Stochastic Convexity for Multidimensional Processes and Its Applications

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Abstract

Consider a multidimensional stochastic process, which is a function of a parametric process. The parametric process may be multidimensional as well. In this paper we compare two such processes that differ only in their parametric processes. We extend known stochastic convexity results for one-dimensional stochastic processes, which are recently obtained by Shaked and Shanthikumar, to multidimensional processes. These results are used to obtain comparison results for various queueing systems that are subject to different parametric processes, which may be the arrival processes, service processes, etc. Based on these comparison results we show how the performances of queueing systems can be affected by the variability of parametric processes.

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I. Introduction

Consider a multidimensional stochastic process whose distribution is a function of a parametric process. The parametric process may be multidimensional as well. In this paper we are interested in comparing the distribution of two such stochastic processes that differ only in their parametric processes. For example, in a queueing system the stochastic process my be the vector of the queue lengths at various stations while the parametric process may be the arrival processes or the service processes.

The comparisons of the distributions of two such processes are based on a form of stochastic dominance usually referred to in the literature as variability ordering or stochastic convex ordering: A nonnegative r.v. $X^{(1)}$ is less variable than a nonnegative r.v. $X^{(2)}$ if $Ef(X^{(1)}) \leq Ef(X^{(2)})$ for all f increasing convex for which the expectation exits. We write $X^{(1)} \leq_{ic} X^{(2)}$ (Stoyan (1983), Ross (1983)). For example, in queueing systems the performance measures to be compared may be the queue length at time t, the number of customers lost by time t, etc.

For many models we do need to analyze and compare underlying stochastic processes which are multidimensional as componentwise comparisons do not provide the structural results necessary. In order to compare such stochastic processes we use certain forms of convexity ordering which are extensions of the stochastic convexity concepts introduced by Shaked and Shanthikumar (1988a, 1988b, 1988c) for one dimensional stochastic processes; we make their convexity ordering applicable to multidimensional stochastic processes by introducing a so-called partial sum ordering on the various component processes. We say that a n dimensional vector is less than another n dimensional vector in the partial sum sense if the summation of the first k components of the first vector is less than or equal to the summation of the first k components of the second vector for all $1 \leq k \leq n$.

In order to compare two stochastic processes with different parametric processes we use so-called segmented processes. The way in which segmented processes are defined enables us to construct two stochastic processes which satisfy our multidimensional convex ordering, using sequence of pairs of random variables which are variably ordered. By approximating stochastic processes with segmented processes we obtain comparison results for arbitrary processes.

Comparisons of multidimensional stochastic processes have numerous applications, especially in queueing theory. In this paper we present three such applications in detail, namely:

(i) A tandem system with finite intermediate buffers subject to batch arrivals. In this model we are interested in how the number of lost customers and queue sizes are affected by the batch sizes, the service times and the dependencies between the batch sizes.

(ii) A single station loss system with batch service. In this model we are interested in how the number of lost customers and the queue size are affected by the batch sizes, the interarrival times, the services times, etc.

(iii) A finite buffer queue with state dependent service rate. In this model the service rate is a special function of an arbitrary parameter and the number of customers present in the system. We are interested in how the blocking probability changes as a function of the parameter, which may be the number of servers at the station, the service rate of each server, etc.

In the discussion we give a description of a number of other models for which we have obtained similar results.

There exists an extensive literature on the monotonicity and convexity of stochastic processes. On monotonicity, see Daley (1968), Kamae, Krengel and O'Brien (1977), Stoyan and Daley (1983), Whitt (1981), Ross (1983) and their references. For the convexity of stochastic processes, see Shaked and Shanthikumar [1888a], [1988b], [1988c], Shanthikumar and Yao (1987), (1988), and Chang, Chao and Pinedo (1988a), (1988b). In the queueing literature, models with batch arrivals, batch services, as well as tandem queues with and without blocking have been considered very often. For surveys of results in these area, see Chaudry and Templeton (1985), Perros and Altiok (1988), Stoyan and Daley (1983), Rolski (1976) and their references.

This paper is organized as follows: in Section II we extend Shaked and Shanthikumar's results for one dimensional stochastic processes to multi-dimensional processes. In Section III we consider the three queueing models described above and show monotonicity and convexity properties for underlying stochastic processes such as the queue length processes, the number of customers lost by time t, etc. These results are then used to obtain stochastic convexity results for the various models. In Section IV we discuss some of the other models for which we have obtained similar results.

II. Convexity Notions for Multidimensional Processes

In this section, we extend the notion of stochastic convexity proposed by Shaked and Shanthikumar (1988a) for one dimensional stochastic processes to multidimensional processes. In the following, "increasing" ("decreasing") means "nondecreasing" ("nonincreasing"). When we write the expectation $Ef(X)$, it is tacitly assumed that the expectation exists.

Consider a family of random vectors $\underline{X}(\underline{\theta})=(X_1(\underline{\theta}),\ X_2(\underline{\theta}),\ \cdots)$ whose distributions are parameterized by vector $\underline{\theta}$, where $\underline{\theta}=(\theta_1,\ \theta_2,\ \cdots)$.

Definition 2.1. $\underline{X}(\underline{\theta})$ is *Stochastically Increasing* (*Decreasing*) for each *Component* in *Sample Path* sense if for every choice of $\underline{\theta}^{(i)}$, i=1,2, with $\underline{\theta}^{(1)} \le \underline{\theta}^{(2)}$, there exist two random vectors $\underline{\hat{X}}^{(i)}$, i=1,2, defined on a common probability space such that

$$\underline{\hat{X}}^{(i)} \stackrel{\text{st}}{=} \underline{X}(\underline{\theta}^{(i)}), \quad i = 1, 2,$$

$$\underline{\hat{X}}^{(1)} \leq (\geq)\ \underline{\hat{X}}^{(2)}, \text{ a.s.}$$

We write $\underline{X}(\underline{\theta}) \in$ SI-CP(sp) (SD-CP(sp)).

Definition 2.2. (Partial Sum Ordering) Consider two vectors $\underline{x}^{(i)}=(x_1^{(i)},\ \cdots,\ x_n^{(i)})$. If $\sum_{l=1}^{j} x_j^{(1)} \leq \sum_{l=1}^{j} x_j^{(2)}$ for all $j = 1, \cdots, n$, it is said that $\underline{x}^{(1)}$ is smaller than $\underline{x}^{(2)}$ for each partial sum. We write $\underline{x}^{(1)} \leq_{ps} \underline{x}^{(2)}$.

Notice that although partial sum ordering and majorization ordering look similar at first glance, they are quite different. In fact, neither one can imply the other. For example, consider two three-dimensional vectors (1,2,3) and (3,2,1). Obviously these two vectors are equal in the majorization ordering sense, but the first vector is strictly less than the second one in partial sum ordering sense.

Definition 2.3. $\underline{X}(\underline{\theta})$ is *Stochastically Increasing* (*Decreasing*) for each *Partial Sum* in *Sample Path* sense if for every choice of $\underline{\theta}^{(i)}$, i=1,2, with $\underline{\theta}^{(1)} \leq \underline{\theta}^{(2)}$, there exist two random vectors $\underline{\hat{X}}^{(i)}$, i=1,2, defined on a common probability space such that

$$\underline{\hat{X}}^{(i)} \stackrel{\text{st}}{=} \underline{X}(\underline{\theta}^{(i)}), \quad i = 1, 2,$$

$$\underline{\hat{X}}^{(1)} \leq_{ps} (\geq_{ps})\ \underline{\hat{X}}^{(2)}, \text{ a.s.}$$

We write $\underline{X}(\underline{\theta}) \in$ SI-PS(sp) (SD-PS(sp)).

Remark 2.4. Clearly, SI-CP(sp) (SD-CP(sp)) implies SI-PS(sp) (SD-PS(sp)). There exist families of stochastic processes which are SD-PS(sp) but not SD-CP(sp). For example, the family of queue length processes $\underline{X}_t(c_1, c_2) = (X_{1,t}(c_1, c_2),\ X_{2,t}(c_1, c_2))$ of the M/M/C→/M/C system, where $X_{1,t}(c_1, c_2)$ ($X_{2,t}(c_1, c_2)$) denotes the number of customers in the first (second) queue and c_1 (c_2) denotes the number of servers in the first (second) queue, is SD-PS(sp) in (c_1, c_2), but not SD-CP(sp) in (c_1, c_2).

Definition 2.5. A function f:$(x_1, \cdots, x_n) \to \mathcal{R}$ is L-superadditive (L-subadditive) if for any $0 \le i < j \le n$ and any nonnegative y_1, y_2 the following inequality holds:

$$f(\cdots, x_i + y_1, \cdots, x_j + y_2, \cdots) + f(\cdots, x_i, \cdots, x_j, \cdots)$$
$$\geq (\leq) f(\cdots, x_i + y_1, \cdots, x_j, \cdots) + f(\cdots, x_i, \cdots, x_j + y_2, \cdots).$$

According to Marshall and Olkin (1979) (Chapter 6), the condition above is equivalent to $\frac{\partial^2 f}{\partial x_i \partial x_j} \geq (\leq) 0$ for all i, j $(i \neq j)$. Clearly, f is L-superadditive iff $-f$ is L-subadditive.

