding back the information of virtual waiting time in the original design of the mailbox switch in [3]. The only information needed is whether a packet is blocked or not. As we will show in the later sections, the throughput of the framed based mailbox switch increases to 100% as the frame size $F \to \infty$. However, this is at the cost of increasing delay in light traffic. On average, every bin needs to wait $NF/2$ time slots in order to be connected to its output port.

III. Generalization of the Pollaczek-Khinchin formula

In this section, we consider the Lindley equation in (1). As long as the average arrival rate $\mathsf{E}[a(1)]$ is smaller than the capacity F, we know from the classical queueing theory (see e.g., [1]) that $q(t)$ converges in distribution to a unique steady state random variable $q(\infty)$ that satisfies

$$q(\infty) =_{st} (q(\infty) - F)^+ + a(1), \tag{14}$$

where $a(1)$ is independent of $q(\infty)$ and $=_{st}$ denotes the stochastic identity, i.e., $X =_{st} Y$ if the two random variables X and Y have the same distribution.

In this section, we solve the Lindley equation in (14) for a general distribution of $a(1)$. Let $\mathrm{i} = \sqrt{-1}$,

$$p_i = \mathsf{P}(q(\infty) = i), \quad i = 0, 1, \ldots \tag{15}$$

and

$$P(z) = \sum_{i=0}^{\infty} p_i z^i = \mathsf{E}[z^{q(\infty)}] \tag{16}$$

be the generating function of $q(\infty)$. Also, let

$$A(z) = \mathsf{E}[z^{a(1)}] \tag{17}$$

be the generating function of $a(1)$,

$$\rho = \frac{\mathsf{E}[a(1)]}{F} \tag{18}$$

be its normalized mean, and

$$\sigma^2 = \frac{\mathsf{E}[(a(1))^2] - (\mathsf{E}[a(1)])^2}{F^2} \tag{19}$$

be its normalized variance.

Theorem 3 (Generalized Pollaczek-Khinchin formula) *If $\rho < 1$, then*

$$P(z) = \frac{(z-1)G(z)}{H(z)}, \tag{20}$$

where

$$H(z) = \frac{z^F}{A(z)} - 1, \tag{21}$$

$$G(z) = F(1-\rho) \prod_{j=1}^{F-1} \frac{(z - \alpha_j)}{(1 - \alpha_j)}, \tag{22}$$

and α_j is the unique root within the unit circle of the following equation:

$$z - e^{\mathrm{i}2\pi j/F} (A(z))^{1/F} = 0, \tag{23}$$

for $j = 0, 1, \ldots, F-1$. Moreover,

$$\mathsf{E}[q(\infty)] = \frac{\sigma^2 F - F\rho^2 - \rho + 2F\rho - F + 1}{2(1-\rho)} + \sum_{j=1}^{F-1} \frac{1}{1 - \alpha_j}. \tag{24}$$

Note that if $F = 1$, then

$$\mathsf{E}[q(\infty)] = \frac{\sigma^2 + \rho - \rho^2}{2(1-\rho)}$$

and (24) reduces to the well-known Pollaczek-Khinchin mean value formula. If $a(1)$ is a Poisson random variable with mean ρF, then

$$A(z) = e^{F\rho(z-1)}. \tag{25}$$

and

$$\sigma^2 = \rho/F. \tag{26}$$

Using (26) in (24), we have the following corollary.

Corollary 4 *If $a(1)$ is a Poisson random variable with mean ρF and $\rho < 1$, then*

$$\mathsf{E}[q(\infty)] = \frac{1 - F(1-\rho)^2}{2(1-\rho)} + \sum_{j=1}^{F-1} \frac{1}{1 - \alpha_j}, \tag{27}$$

where α_j is the unique root within the unit circle of the following equation:

$$z - e^{\mathrm{i}2\pi j/F} e^{\rho(z-1)} = 0, \tag{28}$$

for $j = 0, 1, \ldots, F-1$.

The proof of Theorem 3 is based on a sequence of lemmas. In Lemma 5 below, we first show that there is a unique root within the unit circle for (23). The proof is based on Rouché's theorem and is shown in Appendix A.

