Monotonicity Results for Queues with Doubly Stochastic Poisson Arrivals: Ross's Conjecture

Cheng-Shang Chang, XiuLi Chao** and Michael Pinedo****

July, 1990

ABSTRACT

In this paper, we compare queueing systems that differ only in their arrival processes, which are special forms of Doubly Stochastic Poisson (DSP) processes. We define a special form of stochastic dominance for DSP processes which is based on the well-known variability or convex ordering for random variables. For two DSP processes that satisfy our comparability condition in such a way that the first process is more "regular" than the second process, we show the following three results: (i) If the two systems are $DSP/GI/1$ queues, then $Ef(V^{(1)}) \leq Ef(V^{(2)})$ for all f increasing convex, with $V^{(i)}$, i=1 and 2, representing the work load (virtual waiting time) in system. (ii) If the two systems are $DSP/M(k)/1 \rightarrow /M(k)/1 \rightarrow \ldots \rightarrow /M(k)/1$ tandem systems, with $M(k)$ representing an exponential service time distribution with a rate that is increasing concave in the number of customers, k, present at the station, then $Ef(Q^{(1)}) \leq Ef(Q^{(2)})$ for all f increasing convex, with $Q^{(i)}$, i=1 and 2, being the total number of customers in the two systems. (iii) If the two systems are $DSP/M(k)/1/N$ systems, with N being the size of the buffer, then $P_B^{(1)} \leq P_B^{(2)}$, where $P_B^{(i)}$ denotes the blocking (loss) probability of the two systems. A model considered before by Ross (1978) satisfies our comparability condition; a conjecture stated by him is shown to be true.

ROSS'S CONJECTURE; STOCHASTIC CONVEXITY; DOUBLY STOCHASTIC POISSON PROCESS; VARIABILITY ORDERING

* *IBM Research Division, T.J. Watson Research Center, H2-K06, P.O. Box 704, Yorktown Heights, N.Y. 10598.*

** *Dept. of Industrial and Management Engineering, New Jersey Institute of Technology, Newark, N.J. 07102.*

*** *Dept. of Industrial Engineering and Operations Research, Columbia University, New York, N.Y. 10027.*

1. Introduction

In this paper we study three types of queueing systems that are subject to Doubly Stochastic Poisson (DSP) arrival processes. The three types of systems are (i) the $DSP/GI/1$ queue, (ii) the $DSP/M(k)/1 \to /M(k)/1 \to \ldots \to /M(k)/1$ tandem system, where the service times at a node are exponentially distributed with a rate that is a function of the number of customers, k, present at that node and (iii) the $DSP/M(k)/1/N$ system with finite buffer size N. The performance measures of interest are the expected time in system when the buffer is infinite (cases (i) and (ii)) and the blocking probability when the buffer is finite (case (iii)). Our results are presented in the form of comparison theorems in which two systems of the same type are being compared with one another; the two systems have identical service time distributions but are subject to different arrival processes.

The comparison results obtained are based on a form of stochastic dominance which is often referred to as variability ordering: a nonnegative random variable (r.v.) $X^{(1)}$ is less variable than a nonnegative r.v. $X^{(2)}$ if $Ef(X^{(1)}) \leq Ef(X^{(2)})$ for all increasing convex f for which the expectations exist. (see Stoyan(1983) or Ross(1983)). We write $X^{(1)} \leq_{ic} X^{(2)}$. If $\underline{X}^{(i)}$, i=1,2, are sequences of independent r.v.'s we write $\underline{X}^{(1)} \leq_{ic} \underline{X}^{(2)}$ if $X_n^{(1)} \leq_{ic} X_n^{(2)}$ for all $n \geq 1$. We use the notation $X^{(1)} \stackrel{\text{st}}{=} X^{(2)}$ ($\underline{X}^{(1)} \stackrel{\text{st}}{=} \underline{X}^{(2)}$) if $X^{(1)}$ and $X^{(2)}$ ($\underline{X}^{(1)}$ and $\underline{X}^{(2)}$) have the same (joint) distribution.