Definition 2.6. A function f:$(x_1, \cdots, x_n) \to \mathcal{R}$ is directionally convex (concave) if f is L-superadditive (L-subadditive) and convex (concave) in each variable.

Remark 2.7. It is shown in Shaked and Shanthikumar (1988c) that a function f is directionally convex (concave) if and only if for every combination of four vectors $\underline{x}^{(i)}$=($x_1^{(i)}$, $\cdots$, $x_n^{(i)}$), $\quad i = 1, 2, 3, 4$, with

$$\underline{x}^{(i)} \leq \underline{x}^{(4)}, \quad i = 1, 2, 3,$$
$$\underline{x}^{(2)} + \underline{x}^{(3)} = \underline{x}^{(1)} + \underline{x}^{(4)},$$

the inequality

$$f(\underline{x}^{(2)}) + f(\underline{x}^{(3)}) \leq (\geq) \ f(\underline{x}^{(1)}) + f(\underline{x}^{(4)})$$

holds.

Definition 2.8. $\underline{X}(\underline{\theta})$ is *Stochastically Increasing* and *Directional Convex* (*Concave*) for each *Partial Sum* in *Sample Path* sense on $\underline{\theta}$ if for every choice of $\underline{\theta}^{(i)}$, $\quad i = 1, 2, 3, 4$, with

(c1) $\underline{\theta}^{(i)} \leq \underline{\theta}^{(4)}, \quad i = 1, 2, 3,$

(c2) $\underline{\theta}^{(2)} + \underline{\theta}^{(3)} = \underline{\theta}^{(1)} + \underline{\theta}^{(4)},$

there exist four random vectors $\underline{\hat{X}}^{(i)}$, i=1, 2, 3, 4, defined on a common probability space such that

(c3) $\underline{\hat{X}}^{(i)} \stackrel{\text{st}}{=} \underline{X}(\underline{\theta}^{(i)}), \quad i = 1, 2, 3, 4,$

(c4) $\hat{\underline{X}}^{(i)} \leq \hat{\underline{X}}^{(4)}$, a.s. $\quad i = 1, 2, 3,$

(c5) $\hat{\underline{X}}^{(2)} + \hat{\underline{X}}^{(3)} \leq_{ps} (\geq_{ps})\ \hat{\underline{X}}^{(1)} + \hat{\underline{X}}^{(4)}$ a.s.

We write $\underline{X}(\underline{\theta}) \in$ SI-DCX-PS(sp). It should be understood that it is the distribution of $\underline{X}(\underline{\theta})$ that is SI-DCX-PS(sp). Similarly, we define SD-DCX-PS(sp) and SD-DCV-PS(sp) with (c4) replaced by $\hat{\underline{X}}^{(i)} \leq \hat{\underline{X}}^{(1)}$, i=1, 2, 3. If $\underline{\theta}$=θ is a scalar, we write $\underline{X}(\theta) \in$ SICX-PS(sp) (SICV-PS(sp), SDCX-PS(sp), SDCV-PS(sp)). If $\underline{X}$ is a random variable, we write $X(\underline{\theta}) \in$ SI-DCX(sp) (SI-DCV(sp), SD-DCX(sp), SD-DCV(sp)). Moreover, if $\underline{X}$ is a random variable and $\underline{\theta} = \theta$ is a scalar, we write $X(\theta) \in$ SICX(sp) (SICV(sp), SDCX(sp), SDCV(sp)).

Let $\underline{X}_t(\underline{\theta})$=$(X_{1,t}(\underline{\theta}), X_{2,t}(\underline{\theta}), \cdots)$ be a family of multidimensional stochastic processes whose distributions are parameterized by vector $\underline{\theta}$.

Definition 2.9. $\underline{X}_t(\underline{\theta})$ is said to be SI-DCX-PS(sp) (SI-DCV-PS(sp)) on $\underline{\theta}$ for all t, if $\underline{X}_t(\underline{\theta}^{(i)})$, i=1,2,3,4, satisfy (c3), (c4) and (c5) for all t>0 for any choice of $\underline{\theta}^{(i)}$, i =1,2,3,4, which satisfy (c1) and (c2) and any set of initial conditions $\underline{X}_0(\underline{\theta})$ that is SI-DCX-PS(sp) (SI-DCV-PS-(sp)). Similarly, we define SD-DCX-PS(sp) and SD-DCV-PS(sp) with (c4) replaced by $\hat{\underline{X}}^{(i)} \leq \hat{\underline{X}}^{(1)}$, i=1, 2, 3.

Example 2.10. It is shown in Chang, Chao and Pinedo (1988b) that the family of two-dimensional queue length processes $\underline{X}_t(\lambda) = (X_{1,t}(\lambda), X_{2,t}(\lambda))$ of the M/M/C→M/C system is SICX-PS(sp) on λ for all t, where $X_{1,t}(\lambda)$ $(X_{2,t}(\lambda))$ denotes the number of customers in the first (second) queue and λ is the Poisson arrival rate.

Remark 2.11. If $\underline{X}_t(\underline{\theta})$ is SI-DCX-PS(sp) (SI-DCV-PS(sp)) on $\underline{\theta}$ for all t, then $Ef(\sum_{l=1}^{j} X_{l,t}(\underline{\theta}))$, j=1, 2, $\cdots$, is increasing and directionally convex (concave) in $\underline{\theta}$ for all f increasing convex (concave).

Two properties of increasing and directionally convex (concave) functions used in this paper are the following:

Proposition 2.12. (Preservation property of variability ordering for independent r.v.'s) Let $\underline{X}^{(i)} = (X_1^{(i)}, X_2^{(i)}, \cdots)$, i=1, 2, be two sequences of independent r.v.'s. If $X_n^{(1)} \leq_{ic} (\leq_{iv}) X_n^{(2)}$, $n \geq 1$, then $\mathrm{E}f(X_1^{(1)}, X_2^{(1)}, \cdots) \leq \mathrm{E}f(X_1^{(2)}, X_2^{(2)}, \cdots)$ for all f increasing convex (concave) in each variable.

Proof: See Ross (1983), Proposition 8.5.4. (Although Ross seems to (implicitly) assume joint convexity of f, his proof is valid under the assumption of convexity in each variable).

Proposition 2.13. (Lorentz's inequality) Let $\underline{X}^{(i)}=(X_1^{(i)}, \cdots, X_{n_i}^{(i)})$, $i = 1, \cdots, m$, be m independent random vectors. If the r.v.'s $X_j^{(i)}$, $j = 1, \cdots, n_i$, have a common distribution $F^{(i)}(x) = \mathrm{P}(\ X_j^{(i)} < x)$, then for any L-superadditive function f,

$$\begin{aligned} &Ef(X_1^{(1)}, \cdots, X_{n_1}^{(1)}, X_1^{(2)}, \cdots, X_{n_2}^{(2)}, \cdots, X_1^{(m)}, \cdots, X_{n_m}^{(m)}) \\ &\leq Ef(X_1^{(1)}, \cdots, X_1^{(1)}, X_1^{(2)}, \cdots, X_1^{(2)}, \cdots, X_1^{(m)}, \cdots, X_1^{(m)}). \end{aligned}$$

Proof: See Tchen (1980) Theorem 5A and Rolski (1986) Lemma 5.

Let $\underline{s}=(s_0, s_1, s_2, \cdots)$ be an increasing sequence with s_0=0.

Definition 2.14. (Segmented Processes) A process $\underline{X}_t(\underline{s}, \underline{\theta})$ with $\underline{X}_0(\underline{s}, \theta) \overset{\mathrm{st}}{=} \underline{X}_0$ is called a segmented process of $\underline{Z}_t(\theta)$ with segmentation times $\underline{s}$ and segment parameters $\underline{\theta}$, if

$\underline{X}_t(\underline{s}, \underline{\theta}) \overset{\mathrm{st}}{=} \underline{Z}_{t-s_{n-1}}(\theta_n)$ for $S_{n-1} < t < S_n$ and $\underline{Z}_0(\theta_n) \overset{\mathrm{st}}{=} \underline{X}_{s_{n-1}}(\underline{s}, \underline{\theta})$.

Lemma 2.15. If $\underline{X}_t(\underline{s}, \underline{\theta})$ is a segmented process of $\underline{Z}_t(\theta)$ and $\underline{Z}_t(\theta)$ is SICX-PS(sp), (SICV-PS(sp), SDCX-PS(sp), SDCV-PS(sp)) for all t in θ, then $\underline{X}_t(\underline{s}, \underline{\theta})$ is SI-DCX-PS(sp) (SI-DCV-PS(sp), SD-DCX-PS(sp), SD-DCV-PS(sp)) on $\underline{\theta}$ for all t.