Lemma 5 *If $\rho < 1$, the equation*

$$z - e^{2\pi \mathrm{i} j/F} (A(z))^{1/F} = 0$$

has exactly one root on or within the unit circle for any $j = 0, 1, \ldots, F-1$.

In Lemma 6 below, we derive the generating function of $q(\infty)$ in terms of p_j, $j = 0, 1, \ldots, F-1$.

Lemma 6 *If $\rho < 1$, then*

$$P(z) = \frac{\sum_{i=0}^{F-1} p_i (z^F - z^i)}{H(z)}, \tag{29}$$

where $H(z)$ is defined in (21).

Proof. since $a(1)$ is independent of $q(\infty)$ in (14). Thus we have from (14) that

$$\begin{aligned}P(z) &= \mathsf{E}[z^{q(\infty)}] \\ &= \mathsf{E}[z^{(q(\infty)-F)^+}]\mathsf{E}[z^{a(1)}] \\ &= \mathsf{E}[z^{(q(\infty)-F)^+}]A(z). \end{aligned} \tag{30}$$

Note that

$$\begin{aligned}&\mathsf{E}[z^{(q(\infty)-F)^+}] \\ &= \sum_{i=0}^{\infty} p_i z^{(i-F)^+} \\ &= \sum_{i=F}^{\infty} p_i z^{i-F} + \sum_{i=0}^{F-1} p_i \\ &= z^{-F}\Big(P(z) + \sum_{i=0}^{F-1} p_i(z^F - z^i)\Big).\end{aligned}$$

Using this in (30) yields (29). ■

Lemma 7 *For $i = 0, 1, \ldots, F-1$, let*

$$\tilde{p}_i = \sum_{j=0}^{i} p_j = \mathsf{P}(q(\infty) \le i).$$

Then the following identity holds:

$$\sum_{j=0}^{F-1} p_j(z^F - z^j) = (z-1)\sum_{j=0}^{F-1} \tilde{p}_j z^j. \tag{31}$$

Proof. Note that

$$\begin{aligned}&\sum_{j=0}^{F-1} p_j(z^F - z^j) \\ &= p_0(z^F - 1) + \sum_{j=1}^{F-1}(\tilde{p}_j - \tilde{p}_{j-1})(z^F - z^j) \\ &= p_0(z^F - 1) + \sum_{j=1}^{F-1}\tilde{p}_j(z^F - z^j) \\ &\qquad - \sum_{j=0}^{F-2}\tilde{p}_j(z^F - z^{j+1}) \\ &= p_0(z-1) + \tilde{p}_{F-1}(z^F - z^{F-1}) \\ &\qquad + \sum_{j=1}^{F-2}\tilde{p}_j(z^{j+1} - z^j) \\ &= \sum_{j=0}^{F-1}\tilde{p}_j(z^{j+1} - z^j) \\ &= (z-1)\sum_{j=0}^{F-1}\tilde{p}_j z^j\end{aligned}$$

■

Lemma 8 *The following identity holds:*

$$\sum_{j=0}^{F-1} \tilde{p}_j z^j = G(z), \tag{32}$$

where $G(z)$ is defined in (22).

Proof. In view of (31), we can rewrite (29) as follows:

$$P(z) = \frac{A(z)(z-1)\sum_{j=0}^{F-1}\tilde{p}_j z^j}{z^F - A(z)}. \tag{33}$$

Note that the denominator of (33) can be factorized into

$$z^F - A(z) = \prod_{j=0}^{F-1}\left(z - e^{2\pi \mathrm{i} j/F}(A(z))^{1/F}\right),$$

where $\mathrm{i} = \sqrt{-1}$. According to Lemma 5, there exists a unique root α_j on or within the unit circle for the equation

$$z - e^{2\pi \mathrm{i} j/F}(A(z))^{1/F} = 0.$$

Therefore, the denominator of (33) has F distinct roots on or within the unit circle. It is easy to verify that $\alpha_0 = 1$ and $\alpha_i \neq \alpha_j$, if $i \neq j$. Furthermore, α_j is the complex conjugate of α_{F-j} for any $j = 1, 2, \ldots, F-1$.