Proof: See Chang, Chao and Pinedo (1988b).

For more general comparisons between stochastic systems, we introduce the following definition due to Shanthikumar (1988).

Definition 2.16. Let $\underline{X}$ and $\underline{Y}$ be two random vectors on R^∞. Then $\underline{X}$ is said to be larger than $\underline{Y}$ in the increasing directionally convex ordering (increasing directionally concave ordering, decreasing directionally convex ordering, decreasing directionally concave ordering) if

$$Ef(X(n_1), X(n_2), \cdots, X(n_k)) \quad \geq \quad Ef(Y(n_1), Y(n_2), \cdots, Y(n_k))$$

for all functions f increasing directionally convex (increasing directionally concave, decreasing directionally convex, decreasing directionally concave) and any $n_1 < n_2 < \cdots < n_k$, then we write $\underline{X} \quad \geq_{idcx} (\geq_{idcv}, \geq_{ddcx}, \geq_{ddcv}) \quad \underline{Y}$.

Lemmas 2.17 and 2.18 provide two methods to obtain the idcx ordering.

Lemma 2.17. (Variability Condition) If $\underline{\Theta}^{(i)}$, $i = 1, 2$, are two sequences of *independent* r.v.'s with $\Theta_n^{(1)} \leq_{ic} \Theta_n^{(2)}$ for n=1, 2, $\cdots$, then $\underline{\Theta}^{(1)} \leq_{idcx} \underline{\Theta}^{(2)}$.

Proof: A direct application of Proposition 2.12.

Lemma 2.18. (Subsequence Condition) Consider two sequences of random variables $\underline{\Theta}^{(i)}$, $i = 1, 2$. $\underline{\Theta}^{(1)}$ consists of a sequence of independent random vectors

$$\{(\Theta_1^{(1)}, \ldots, \Theta_{n_1}^{(1)}), (\Theta_1^{(2)}, \ldots, \Theta_{n_2}^{(2)}), \ldots\}$$

with the r.v.'s $\Theta_j^{(l)}$, $j = 1, \ldots, n_l$, having a common distribution for $l = 1, 2, \ldots$. $\underline{\Theta}^{(2)}$ consists of a sequence of independent random vectors

$$\{(\Theta_1^{(1)}, \ldots, \Theta_1^{(1)}), (\Theta_1^{(2)}, \ldots, \Theta_1^{(2)}), \ldots\}.$$

Then $\underline{\Theta}^{(1)} \leq_{idcx} \underline{\Theta}^{(2)}$.

Proof: A direct application of Proposition 2.13.

Lemma 2.19. (idcx Ordering on $\underline{\Theta}$) If

(v1) $\underline{\Theta}^{(1)} \leq_{idcx} \underline{\Theta}^{(2)}$ ($\underline{\Theta}^{(1)} \leq_{idcv} \underline{\Theta}^{(2)}$), (v2) $\underline{S}^{(1)} \overset{\mathrm{st}}{=} \underline{S}^{(2)}$,

(v3) the two processes $\underline{X}_t(\underline{S}^{(i)}, \underline{\Theta}^{(i)})$ with identical initial conditions $\underline{X}_0(\underline{S}^{(1)}, \underline{\Theta}^{(1)}) = \underline{X}_0(\underline{S}^{(2)}, \underline{\Theta}^{(2)})$ are segmented processes of $\underline{Z}_t(\theta)$, where $\underline{Z}_t(\theta)$ is SICX-PS(sp) (SICV-PS(sp)) on θ for all t,

then

(v4) $\sum_{l=1}^{j} X_{l,t}(\underline{S}^{(1)}, \underline{\Theta}^{(1)}) \leq_{ic} (\leq_{iv}) \sum_{l=1}^{j} X_{l,t}(\underline{S}^{(2)}, \underline{\Theta}^{(2)}), \quad j \geq 1$ for all t.

Proof: Applying Lemma 2.15 and Definition 2.16 completes the proof.

Analogous to Definition 2.8. we define a stochastic process $\underline{X}_t$ to be SICX-PS(sp) (SICV-PS(sp)) in t as follows:

Definition 2.20. (Temporal Stochastic Convexity) A stochastic process $\underline{X}_t$ is SICX-PS(sp) (SICV-PS(sp)) in t, if for every choice of t_i, i=1, 2, 3, 4, with

(c1) $t_i \leq t_4$, i=1, 2, 3.

(c2) $t_2 + t_3 = t_1 + t_4$

there exist four random vectors $\underline{\hat{X}}^{(i)}$ on a common probability space such that

(c3) $\underline{\hat{X}}^{(i)} \quad \overset{\mathrm{st}}{=} \quad \underline{X}_{t_i}, \quad i = 1, 2, 3, 4.$

(c4) $\underline{\hat{X}}^{(i)} \quad \leq \quad \underline{\hat{X}}^{(4)} \quad a.s., i = 1, 2, 3.$

(c5) $\underline{\hat{X}}^{(2)} + \underline{\hat{X}}^{(3)} \quad \leq_{ps} (\geq_{ps}) \quad \underline{\hat{X}}^{(1)} + \underline{\hat{X}}^{(4)}$ a.s.

We can also define SDCX-PS(sp) (SDCV-PS(sp)) in t with (c4) replaced by $\underline{\hat{X}}^{(i)} \leq \underline{\hat{X}}^{(1)} a.s., \quad i = 2, 3, 4.$

Lemma 2.21. (Variability Condition on $\underline{S}$) If

(t1) $\underline{\Theta}^{(1)} \overset{\mathrm{st}}{=} \underline{\Theta}^{(2)}$

(t2) $\{T_n^{(i)} \overset{\mathrm{def}}{=} S_n^{(i)} - S_{n-1}^{(i)}, n \geq 1\}$ are two sequences of independent r.v.'s with $T_n^{(1)} \leq_{ic} T_n^{(2)}$ $(-T_n^{(1)} \leq_{ic} - T_n^{(2)})$.

(t3) the two processes $\underline{X}_t(\underline{S}^{(i)}, \underline{\theta}^{(i)})$ with identical initial conditions $\underline{X}_0(\underline{S}^{(1)}, \underline{\theta}^{(1)}) = \underline{X}_0(\underline{S}^{(2)}, \underline{\theta}^{(2)})$ are segmented processes of $\underline{Z}_t(\theta)$, where $\underline{Z}_t(\theta)$ is SICX-PS(sp) (SDCX-PS(sp)) in t for all θ and either SICX-PS(sp) or SDCX-PS(sp) on θ for all t,

then

$$\{\sum_{l=1}^{j} X_{l,S_n^{(1)}}(\underline{S}^{(1)}, \underline{\Theta}^{(1)})\} \quad \leq_{ic} \quad \{\sum_{l=1}^{j} X_{l,S_n^{(2)}}(\underline{S}^{(2)}, \underline{\Theta}^{(2)})\}, \quad n \geq 1$$

Proof: Since the proofs of the two cases are similar, we only show the case for SICX-PS(sp). Because $\underline{\Theta}^{(i)}$ are independent of $\underline{S}^{(i)}$, we may condition on $\underline{\Theta}^{(i)} = \underline{\theta}$, i=1,2. Consider four processes $\underline{X}_t^{(i)}$ with $T_n^{(i)} = T_n$ for $n \neq k$ and $T_k^{(i)} = t_i$ with t_i satisfying (c1) and (c2). Observe that $\underline{X}_t^{(i)}$ are stochastically identical before S_{k-1}. Since $\underline{Z}_t(\theta)$ is SICX-PS(sp) in t for all θ, we can construct $\underline{X}^{(i)}_{S_k^{(i)}} \stackrel{\text{st}}{=} \underline{Z}_{t_i}(\theta_k)$ such that

$$\underline{X}^{(2)}_{S_k^{(2)}} + \underline{X}^{(3)}_{S_k^{(3)}} \quad \leq_{ps} \quad \underline{X}^{(1)}_{S_k^{(1)}} + \underline{X}^{(4)}_{S_k^{(4)}} \quad a.s.$$

and

$$\underline{X}^{(i)}_{S_k^{(i)}} \quad \leq \quad \underline{X}^{(4)}_{S_k^{(4)}} \quad a.s., \quad i = 1, 2, 3.$$

From the condition that $\underline{Z}_t(\theta)$ is either SICX-PS(sp) or SDCX-PS(sp) on θ for all t, we can construct $\underline{X}^{(i)}_{S_n^{(i)}} \stackrel{\text{st}}{=} \underline{Z}_{T_n}(\theta_n)$ such that

$$\underline{X}^{(2)}_{S_n^{(2)}} + \underline{X}^{(3)}_{S_n^{(3)}} \quad \leq_{ps} \quad \underline{X}^{(1)}_{S_n^{(1)}} + \underline{X}^{(4)}_{S_n^{(4)}} \quad a.s.$$

and

$$\underline{X}^{(i)}_{S_n^{(i)}} \quad \leq \quad \underline{X}^{(4)}_{S_n^{(4)}}, \quad a.s. \quad i = 1, 2, 3. \quad \text{and} \quad n > k$$

From the inequalities above, it follows that $E[f(\sum_{l=1}^{j} X_{l,S_n}(\underline{S}, \underline{\theta}) \mid T_k]$ is increasing convex in T_k for any f increasing convex and $n \geq 1$. Since $T_n^{(i)}$ are independent r.v.'s, the result is obtained by Proposition 2.12.

Remark 2.22. The result does not hold for arbitrary t. Observe that the comparisons are made only at the segmentation times. We will show in the next section that these segmentation times could be the arrival epochs of the customers in some queueing systems.

III. Multidimensional Stochastic Convex Processes

In this section, we consider the three queueing systems described in Section I and show that several underlying multidimensional stochastic processes are stochastically convex in the partial sum sense. Using these properties we then obtain the variability ordering results for these models.

The first model we consider is the tandem system with batch arrivals, i.e., $G^{[X]}/M(n)/1/N \rightarrow /M(n)/1/N \rightarrow \cdots \rightarrow /M(n)/1/N$. In this p-stage tandem system, batches of customers arrive according to an arbitrary arrival process $A(t)$. Let $\{\tau_n, \quad n \geq 1,\}$ be the arrival epoch of the n-th batch and let $X_n^{(i)}$ be the number of customers in this batch. Assume that these two sequences are independent stochastic processes and are also independent of the system. The service rate at the j-th stage is $\mu_j(k)$ when k customers are present at that stage. Assume that the buffer size at the j-th stage is N_j (N_j may be infinite). Customers arriving at the first stage are lost if there are N_1 customers present at the first station (some of the customers in a batch may be lost while others may be accepted). If a service completion occurs while the buffer downstream is full, communication blocking will occur, i.e., it will restart its service at that station. Let B_t denote the number of customers lost by time t and let $Q_{j,t}$ be the number of customers at the j-th queue at time t. Let $\underline{Q}_t = (B_t, Q_{1,t}, \cdots, Q_{p,t})$ and let $\underline{X} = (X_1, X_2, \cdots)$ be the batch sizes process.

Lemma 3.1. If $\mu_j(k)$ is increasing concave in k and $sup_{j,k}\ \mu_j(k) = \eta < \infty$, then $\underline{Q}_t(\underline{X}) \in$ SI-DCX-PS(sp) on $\underline{X}$ for all t.

Proof: Consider the stochastic processes $\underline{Q}_t^{(i)}(\underline{X}^{(i)})$ of four tandem queues with batch sizes $\underline{X}^{(i)}$, i=1, 2, 3, 4, and arrival epochs of batches $\{\tau_n, n \geq 1\}$. Let $\tau_0 = 0$ and $\underline{\tau} = (\tau_0, \tau_1, \tau_2, \cdots)$. Then $\underline{Q}_t^{(i)}(\underline{\tau}, \underline{X}^{(i)})$ can be modeled as a segmented process with segmentation times $\underline{\tau}$ and segment parameter $\underline{X}^{(i)}$. According to Lemma 2.15, it suffices to show that the process between two successive arrival epochs is SICX-PS(sp) on the batch size for all t.

Let $\underline{Z}_t(\underline{x}^{(i)}) = (B_t(\underline{x}^{(i)}), Q_{1,t}(\underline{x}^{(i)}), \cdots, Q_{p,t}(\underline{x}^{(i)}))$ be the process between two arrival epochs. According to Definition 2.9, we have to show that we can construct four processes $\hat{\underline{Z}}_t(x^{(i)})$ that satisfy (c3), (c4) and (c5) for all t given that $x^{(i)}, \quad i = 1, 2, 3, 4,$ satisfy (c1), (c2) and that $\hat{\underline{Z}}_0(x^{(i)})$ satisfies (c3), (c4) and (c5).

Observe that a batch of size $x^{(i)}$ arrives at the ith queueing system immediately after time 0 (at 0+), i.e.,

(a1) $\hat{B}_{0+}^{(i)} = \hat{B}_0^{(i)} + (\hat{Q}_{1,0}^{(i)} + x^{(i)} - N_1)^+$

(a2) $\hat{Q}_{1,0+}^{(i)} = min(\hat{Q}_{1,0}^{(i)} + x^{(i)}, N_1)$

(a3) $\hat{Q}_{l,0+}^{(i)} = \hat{Q}_{l,0}^{(i)} \quad l = 2, 3, \cdots, p.$

Observe that

$$\hat{B}_{0+}^{(i)} + \hat{Q}_{1,0+}^{(i)} = \hat{B}_0^{(i)} + \hat{Q}_{1,0}^{(i)} + x^{(i)}.$$

Conditions (c4) and (c5) at time 0+ can be verified easily by the induction hypothesis except

(a4) $\hat{B}_{0+}^{(2)} + \hat{B}_{0+}^{(3)} \leq \hat{B}_{0+}^{(1)} + \hat{B}_{0+}^{(4)}$.

Since

$$\hat{B}_0^{(2)} + \hat{B}_0^{(3)} \leq \hat{B}_0^{(1)} + \hat{B}_0^{(4)},$$

we can assume that

$$\hat{B}_0^{(2)} + \hat{B}_0^{(3)} + k = \hat{B}_0^{(1)} + \hat{B}_0^{(4)}$$

for some $k \geq 0$. From the induction hypothesis,

$$\hat{B}_0^{(2)} + \hat{Q}_{1,0}^{(2)} + \hat{B}_0^{(3)} + \hat{Q}_{1,0}^{(3)} \leq \hat{B}_0^{(1)} + \hat{Q}_{1,0}^{(1)} + \hat{B}_0^{(4)} + \hat{Q}_{1,0}^{(4)},$$

it follows that

$$k + \hat{Q}^{(1)}_{1,0} + \hat{Q}^{(4)}_{1,0} \geq \hat{Q}^{(2)}_{1,0} + \hat{Q}^{(3)}_{1,0}.$$

There exists an integer k_1 ($k_1 \leq k$) such that

$$k_1 + \hat{Q}^{(1)}_{1,0} + \hat{Q}^{(4)}_{1,0} = \hat{Q}^{(2)}_{1,0} + \hat{Q}^{(3)}_{1,0}.$$

If $k_1 < 0$, then $\hat{Q}^{(1)}_{1,0} + \hat{Q}^{(4)}_{1,0} > \hat{Q}^{(2)}_{1,0} + \hat{Q}^{(3)}_{1,0}$ and inequality (a4) follows easily from the induction hypothesis and (a1). In case $k_1 \geq 0$ consider the virtual queue length $\tilde{Q}^{(1)}_{1,0}$ where $\tilde{Q}^{(1)}_{1,0} = \hat{Q}^{(1)}_{1,0} + k_1$. Clearly, $\tilde{Q}^{(1)}_{1,0} \leq \hat{Q}^{(2)}_{1,0}, \quad \hat{Q}^{(3)}_{1,0} \leq \hat{Q}^{(4)}_{1,0}$ and

$$\tilde{Q}^{(1)}_{1,0} + \hat{Q}^{(4)}_{1,0} = \hat{Q}^{(2)}_{1,0} + \hat{Q}^{(3)}_{1,0}.$$

Define

$$\Delta\tilde{\beta}^{(1)} \stackrel{\text{def}}{=} (\tilde{Q}^{(1)}_{1,0} + x^{(1)} - N)^+,$$

and

$$\Delta\beta^{(i)} \stackrel{\text{def}}{=} (\hat{Q}^{(i)}_{1,0} + x^{(i)} - N)^+ \quad i = 1, 2, 3, 4.$$

Notice that $(\tilde{Q}^{(1)}_{1,0} + x^{(1)} - N)^+$ is directionally convex in $(\tilde{Q}^{(1)}_{1,0}, x^{(1)})$, it then follows from Remark 2.7 and the induction hypothesis that

$$\Delta\tilde{\beta}^{(1)} + \Delta\beta^{(4)} \geq \Delta\beta^{(2)} + \Delta\beta^{(3)}.$$

Since $\Delta\tilde{\beta}^{(1)} \leq \Delta\beta^{(1)} + k_1 \leq \Delta\beta^{(1)} + k$, we have

$$k + \Delta\beta^{(1)} + \Delta\beta^{(4)} \geq \Delta\beta^{(2)} + \Delta\beta^{(3)}.$$

Thus

$$\begin{aligned}\hat{B}^{(1)}_{0+} + \hat{B}^{(4)}_{0+} &= \hat{B}^{(2)}_{0} + \hat{B}^{(3)}_{0} + k + \Delta\beta^{(1)} + \Delta\beta^{(4)} \\ &\geq \hat{B}^{(2)}_{0+} + \hat{B}^{(3)}_{0+}\end{aligned}$$

and we conclude that $\hat{Z}_t(x^{(i)})$ satisfies (c3), (c4) and (c5) after time 0. By an argument that is similar to the proof of Lemma 3.3 in Chang, Chao and Pinedo (1988b), it can be shown that $\hat{Z}_t(x^{(i)})$ satisfies (c3), (c4) and (c5) for all t.