Since $P(z)$ is analytic on or within the unit circle, the numerator on the right hand side of (33) must vanish when z equals any one of the roots $\alpha_0, \alpha_1, \ldots, \alpha_{F-1}$. Since $\alpha_0 = 1$, we then have

$$\sum_{j=0}^{F-1}\tilde{p}_j z^j = \gamma\prod_{j=1}^{F-1}(z - \alpha_j), \tag{34}$$

where γ is the normalization constant. To find out the normalization constant γ, we use the fact that $P(1) = 1$. Note that

$$\begin{aligned}A(1) &= 1, \\ A'(1) &= F\rho.\end{aligned}$$

Applying L'Hospital's rule, we then obtain

$$\gamma = \frac{F(1-\rho)}{\prod_{j=1}^{F-1}(1-\alpha_j)}. \tag{35}$$

Using this in (34) yields (32). ■

Proof. (Theorem 3) Note that (20) is a direct consequence of Lemma 6, Lemma 7 and Lemma 8.

To derive the mean queue length in (27), we use the fact that $\mathsf{E}[q(\infty)] = P'(1)$. Recall from (20) that

$$P(z) = \frac{(z-1)G(z)}{H(z)}.$$

Thus,

$$P'(z) = \frac{(z-1)G'(z)}{H(z)} + \frac{G(z)(H(z)-(z-1)H'(z))}{(H(z))^2}. \tag{36}$$

Note that

$$\begin{aligned} A(1) &= 1, \\ A'(1) &= F\rho, \\ A''(1) &= F^2\sigma^2 + (F\rho)^2 - F\rho. \end{aligned}$$

It is easy to verify that

$$G(1) = F(1-\rho), \tag{37}$$

$$\begin{aligned} G'(1) &= F(1-\rho)\sum_{j=1}^{F-1}\frac{1}{1-\alpha_j}, \\ H(1) &= 0, \\ H'(1) &= F(1-\rho), \\ H''(1) &= -F(\sigma^2 F - F\rho^2 - \rho + 2F\rho - F + 1) \end{aligned}$$

Applying L'Hospital's rule, we then obtain

$$\begin{aligned} &\lim_{z\to 1}\frac{(z-1)G'(z)}{H(z)} \\ &= \lim_{z\to 1}\frac{(z-1)}{H(z)}G'(1) \\ &= \lim_{z\to 1}\frac{1}{H'(z)}G'(1) \\ &= \frac{G'(1)}{H'(1)}. \end{aligned}$$

Also,

$$\begin{aligned} &\lim_{z\to 1}\frac{G(z)(H(z)-(z-1)H'(z))}{(H(z))^2} \\ &= G(1)\lim_{z\to 1}\frac{H(z)-(z-1)H'(z)}{(H(z))^2} \\ &= -G(1)\lim_{z\to 1}\frac{(z-1)H''(z)}{2H(z)H'(z)} \\ &= -G(1)\frac{H''(1)}{2H'(1)}\lim_{z\to 1}\frac{(z-1)}{H(z)} \\ &= -\frac{G(1)H''(1)}{2(H'(1))^2}. \end{aligned}$$

Thus,

$$\mathsf{E}[q(\infty)] = P'(1) = \frac{G'(1)}{H'(1)} - \frac{G(1)H''(1)}{2(H'(1))^2} \tag{38}$$

$$= \sum_{j=1}^{F-1}\frac{1}{1-\alpha_j} + \frac{\sigma^2 F - F\rho^2 - \rho + 2F\rho - F + 1}{2(1-\rho)}.$$

■

IV. Throughput analysis for input-buffered switches

As described in Section I and Section II, the number of head-of-line (HOL) packets destined for a particular output port satisfies the Lindley equation

$$q(t+1) = (q(t) - F)^+ + a(t),$$

where $\{a(t), t \geq 1\}$ is a sequence of i.i.d. *Poisson* random variable with mean ρF. As long as $\rho < 1$, we have

$$q(\infty) =_{st} (q(\infty - F)^+ + a(1), \tag{39}$$

where $a(1)$ is a Poisson random variable with mean ρF and it is independent of $q(\infty)$. For such a queueing system, we have derived a generalized Pollaczek-Khinchin formula for $\mathsf{E}[q(\infty)]$ in Corollary 4. To find the maximum throughput, we need to solve ρ from the fixed point equation in (9), i.e.,

$$\mathsf{E}[q(\infty)] = sF.$$

A. An algorithm for solving the fixed point problem

Here we provide an algorithm for solving the fixed point problem in (9). For this, we first obtain a lower bound and an upper bound for the maximum throughput ρ.

Lemma 9 *The maximum throughput ρ that satisfies (39) and (9) has the following lower and upper bounds:*

$$1 - \frac{1}{\sqrt{((s-1)F+1)^2 + F} + (s-1)F + 1} \leq \rho$$

$$\leq 1 - \frac{2}{\sqrt{((2s-1)F+1)^2 + 4F} + (2s-1)F + 1}. \tag{40}$$

In particular, if $s = 1$,

$$1 - \frac{1}{\sqrt{F+1}+1} \leq \rho \leq 1 - \frac{2}{\sqrt{(F+1)^2 + 4F} + F + 1}. \tag{41}$$

Note that for $F = 1$ the upper bound is the same as the lower bound and we have

$$\rho = s + 1 - \sqrt{s^2+1}. \tag{42}$$

Moreover, for $s = 1$, the lower bound for the throughput tends to 1 as $F \to \infty$. This shows that the throughput of the frame based mailbox switch in Example 2 can be increased to 100% by increasing the frame size F to ∞.

Proof. Note from Lemma 7 and Lemma 8 that

$$\sum_{j=0}^{F-1} p_j(z^F - z^j) = (z-1)G(z). \tag{43}$$

Differentiating (43) and applying (37), we have that

$$G(1) = \sum_{j=0}^{F-1} p_j(F-j) = (1-\rho)F \tag{44}$$

and

$$G'(1) = -\frac{1}{2}F(1-\rho) + \frac{1}{2}\sum_{j=0}^{F-1} p_j(F^2 - j^2). \tag{45}$$

Substituting (45) into (38) and using the identity $\sigma^2 = \rho/F$, we have another form of $\mathsf{E}[q(\infty)]$ as follows:

$$\mathsf{E}[q(\infty)] = \frac{F\rho - F^2(1-\rho)^2 + \sum_{j=0}^{F-1} p_j(F^2-j^2)}{2F(1-\rho)}. \tag{46}$$

Let

$$\beta = \sum_{j=0}^{F-1} p_j(F^2 - j^2). \tag{47}$$

It then follows from (46) that the maximum throughput ρ satisfies the following equation:

$$sF = \mathsf{E}[q(\infty)] = \frac{F\rho - F^2(1-\rho)^2 + \beta}{2F(1-\rho)}. \tag{48}$$

Note from (44) that

$$F^2(1-\rho) \le \sum_{j=0}^{F-1} p_j(F-j)(F+j) \le (2F-1)F(1-\rho). \tag{49}$$

Thus, we have from (47) and (49) that

$$F^2(1-\rho) \le \beta \le (2F-1)F(1-\rho). \tag{50}$$

Replacing β in (48) by its upper bound in (50) yields the lower bound for ρ in (40). Similarly, replacing β in (48) by its lower bound in (50) yields the upper bound for ρ in (40). The bounds in (41) are then derived by setting $s = 1$ in (40). ■

From the well-known stochastic comparison theorem (see e.g., [11], [9]), the expect queue length $\mathsf{E}[q(\infty)]$ in (27) is an increasing function in ρ. As such, we can then apply the binary search algorithm to find the maximum throughput by the lower bound and the upper bound in Lemma 9. The algorithm is outlined as follows:

Algorithm 1:

(i) Let

$$\rho^L = 1 - \frac{1}{\sqrt{((s-1)F+1)^2 + F} + (s-1)F + 1}$$

and

$$\rho^U = 1 - \frac{2}{\sqrt{((2s-1)F+1)^2 + 4F} + (2s-1)F + 1}.$$

(ii) Let $\rho = (\rho^L + \rho^U)/2$. Solve each α_j for $j = 0, 1, \ldots, F-1$ in (28) for this ρ.