Remark 3.2. If $N_j = \infty$, j=1,2,$\cdots$, then the queueing system is a tandem queue with infinite buffers. Clearly $B_t = 0$ and $\underline{Q}_t = (Q_{1,t}, Q_{2,t}, \cdots, Q_{p,t})$.

Using Lemmas 3.1 and 2.19, we obtain the following result.

Theorem 3.3. Suppose that the batch sizes $\underline{X}^{(i)}$, i=1,2, satisfy $\underline{X}^{(1)} \leq_{idcx} \underline{X}^{(2)}$, then the following inequalities hold:

(i) $B_t^{(1)} \leq_{ic} B_t^{(2)}$, if $N_1 < \infty$,

(ii) $\sum_{l=1}^{p} Q_{l,t}^{(1)} \leq_{ic} \sum_{l=1}^{p} Q_{l,t}^{(2)}$, if $N_1 = \infty$.

In particular, if the batch sizes are independent and $X_n^{(1)} \leq_{ic} X_n^{(2)}$, then it follows from Lemma 2.20 that $\underline{X}^{(1)} \leq_{idcx} \underline{X}^{(2)}$ and the results hold.

Remark 3.4.

(i) If the arrival process is renewal and the batch sizes are i.i.d. r.v.'s with $EX_n^{(1)} = EX_n^{(2)}$, $E(X_n^{(2)})^2 < \infty$, then the blocking probabilities exist and $P_B^{(1)} \leq P_B^{(2)}$.

(ii) If $N_1 = \infty$, then

$$\sum_{l=1}^{p} Q_{l,t}^{(1)} \leq_{ic} \sum_{l=1}^{p} Q_{l,t}^{(2)}$$

and

$$EX_n^{(1)} = EX_n^{(2)}$$

imply that $EW^{(1)} \leq EW^{(2)}$ by Little's formula, where $W^{(1)}(W^{(2)})$ denotes the total sojourn time in the first (second) system. This extends Chang, Chao and Pinedo's (1988c) result for $M^{[X]}/M/C$ systems.

(iii) If $EX_n^{(1)} = EX_n^{(2)}$, then $EC_t^{(1)} \geq EC_t^{(2)}$, where $C_t^{(i)}$ denotes the number of customers that have completed their services by time t.

The second system we consider is a single stage finite capacity queue with batch services, i.e., G/$M^{[Y]}$/1/N. At this system customers arrive according to an arbitrary point process A(t). Let $\tau'_n, \quad n \geq 1$, be the arrival epoch of the n-th customer. Assume that the arrival process is independent of the service process, and let μ be the rate at which a batch of customers is served. Let $Y_n, n \geq 1$, be the maximum number of customers that are allowed to be served in the n-th batch and suppose that the sequence $\{Y_n, n \geq 1\}$ is also independent of the system. An arrival is lost if there are already N customers in the system. Let B_t be the number of customers lost by time t and let Q_t be the number of customers in the system at time t. Denote $\underline{Q}_t$ =(B_t, Q_t) and $\underline{Y} = (Y_1, Y_2, \cdots)$.

Lemma 3.5. $\underline{Q}_t(\underline{Y}) \in$ SD-DCX-PS(sp) on the batch sizes, $\underline{Y}$, for all t.

Proof: We use standard coupling and uniformization techniques (see Keilson (1979), Ross (1983) and Shaked and Shanthikumar (1988a), (1988b), (1988c)). Construct a Poisson process with rate μ (the service rate of a batch) and let $\{\tau''_n, n \geq 1\}$ be its arrival epochs. Let $\{\tau_n, n \geq 1\} = \{\tau'_n, n \geq 1\} \bigcup \{\tau''_n, n \geq 1\}$.

According to Definition 2.9, we assume that given four numbers $y^{(1)}, y^{(2)}, y^{(3)}$ and $y^{(4)}$ which satisfy (c1) and (c2), and given $\underline{Q}_0(y^{(i)}) = (B_0(y^{(i)}), Q_0(y^{(i)}))$ that is SI-DCX-PS(sp), there exist four random vectors $\hat{\underline{Q}}_0^{(i)}$ on a common probability space such that

(c3) $\hat{\underline{Q}}_0^{(i)} \stackrel{\text{st}}{=} \underline{Q}_0(y^{(i)}) \quad i$=1, 2, 3, 4.

(c4) $\hat{\underline{Q}}_0^{(i)} \leq \hat{\underline{Q}}_0^{(1)}$ a.s. i= 2, 3, 4.

(c5) $\hat{\underline{Q}}_0^{(2)} + \hat{\underline{Q}}_0^{(3)} \quad \leq_{ps} \quad \hat{\underline{Q}}_0^{(1)} + \hat{\underline{Q}}_0^{(4)} \qquad a.s.$

We want to show that we can construct $\hat{\underline{Q}}_t^{(i)}$ on a common probability space such that (c3), (c4) and (c5) are satisfied for all t. Clearly, the case of arrivals can be verified by a similar argument to the proof of arrivals in Lemma 3.1. Therefore, we need only to consider departures at service completion times. Observe that the number of customers blocked by time t, B_t, does not change and the queue length is governed by the recursive equation at service completion epochs

$$\hat{Q}^{(i)}_{\tau_n+} = (\hat{Q}^{(i)}_{\tau_n} - y^{(i)})^+.$$

It is easy to see that (c4) can be verified easily. If at time τ_n,

$$\hat{Q}^{(1)}_{\tau_n} + \hat{Q}^{(4)}_{\tau_n} \geq \hat{Q}^{(2)}_{\tau_n} + \hat{Q}^{(3)}_{\tau_n},$$

then

$$\hat{Q}^{(1)}_{\tau_n+} + \hat{Q}^{(4)}_{\tau_n+} \geq \hat{Q}^{(2)}_{\tau_n+} + \hat{Q}^{(3)}_{\tau_n+}$$

due to the fact that $(\underline{\hat{Q}}^{(i)}_{\tau_n} - y^{(i)})^+$ is directionally convex in $(\underline{\hat{Q}}^{(i)}_{\tau_n}, -y^{(i)})$ (Remark 2.7). From the induction hypothesis (c5), $\hat{B}^{(1)}_{\tau_n} + \hat{B}^{(4)}_{\tau_n} \geq \hat{B}^{(2)}_{\tau_n} + \hat{B}^{(3)}_{\tau_n}$, it follows that

$$\hat{B}^{(1)}_{\tau_n+} + \hat{Q}^{(1)}_{\tau_n+} + \hat{B}^{(4)}_{\tau_n+} + \hat{Q}^{(4)}_{\tau_n+} \geq \hat{B}^{(2)}_{\tau_n+} + \hat{Q}^{(2)}_{\tau_n+} + \hat{B}^{(3)}_{\tau_n+} + \hat{Q}^{(3)}_{\tau_n+}.$$

If

$$\hat{Q}^{(1)}_{\tau_n} + \hat{Q}^{(4)}_{\tau_n} < \hat{Q}^{(2)}_{\tau_n} + \hat{Q}^{(3)}_{\tau_n},$$

we may rewrite the recursive equation as follows:

$$\hat{Q}^{(i)}_{\tau_n+} = \hat{Q}^{(i)}_{\tau_n} - min(\hat{Q}^{(i)}_{\tau_n}, y^{(i)}) \stackrel{\text{def}}{=} \hat{Q}^{(i)}_{\tau_n} - \beta^{(i)}_n.$$

Observe that $\beta^{(i)}_n$ is directionally concave in $(Q^{(i)}_{\tau_n}, y^{(i)})$. It follows from Remark 2.7 that $\beta^{(1)}_n + \beta^{(4)}_n \leq \beta^{(2)}_n + \beta^{(3)}_n$.