(iii) Compute $\mathsf{E}[q(\infty)]$ in (27). If $\mathsf{E}[q(\infty)] > sF$, then set $\rho^U = \rho$. Otherwise, set $\rho^L = \rho$.

(iv) Go back to Step (ii) until the difference of ρ^U and ρ^L falls within a satisfactory range.

B. A better asymptotic lower bound

Numerical computation for Algorithm 1 becomes difficult for large F as it requires to solve F equations in each iteration. It this section, we derive a better asymptotic lower bound than that in Lemma 9.

Lemma 10 *The maximum throughput ρ that satisfies (39) and (9) has the following lower bound:*

$$\rho \ge 1 - \frac{2}{\sqrt{(2(s-1)F+1)^2 + 8F} + 2(s-1)F + 1}. \tag{51}$$

In particular, if we set $s = 1$ in (51), we have

$$\rho \ge 1 - \frac{2}{\sqrt{8F+1} + 1}. \tag{52}$$

Proof. Let $Z = q(\infty) - F$ and $Y = a(1) - F$. From (39), it follows that

$$Z =_{st} Z^+ + Y, \tag{53}$$

and Y is independent of Z^+. Note that $Z = Z^+ - Z^-$ (with $Z^- = \max(0, -Z)$). Taking expectation on both sides of (53) yields

$$\mathsf{E}Z^- = -\mathsf{E}Y. \tag{54}$$

Also, note that $Z^2 = (Z^+)^2 + (Z^-)^2$ as $Z^+Z^- = 0$. Thus, squaring both sides of (53) and taking expectation yields

$$\mathsf{E}(Z^-)^2 = 2\mathsf{E}Z^+\mathsf{E}Y + \mathsf{E}Y^2. \tag{55}$$

As the variance of a random variable is nonnegative, we then have from (55) and (54) that

$$\begin{aligned} 0 &\le \mathsf{E}(Z^-)^2 - (\mathsf{E}Z^-)^2 \\ &= 2\mathsf{E}Z^+\mathsf{E}Y + \mathsf{E}Y^2 - (\mathsf{E}Z^-)^2 \\ &= 2\mathsf{E}Z^+\mathsf{E}Y + \mathsf{E}Y^2 - (\mathsf{E}Y)^2 \end{aligned}$$

Thus,

$$\mathsf{E}Z^+ \le -\frac{\mathsf{E}Y^2 - (\mathsf{E}Y)^2}{2\mathsf{E}Y}. \tag{56}$$

As $a(1)$ is a Poisson random variable with mean ρF, we have $\mathsf{E}Y = \rho F - F$ and $\mathsf{E}Y^2 - (\mathsf{E}Y)^2 = \rho F$. From (53) and (56), it then follows

$$\begin{aligned} \mathsf{E}q(\infty) &\le -\frac{\mathsf{E}Y^2 - (\mathsf{E}Y)^2}{2\mathsf{E}Y} + \mathsf{E}Y + F \\ &= \rho F + \frac{\rho}{2(1-\rho)}. \end{aligned} \tag{57}$$

Since $\mathsf{E}q(\infty) = sF$ in (9), it then follows from (57) that

$$2F\rho^2 - (1 + 2(s+1)F)\rho + 2sF \le 0. \tag{58}$$

Solving (58) yields the lower bound in (51). ■

C. Poisson Approximation

For the case $s = 1$, it is possible to obtain a better approximation for the throughput than the asymptotic lower bound in (52). The idea is that the steady state queue length of $M/GI/\infty$ queues has a Poisson distribution and we may approximate $q(\infty)$ by a Poisson random variable for large F. Since we are looking for the value of ρ such that (9) holds, we assume that $q(\infty)$ is a Poisson random variable with mean F (as $s = 1$). Since a Poisson random variable with mean F can be decomposed into a sum of F independent Poisson random variables with mean 1, from central limit theorem, we have

$$\lim_{F\to\infty} \frac{q(\infty) - F}{\sqrt{F}} = N(0,1). \tag{59}$$

Taking expectation on (1) and using (59) to compute $\mathsf{E}(q(\infty) - F)^+$, we have

$$F = \sqrt{\frac{F}{2\pi}} + \rho F,$$

from which we obtain

$$\rho = 1 - \frac{1}{\sqrt{2\pi F}}. \tag{60}$$

D. Normal Approximation

Although the Poisson approximation method works reasonably well, we can improve its performance by allowing freedom in the choice of the variance in (59). Specifically, we assume that the distribution of $(q(\infty) - F)/\sqrt{F}$ is approximately normal with zero mean and variance δ^2, when F is large. Let $\tilde{Z} = (q(\infty) - F)^+/\sqrt{F}$. We can rewrite (39) into

$$\tilde{Z} =_{st} \tilde{Z}^+ + \frac{a(1) - F}{\sqrt{F}}. \tag{61}$$

Since

$$\mathsf{E}[\tilde{Z}^+] = \frac{\delta}{\sqrt{2\pi}},$$

taking expectation on (61) yields

$$0 = \frac{\delta}{\sqrt{2\pi}} + (\rho - 1)\sqrt{F}. \tag{62}$$

Moreover,

$$\mathsf{E}[(\tilde{Z}^+)^2] = \frac{\delta^2}{2}.$$

Taking expectation on the square of (61), we have

$$\delta^2 = \frac{\delta^2}{2} + (\rho + F(1-\rho)^2) + 2 \cdot \sqrt{\frac{F}{2\pi}}(\rho - 1)\delta. \tag{63}$$

Solving (62) and (63) for ρ, we have

$$\rho = 1 - \frac{2}{\sqrt{4(\pi+1)F + 1} + 1} \tag{64}$$

$$\delta = (1-\rho)\sqrt{2\pi F}. \tag{65}$$

V. Numerical Studies and Simulations

In this section we report our numerical study of the queueing results and the approximation methods presented in Section IV. We simulate the frame based mailbox switches described in Example 2 in Section II. Two cases are considered. In the first case, $N = 128$ and in the second case, $N = 500$. The maximum throughput of these two cases is shown in Table I. We compare the simulation results with the analytical results. The column with label "exact" corresponds to the numerical solution of the algorithm presented in Section IV-A. From this table, we see that the maximum throughput increases with F. With $F = 30$, the throughput is more than 90%. Also, we note that as F becomes large, the numerical evaluation of the queueing result obtained in section Section III becomes too time consuming and impractical. The lower bound in (52) is a little bit conservative comparing with the simulations results. However, both the Poisson approximation in (60) and the normal approximation in (64) work quite well, even for small F.

VI. Conclusions

In this paper, we showed that the number of HOL packets in several input-buffered switches (including input-buffered switches with line grouping in Example 1 and the frame based mailbox switches in Example 2) can be characterized by the Lindley equation for a discrete-time queue, i.e., $q(t+1) = (q(t) - F)^+ + a(t)$, where $q(t)$ is the number of HOL packets at time t, $a(t)$ is the number of new HOL packets at time t, and F is the maximum number of HOL packets that can depart per unit of time. As the total number of HOL packets in an input-buffered switch is bounded, it places an upper limit on the expected number of HOL packets. Thus, the maximum throughput is utilization that makes the expected HOL packets equal to the upper limit. For the case with $F = 1$, the expected number of HOL packets can be found via the Pollaczek-Khinchin formula and the maximum throughput can be solved by a quadratic equation as reported in [5], [7], [13]. One of the main contributions of this paper is that we derived a generalized Pollaczek-Khinchin formula for the case $F > 1$. Based on such a formula, we presented an algorithm for finding the maximum throughput of input-buffered switches. For the case $F >> 1$, numerical computation of the maximum throughput becomes difficult. For large F, we present several bounds and approximations for the throughput. Numerical studies and simulation results confirm that the approximation methods work well.

APPENDIX

In this appendix we prove Lemma 5. We will show that

$$z - e^{2\pi \mathrm{i} j/F}(A(z))^{1/F} = 0, \quad j = 0, 1, \ldots, F-1, \tag{66}$$

has exactly one root on or within the unit circle, where $A(z) = \mathsf{E}[z^{a(1)}]$ is analytic on the unit circle and does not vanish on the unit circle. We further assume that $A'(1) = \rho F < F$.