Then

$$\begin{aligned}
&\hat{B}^{(1)}_{\tau_n+} + \hat{Q}^{(1)}_{\tau_n+} + \hat{B}^{(4)}_{\tau_n+} + \hat{Q}^{(4)}_{\tau_n+} \\
&= \hat{B}^{(1)}_{\tau_n} + \hat{Q}^{(1)}_{\tau_n} + \hat{B}^{(4)}_{\tau_n} + \hat{Q}^{(4)}_{\tau_n} - \beta^{(1)}_n - \beta^{(4)}_n \\
&\geq \hat{B}^{(2)}_{\tau_n} + \hat{Q}^{(2)}_{\tau_n} + \hat{B}^{(3)}_{\tau_n} + \hat{Q}^{(3)}_{\tau_n} - \beta^{(2)}_n - \beta^{(3)}_n \\
&= \hat{B}^{(2)}_{\tau_n+} + \hat{Q}^{(2)}_{\tau_n+} + \hat{B}^{(3)}_{\tau_n+} + \hat{Q}^{(3)}_{\tau_n+}.
\end{aligned}$$

Applying Lemmas 3.5 and Lemmas 2.19, we have the following theorem:

Theorem 3.6. If the batch sizes $\underline{Y}^{(i)} = \{Y_n^{(i)} n \geq 1\}$ are i.i.d. r.v.'s, i=1, 2, $\underline{Y}^{(1)}$ and $\underline{Y}^{(2)}$ are independent and $-{Y_1}^{(1)} \leq_{ic} -{Y_1}^{(2)}$, then $B_t^{(1)} \quad \leq_{ic} \quad B_t^{(2)}$ and $-C_t^{(1)} \leq_{ic} -C_t^{(2)}$, where $C_t^{(i)}$ is the number of customers that have finished service by time t.

The third system we consider is a finite capacity queue with state dependent service rate G/M(n)/1/N. Consider a family of these systems at which customers arrive according to an arbitrary point process A(t). Let τ_n', $n \geq 1$ be the arrival epoch of the n-th customer and assume that the arrival process is independent of the system. Let $\mu(j,k)$ be the service rate of the j-th system when k customers are present. Again an arrival is lost if there are N customers present in the system. Denote by B_t the number of customers lost by time t and Q_t the number of customers present in system at time t. Let $\underline{Q}_t = (B_t, Q_t)$.

Lemma 3.7. If $\mu(j,k)$ is increasing and directionally concave and $sup_{j,k}\ \mu(j,k) = \eta < \infty$, then $\underline{Q}_t(j) \in$ SDCX-PS(sp) on j for all t.

Proof: Again, we use the standard coupling and uniformization technique. Generate a Poisson process with rate 2η and let $\{\tau_n'', n \geq 1\}$ be its arrival epochs. Let $\{\tau_n, n \geq 1\} = \{\tau_n', n \geq 1\} \bigcup \{\tau_n'', n \geq 1\}$. Generate a sequence of i.i.d. r.v's $\{U_n, n \geq 1\}$ uniformly distributed over [0,1]. Construct the uniformized Markov Chain $\underline{Q}_t^{(i)}$, i=1,2,3,4. as follows:

Case (i): Arrivals.

$$\hat{B}^{(i)}_{\tau_n+} = \hat{B}^{(i)}_{\tau_n} + \mathbf{1}\{\hat{Q}^{(i)}_{\tau_n} = N\}$$

$$\hat{Q}^{(i)}_{\tau_n+} = \hat{Q}^{(i)}_{\tau_n} + \mathbf{1}\{\hat{Q}^{(i)}_{\tau_n} < N\}$$

Conditions (c4) and (c5) can be verified by an argument similar to the proof of Lemma 3.1 with batch sizes equal to 1.

Case (ii): Departures.

$$\hat{Q}^{(i)}_{\tau_n+} = \hat{Q}^{(i)}_{\tau_n} - \beta^{(i)}_n \quad i = 1, 2, 3, 4,$$

$$\hat{B}^{(i)}_{\tau_n+} = \hat{B}^{(i)}_{\tau_n},$$

where if $\hat{Q}^{(1)}_{\tau_n} + \hat{Q}^{(4)}_{\tau_n} \leq \hat{Q}^{(2)}_{\tau_n} + \hat{Q}^{(3)}_{\tau_n}$, then $\beta^{(i)}_n$ is given by

$$\begin{aligned}
\beta^{(1)}_n &= \mathbf{1}\{\frac{\eta}{2\eta} \leq U_n < \frac{\eta + \mu(j_1, \hat{Q}^{(1)}_{\tau_n})}{2\eta}\} \\
\beta^{(2)}_n &= \mathbf{1}\{\frac{\eta + \mu(j_1, \hat{Q}^{(1)}_{\tau_n}) - \mu(j_2, \hat{Q}^{(2)}_{\tau_n})}{2\eta} \leq U_n < \frac{\eta + \mu(j_1, \hat{Q}^{(1)}_{\tau_n})}{2\eta}\} \\
\beta^{(3)}_n &= \mathbf{1}\{\frac{\eta}{2\eta} \leq U_n < \frac{\eta + \mu(j_3, \hat{Q}^{(3)}_{\tau_n})}{2\eta}\} \\
\beta^{(4)}_n &= \mathbf{1}\{\frac{\eta + \mu(j_3, \hat{Q}^{(3)}_{\tau_n}) - \mu(j_4, \hat{Q}^{(4)}_{\tau_n})}{2\eta} \leq U_n < \frac{\eta + \mu(j_3, \hat{Q}^{(3)}_{\tau_n})}{2\eta}\}
\end{aligned}$$

and if $\hat{Q}^{(1)}_{\tau_n} + \hat{Q}^{(4)}_{\tau_n} > \hat{Q}^{(2)}_{\tau_n} + \hat{Q}^{(3)}_{\tau_n}$, then $\beta^{(i)}_n$ is given by

$$\beta^{(i)}_n = \mathbf{1}\{\frac{\eta}{2\eta} \leq U_n < \frac{\eta + \mu(j_i, \hat{Q}^{(i)}_{\tau_n})}{2\eta}\} \quad i = 1, 2, 3, 4$$

where j_1, j_2, j_3 and j_4 satisfy (c1) and (c2).

If $\beta^{(2)}_n \geq \beta^{(1)}_n$, then $\hat{Q}^{(2)}_{\tau_n+} \leq \hat{Q}^{(1)}_{\tau_n+}$. If $\beta^{(2)}_n < \beta^{(1)}_n$, then $\mu(j_2, \hat{Q}^{(2)}_{\tau_n}) < \mu(j_1, \hat{Q}^{(1)}_{\tau_n})$. Since $j_1 < j_2$ and $\mu(j, k)$ is increasing, we conclude $\hat{Q}^{(1)}_{\tau_n} > \hat{Q}^{(2)}_{\tau_n}$, so we still have $\hat{Q}^{(2)}_{\tau_n+} \leq \hat{Q}^{(1)}_{\tau_n+}$. By a similar argument $\hat{Q}^{(3)}_{\tau_n+} \leq \hat{Q}^{(1)}_{\tau_n+}$ and $\hat{Q}^{(4)}_{\tau_n+} \leq \hat{Q}^{(1)}_{\tau_n+}$ if $\hat{Q}^{(1)}_{\tau_n} + \hat{Q}^{(4)}_{\tau_n} > \hat{Q}^{(2)}_{\tau_n} + \hat{Q}^{(3)}_{\tau_n}$. If $\hat{Q}^{(1)}_{\tau_n} + \hat{Q}^{(4)}_{\tau_n} \leq \hat{Q}^{(2)}_{\tau_n} + \hat{Q}^{(3)}_{\tau_n}$ then $\hat{Q}^{(4)}_{\tau_n} \leq \hat{Q}^{(3)}_{\tau_n}$. Applying the same argument, we have

$$\hat{Q}^{(4)}_{\tau_n+} \leq \hat{Q}^{(3)}_{\tau_n+}, \hat{Q}^{(3)}_{\tau_n+} \leq \hat{Q}^{(2)}_{\tau_n+}, \text{and} \quad \hat{Q}^{(2)}_{\tau_n+} \leq \hat{Q}^{(1)}_{\tau_n+}.$$

Therefore $(c4)$ is satisfied.

Clearly, if

$$\hat{Q}^{(1)}_{\tau_n} + \hat{Q}^{(4)}_{\tau_n} \geq \hat{Q}^{(2)}_{\tau_n} + \hat{Q}^{(3)}_{\tau_n} + 2,$$

we have

$$\hat{Q}^{(1)}_{\tau_n+} + \hat{Q}^{(4)}_{\tau_n+} \geq \hat{Q}^{(2)}_{\tau_n+} + \hat{Q}^{(3)}_{\tau_n+}$$

and therefore (c5) is satisfied. If $\hat{Q}^{(1)}_{\tau_n} + \hat{Q}^{(4)}_{\tau_n} = \hat{Q}^{(2)}_{\tau_n} + \hat{Q}^{(3)}_{\tau_n} + 1$ then $\hat{Q}^{(4)}_{\tau_n} - 1 \leq \hat{Q}^{(i)}_{\tau_n}, \quad i = 2, 3, 4.$ If $\hat{Q}^{(4)}_{\tau_n} = 0$, then $\beta^{(4)}_n$ =0. Thus

$$\beta^{(1)}_n + \beta^{(4)}_n \leq \beta^{(2)}_n + \beta^{(3)}_n + 1$$

and (c5) follows easily. If $\hat{Q}^{(4)}_{\tau_n} > 0$, then

$$\mu(j_1, \hat{Q}^{(1)}_{\tau_n}) + \mu(j_4, \hat{Q}^{(4)}_{\tau_n} - 1) \quad \leq \quad \mu(j_2, \hat{Q}^{(2)}_{\tau_n}) + \mu(j_3, \hat{Q}^{(3)}_{\tau_n})$$

due to the fact $\mu(j,k)$ is increasing and directionally concave. Therefore $\beta^{(1)}_n + \beta^{(4)}_n \leq \beta^{(2)}_n + \beta^{(3)}_n + 1$ and (c5) is satisfied.