We first quote some results of arithmetic random variables from Feller [4, pp. 138,500,501]. Let $A(z)$ be the probability generating function of the discrete probability distribution $\{a_n, n = 0, 1, \ldots\}$. Let S be the set of indices n such that $a_n > 0$. Let c be the largest common divisor of the numbers in the set $S - \{0\}$. c is called the span of $\{a_n\}$. From Feller [4, p501], $A(e^{\mathrm{i}2\pi j/c}) = 1$ for $j = 0, 1, \ldots, c-1$ and $|A(e^{\mathrm{i}\theta})| < 1$ for $\theta \neq 2\pi j/c$.

Define

$$h(z) = z - e^{2\pi \mathrm{i} j/F}(A(z))^{1/F}.$$

Denote by $\Delta \arg h(z)$ the increase of $\arg h(z)$ when z traverses the unit circle exactly once. We have

$$\Delta \arg h(z) = \Delta \arg z + \Delta \arg \left\{ 1 - \frac{e^{2\pi \mathrm{i} j/F}}{z}(A(z))^{1/F} \right\}.$$

If $2\pi j/F \neq 2\pi k/c$ for any $k = 0, 1, \ldots, c-1$, it is easy to see that $|\frac{e^{2\pi \mathrm{i} j/F}}{z}(A(z))^{1/F}| \leq 1$ and $\frac{e^{2\pi \mathrm{i} j/F}}{z}(A(z))^{1/F} \neq 1$. Hence, $u(z)$ never passes the origin and is always within or on the circle with radius 1 and center at 1. This shows that $h(z)$ has exactly one root in $|z| = 1$

Now consider the case in which $2\pi j/F = 2\pi k/c$ for some $k = 0, 1, \ldots, c-1$. For any $|z| < 1$, we have

$$\begin{aligned} \left| \frac{d}{dz}(A(z))^{1/F} \right| &= \left| A'(z)(A(z))^{(1-F)/F}/F \right| \\ &\leq F\rho/F \\ &< 1. \end{aligned} \tag{67}$$

To simplify notation, we let $z_2 = \exp(2\pi \mathrm{i} j/F)$. Note that $A(z_2) = 1$. Thus, Eq. (67) implies that

$$\begin{aligned} |z_2(1 - (A(z))^{1/F})| &= \left| z_2 \int_z^{z_2} \frac{d}{dv}(A(v))^{1/F}\, dv \right| \\ &\leq \int_z^{z_2} \left| \frac{d}{dv}(A(v))^{1/F} \right| |dv| \\ &< \int_z^{z_2} |dv| \\ &= |z_2 - z|, \end{aligned} \tag{68}$$

where the integration is carried out along the straight line that connects z and z_2. Hence, we choose a circle $D = \{z : |z| = 1 - \epsilon\}$ for small ϵ and define

$$\begin{aligned} f(z) &= z - z_2 \\ g(z) &= z_2(1 - (A(z))^{1/F}). \end{aligned}$$

Eq. (68) implies that $|f(z)| > |g(z)|$ for each point on D. It follows from Rouché's theorem that $f(z)$ and $f(z) + g(z)$ has the same number of roots inside D. Since $f(z)$ has no root in D, it follows that (66) has no root in D. Hence, (66) has no root on or within $|z| = 1$, except for $z = z_2$. From (68), we can see that $z = z_2$ is a simple root.

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F	Eq. (60)	Eq. (64)	Eq.(52)	exact	simulation (N=128)	simulation (N=500)
1	0.601	0.615	0.500	0.586	0.589	0.587
3	0.770	0.754	0.667	0.736	0.740	0.737
5	0.822	0.803	0.730	0.790	0.793	0.790
7	0.849	0.831	0.766	0.820	0.823	0.820
10	0.874	0.856	0.800	0.848	0.850	0.848
30	0.927	0.914	0.879	NA	0.912	0.910
60	0.948	0.939	0.913	NA	0.938	0.936
90	0.958	0.950	0.928	NA	0.950	0.948

TABLE I

COMPARISON OF BOUND AND APPROXIMATIONS WITH SIMULATION RESULTS FOR $N \times N$ FRAME BASED MAILBOX SWITCHES.