If $\hat{Q}^{(1)}_{\tau_n} + \hat{Q}^{(4)}_{\tau_n} \leq \hat{Q}^{(2)}_{\tau_n} + \hat{Q}^{(3)}_{\tau_n}$ then

$$\mu(j_1, \hat{Q}^{(1)}_{\tau_n}) + \mu(j_4, \hat{Q}^{(4)}_{\tau_n}) \leq \mu(j_2, \hat{Q}^{(2)}_{\tau_n}) + \mu(j_3, \hat{Q}^{(3)}_{\tau_n})$$

by the same property of $\mu(j,k)$. Clearly, $\beta^{(1)}_n + \beta^{(4)}_n \leq \beta^{(2)}_n + \beta^{(3)}_n$ and (c5) is satisfied.

Lemma 3.8. In the G/M/K/N loss system, the blocking probability is decreasing convex in the number of servers K as well as in the service rate μ.

Proof: It follows from Lemma 3.4 and the fact that $\mu\ min(K,n)$ is an increasing directionally concave function of both (μ, n) and (K, n), given the other variable fixed (Shaked and Shanthikumar (1988c)).

Remark 3.9. When $K = N$ and the arrival process is Poisson, the Erlang formula is decreasing convex in the number of servers is proved by Messerli (1972), and when the arrival process is renewal, the result is proved by Yao (1986) using Laplace transform method.

Remark 3.10. The result in Lemma 3.4 can also be used to study the heterogeneous ordered-entry system with K servers in parallel with service rates μ_1, μ_2, $\cdots$, μ_K respectively (Yao (1986)). A customer will always select an available server according to the index order. When all servers are busy, an arriving customer join the queue if there is space in the buffer and is lost otherwise. We also assume that when a customer, say the one at the k-th server, finishes service, the customer being served on the $(k+1)$-st server will move to the k-th server, the customer being served on the $(k+2)$-nd server will move to the $(k+1)$-st server, $\cdots$, etc, hence the total service rate equals $\mu_1 + \mu_2 + \cdots + \mu_n$ if the number of customers in the system $n \le K$ and $\mu_1 + \mu_2 + \cdots + \mu_K$ if $n > K$. We can show that this function satisfies the condition in Lemma 4 and therefore, the blocking probability is decreasing convex in the number of servers in the heterogeneous ordered-entry system.

In the following lemma, we show the temporal convexity result of G/M(n)/1/N queue.

Lemma 3.11. If $\mu(k)$ is increasing concave and $sup_k(k) = \eta < \infty$ then $\underline{Q}_t \in$ SDCX-PS(sp) in t where $t \in [\tau'_n, \tau'_{n+1})$ $(\tau'_0 = 0)$.

Proof: Consider the stochastic process between arrival epochs τ'_n and τ'_{n+1}. At time 0, there is an arrival and the state changes as follows:

$$\hat{B}_{0+} = \hat{B}_0 + \mathbf{1}\{\hat{Q}_0 = N\}$$

and

$$\hat{Q}_{0+} = \hat{Q}_0 + \mathbf{1}\{\hat{Q}_0 < N\}.$$

Hence at time 0+ it can be verified by a similar argument to the proof of arrivals in Lemma 3.1. Observe that $\hat{B}_t$ does not change after time 0+ in $t \in [\tau'_0+, \tau'_1)$, so that $\hat{Q}_t$ is a pure death process in this interval. According to Shanthikumar and Yao (1988) Theorem 3.18, we can construct four random variables $\underline{\hat{Q}}_{t_i}$, $i = 1, 2, 3, 4$, such that

$$\underline{\hat{Q}}_{t_i} \stackrel{\text{st}}{=} \underline{\hat{Q}}_{t_i}, \quad i = 1, 2, 3, 4,$$
$$\underline{\hat{Q}}_{t_i} \le \underline{\hat{Q}}_{t_4}$$

and

$$\underline{\hat{Q}}_{t_2} + \underline{\hat{Q}}_{t_4} \le_{ps} \underline{\hat{Q}}_{t_1} + \underline{\hat{Q}}_{t_4}.$$

with t_i, $\quad i = 1, 2, 3, 4$, satisfying (c1) and (c2). Hence conditions (c3), (c4) and (c5) in Definition 2.20 are easily verified.

Consider GI/M/C/N queues with interarrival times $T_n^{(i)}, \quad i = 1, 2.$

Theorem 3.12. If $-T_n^{(1)} \leq_{ic} -T_n^{(2)}$, then $P_B^{(1)} \leq P_B^{(2)}$, where $P_B^{(i)}, \quad i = 1, 2.$ represent the blocking probabilities.

Proof: Applying Lemmas 2.19, 2.21 (Variability Condition on S) and Lemma 3.11, assuming that the parameter Θ is a constant, proves the theorem.

IV. Discussion

In this discussion we give a brief overview of the basic ideas used in this paper. Then we present additional examples.

The two basic ideas used in this paper are the following:

In our comparisons of multidimensional stochastic processes we need to compare partial sums of the random vectors as componentwise comparisons often do not yield any results. It should be noted that partial sum ordering is somewhat similar but not equivalent to majorization ordering. In fact neither one can implies the other. The partial sum ordering has proven also to be very useful in the analysis of stochastic scheduling problems (Chang, Chao and Pinedo (1989)).

Furthermore, in our comparisons of systems with different parametric processes we use the so-called segmented processes which turns out to very useful in our comparisons. Segmented processes give us a very useful tool to compare stochastic processes since any stochastic processes, if smooth enough, can be approximated by a sequence of segmented processes and this enables us to first obtain convexity result of some underlying stochastic processes with respect to some parameters instead of processes. For example, it was shown in Chang, Chao and Pinedo (1988b) that for tandem systems with doubly stochastic processes, a similar upper bound can be obtained as that of Rolski (1986) for DSP/G/1 queues. Shanthikumar (1988) developed a theory on regularity of stochastic processes in which the concept of segmented processes fits quite well. He presented a mathematically elegant definition to measure the regularity among stochastic processes according to which different segmented processes can be ordered easily.

In Section III we present comparison results for the three queueing systems in which the Variability Condition (Lemma 2.17) plays a key role. In the remaining part of this section we present the other models where we use the theory developed in this paper (Subsequence Condition Lemma 2.18)) to obtain comparisons between the relevant stochastic processes. The coupling technique for two stochastic processes to be constructed in such a way that they satisfy the Subsequence Condition can be obtained similar to that of Chang, Chao and Pinedo (1988a) for discrete time (Examples 4.1 and 4.3) and that of Chang, Chao and Pinedo (1988b) for continuous time (Example 4.2).

Example 4.1. (Markovian Routing Problem)

Consider two queueing systems subject to an arbitrary arrival process A(t). An arrival is routed to the first queue with probability p_1 (respectively $1 - p_2$) if the previous one is routed to the first (respectively second) queue. Clearly, if $p_1 = p_2$, the routing scheme is the so-called geometric routing. We can also consider this problem as point process thinning (see Bremaud (1983) Chapter 7); that is, each customer, upon arrival, will join the queue with a certain probability. We are interested in how correlation coefficient affects system performance when the average arrival rate is kept constant. In what follows, we provide some results for positively correlated routing.

Let $\underline{X} = (X_1, X_2, \cdots)$ be a sequence of r.v.'s. If $X_n = 1$ (0), then the n-th arrival is routed to the first (second) queue. According to the routing scheme described above, $\underline{X}$ is a Markov Chain with the following transition probabilities:

$$P(X_{n+1} = 1 | X_n = 1) = p_1$$

$$P(X_{n+1}=0|X_n=1)=1-p_1$$

$$P(X_{n+1}=0|X_n=0)=p_2$$

$$P(X_{n+1}=1|X_n=0)=1-p_2.$$

The stationary distribution of this Markov Chain is:

$$P(X_n=1)=\frac{1-p_2}{2-p_1-p_2}\stackrel{\text{def}}{=}\pi_1$$

$$P(X_n=0)=\frac{1-p_1}{2-p_1-p_2}=1-\pi_1.$$

Observe that π_1 is the percentage of arrivals which are routed to the first queue. Define the correlation coefficient

$$\omega\stackrel{\text{def}}{=}\frac{EX_nEX_{n+1}-EX_nEX_{n+1}}{\sqrt{var(X_n)var(X_{n+1})}}=p_1+p_2-1.$$

Obviously the Markovian routing mechanisms can be characterized by π_1 and ω.

If the two queues are the tandem systems described in Lemma 3.1, then the following result, which we state without proof, can be obtained.

Consider two Markovian routing mechanisms with parameters $\pi_1^{(i)}$ and $\omega^{(i)}$. Let $\underline{Q}_{t,r}^{(i)}=(B_{t,r}^{(i)},Q_{1,t,r}^{(i)},\cdots,Q_{p_r,t,r}^{(i)})$, r=1,2, where p_r represents the number of stages of the r-th tandem system and $Q_{l,t,r}^{(1)}(Q_{l,t,r}^{(2)})$ represents the number of customers at the l-th stage of the r-th tandem system at time t under the first (second) router. If $\pi_1^{(1)}=\pi_1^{(2)}$ and $0\le\omega^{(1)}\le\omega^{(2)}$, then

(i) $B_{t,r}^{(1)}\le_{ic}B_{t,r}^{(2)},\quad r=1,2$, where $B_{t,r}^{(1)}$ $(B_{t,r}^{(2)})$ represents the number of customers routed to r-th system under the first (second) routing mechanism by time t and which are blocked.

(ii) $EC_{t,r}^{(1)}\ge EC_{t,r}^{(1)}$, where $C_{t,r}^{(i)}$ is the number of customers that have completed their services from the r-th tandem system by time t

If, furthermore, $N_{1,r} = \infty, \quad r = 1, 2$, then $\sum_{l=1}^{p_r} Q^{(1)}_{l,t,r} \leq_{ic} \sum_{l=1}^{p_r} Q^{(2)}_{l,t,r}$

If $p_r = 1$ and $N_1 = \infty$, the comparison result for the Markovian routers with positive correlations holds for the virtual waiting time processes in queues.

Example 4.2. (Queues with Service Interruptions)

Consider a M/M/C/N queue with a Poisson arrival rate λ_D and exponential service times with mean $1/\mu_D$. A server functions for an amount of time which is exponentially distributed with mean $1/\lambda_V$. When a server breaks down, the job being served , if any, is returned to the queue and an independent service time with the same mean is generated for this job. The server is then sent for repair and the repair time is exponentially distributed with mean $1/\mu_V$. Define $\alpha = \frac{\mu_D}{\mu_V}$, $\varrho_V = \frac{\lambda_V}{\mu_V}$. The following result, again presented without proof, can be obtained: If λ_D, μ_D and ϱ_V are fixed, the blocking probability is increasing in α. If N=∞, the average sojourn time is increasing in α.

The problem of service interruptions was first studied by Mitrani and Avi-Itzhak (1968). In a telecommunications context, the problem is often referred to as a voice/data integration problem with a movable boundary scheme. Such models have been studied extensively using numerical approaches (see Kraimeche and Schwartz (1986) and their references).

In this model the number α regulates how fast the server changes between the two states, with the proportion of time that the machine is functioning being a constant. It can be shown that when α goes to 0, the machine becomes an perfect machine with that constant service rate (Grandell (1976)). The above corollary shows that Ross's conjecture, which states the monotonicity property for arrival processes (Chang, Chao and Pinedo (1988b)), holds for the service processes as well. In fact from Lemma 2.19 we can see that it holds for more general parametric processes.

Example 4.3. (Correlation Between Batch sizes)

We can also use our methods to study the effect of correlations between batch sizes. For example, consider an arrival mechanism with dependent batch sizes. Suppose we have two types of products, say products A and B. Both arrive at the same service facility in batches whose size have the same distribution. If a batch is from type A (B) at an arrival epoch, then the next batch will be either from type A (B) with probability p_A (p_B) with exactly the same batch size as the previous one , or from type B (A) with probability $1-p_A$ ($1-p_B$) with a batch size that is independent of the previous batch sizes. The batch arrival process is thus governed by a two state Markov chain. Keeping the proportion of batches from types A and B constant, that is, $\frac{1-p_A}{2-p_A-p_B}$ is a constant, we investigate how the correlation coefficient of the Markov Chain affects the performance of the system. We consider again the tandem system and assume that the correlation coefficient

$$\omega = \frac{cov(X_n, X_{n+1})}{\sqrt{varX_n varX_{n+1}}} = 1 - p_A - p_B \geq 0.$$

For this model we have the following result: if $0 \leq \omega_1 \leq \omega_2$, then

(i) $B_t(\omega_1) \quad \leq_{ic} \quad B_t(\omega_2)$

(ii) $EC_t(\omega_1) \quad \geq \quad EC_t(\omega_2)$, where $C_t(\omega_i)$ is the number of customers that have completed service by time t.

Similar result can be obtained regarding to the correlations between the batch sizes in the batch service models.

References:

[1] Brémaud, P.(1981) Point Processes and Queues. Springer-Verlag, New York.

[2] Chang,C.S., Chao, X. and Pinedo,M.(1988a) Intergration of Discrete-time Correlated Markov Processes in a TDM System: Structural Results, to appear in the Probability in Engineering and Information Sciences.

[3] Chang, C.S., Chao, X. and Pinedo, M.(1988b) Monotonicity Results for Queues with Doubly Stochastic Poisson Arrivals: Ross's Conjecture. Submitted for Publication.

[4] Chang, C.S., Chao, X. and Pinedo, M.(1988c) Some Inequalities for Queues with Batch Poisson Arrivals. Technical Report, Dept. of IE/OR, Columbia University, New York.

[5] Chang, C.S., Chao, X. and Pinedo, M.(1989) A note on $c\mu$ rule for parallel machines. Submitted for publication.

[6] Chaudhry, M.L. and Templeton, J.G.C. A First Course in Bulk Queues. John Wiley & Sons. New York.

[7] Daley, D.J.(1968)Stochastic Monotone Markov Processes. Z. Wahrsch. Verw Gebiete, 10,305-317.

[8] Kamae, T.Krengel, U. and O'Brien, G. (1977) Stochastic Inequalities on Partial Ordered Spaces. Annals of Prob. 5, 899-912.

[9] Keilson,T.(1979) Markov Chain Models-Rarity and Exponentiality, Springer-Verlag, New York.

[10] Kraimeche, B. and Schwarz, M.(1986) Bandwith Allocation Strategies in Wide-Band Integrated Networks, IEEE J. Select. Areas Commun., Vol. SAC-4,869-878.

[11] Marshall, A.W. and Olkin,I.(1979) Inequalities: Theory of Majorization and Its Applications, Academic Press, New York.

[12] Mitrani,I.L. and Avi-Itzhak,B.(1968) A Many Server Queue with Server Interruptions,Oper. Res., Vol.16,No.3,628-638.

[13] Messerli, E.J.(1972) Proof of Convexity of the Erlang B-formula, Bell Sys. Tech. J. 51, 951-953.

[14] Perros, H.G. and Altiok, T.(1988) First International Workshop on Queueing Networks with Blocking. Pre-Conference Proceedings, North Carolina State University, Raleigh, May 20-21.

[15] Rolski, T. (1976) Order Relations in the Set of Probability Distribution Functions and Their Applications in Queueing Theory. Dissertationes Mathematicae, Polska Akademia Nauk, Instytut Matematyczny.

[16] Rolski, T.(1986) Upper Bounds for Single Server Queues with Doubly Stochastic Poisson Arrivals. Math. of Oper. Res. Vol. 11, No. 3, 442-450.

[17] Ross, S.M.(1978) Average Delay in Queues with Non-stationary Poisson Arrivals. J. Appl. Prob. 15, 602-609.

[18] Ross, S.M.(1983) Stochastic Processes. J. Wiley &Sons. New York.

[19] Shaked, M and Shanthikumar, J.G.(1988a) Temporal Stochastic Convexity and Concavity, Stochastic Processes and its Applications, 27, 1-20.

[20] Shaked, M and Shanthikumar, J.G.(1988b) Stochastic Convexity and Its Applications, Adv. Appl. Prob. 20, 427-446.

[21] Shaked, M and Shanthikumar, J.G.(1988c) Parametric Stochastic Convexity and concavity of Stochastic Processes. School of Business Administration, University of California at Berkeley.

[22] Shanthikumar, J.G. (1988) Regularity of Stochastic Processes: A Theory Based on Stochastic Convexity. Research Report, School of Business Administration, University of California at Berkeley.

[23] Shanthikumar, J.G. and Yao, D.D. Spatiotemporal Convexity of Stochastic Processes and Applications. Research Report, School of Business Administration, University of California at Berkeley.

[24] Shanthikumar, J.G. and Yao, D.D. Strong Stochastic Convexity: Closure Properties and Applications. Research report, School of Business Administration, University of California at Berkeley.

[25] Tchen, A.H.(1980) Inequalities for Distributions with Given Marginals, Ann. Prob. Vol.8 No.4, 814-827.

[26] Yao, D.D.(1986) Convexity Properties of the Overflow in a Ordered-Entry System with Heterogeneous Servers, Operations Rearch Letters, Vol.5, No.3