The intensity processes of the DSP processes are constructed as follows: Consider two sequences of r.v.'s $\underline{S}$ and $\underline{\Lambda}$, where $\underline{S}=(S_0, S_1, S_2, ...)$ is an increasing sequence with S_0=0 and $\underline{\Lambda} =(\Lambda_1, \Lambda_2, ...)$ is an arbitrary sequence of nonnegative r.v.'s. The sequences $\underline{S}$ and $\underline{\Lambda}$ are independent. A stochastic process λ_t (with λ_0 given) is constructed in such a way that $\lambda_t=\Lambda_n$ for $S_{n-1} < t \leq S_n$ (see Fig. 1). We call such a process segmentable with segmentation times $\underline{S}$ and segment variables $\underline{\Lambda}$. Some examples of segmentable processes are given in Section IV (see Examples 4.1., 4.3. and 4.4.). We refer to this class of segmentable processes as class $\mathcal{G}$.

*** Insert Fig. 1 here ***

FIGURE 1 A segmentable process

We say that two segmentable processes $\lambda_t^{(i)}$ satisfy

(i) the Variability Condition if

(v1) $\underline{\Lambda}^{(1)}$ and $\underline{\Lambda}^{(2)}$ are two sequences of *independent* r.v.'s with $\underline{\Lambda}^{(1)} \leq_{ic} \underline{\Lambda}^{(2)}$ and

(v2) $\underline{S}^{(1)} \stackrel{\text{st}}{=} \underline{S}^{(2)}$, and

(ii) the Subsequence Condition if

(s1) $\underline{S}^{(2)} \stackrel{\text{st}}{=} (S_{n_0}^{(1)}, S_{n_1}^{(1)}, S_{n_2}^{(1)}, \ldots)$, $0 = n_0 < n_1 < n_2 < \ldots$, is a subsequence of $\underline{S}^{(1)}$ and

(s2) $\underline{\Lambda}^{(2)}$ is a sequence of independent r.v.'s and the random vectors $(\Lambda_{n_{m-1}+1}^{(1)}, \ldots, \Lambda_{n_m}^{(1)})$, $m \geq 0$, are independent with $\Lambda_n^{(1)} \stackrel{\text{st}}{=} \Lambda_m^{(2)}$, $n_{m-1} + 1 \leq n \leq n_m$.

We say that two processes $\lambda_t^{(i)} \in \mathcal{G}$ satisfy the *Comparability Condition* if they satisfy either the Variability Condition or the Subsequence Condition.

Our main results are presented in three theorems. In the first theorem, we consider two $DSP/GI/1$ queues with arrival intensity processes $\lambda_t^{(i)} \in \mathcal{G}$ and with an identical service time distribution G. The virtual waiting time is denoted by $V_t^{(i)}$, i.e., $V_t^{(i)}$ equals the workload in system i at time t.

Theorem 1. If (i) $\lambda_t^{(1)}$ and $\lambda_t^{(2)}$ satisfy the Comparability Condition and (ii) $V_0^{(1)} \leq_{ic} V_0^{(2)}$, then $V_t^{(1)} \leq_{ic} V_t^{(2)}$ for all t$\geq$0.

Our second result concerns two p-stage tandem systems $DSP/M(k)/1 \rightarrow /M(k)/1 \rightarrow \ldots \rightarrow /M(k)/1$ with identical exponential service rates $\mu_j(k)$ at stage j when k customers are present, but subject to different DSP arrival processes $\lambda_t^{(i)} \in \mathcal{G}$. We consider the total number of customers in the two systems at time t, i.e., $Q_t^{(i)}$, i=1,2.

Theorem 2. If (i) $\lambda_t^{(1)}$ and $\lambda_t^{(2)}$ satisfy the Comparability Condition, (ii) both systems start at time zero with the same number of customers at each stage, (iii) $\sup_{j,k} \mu_j(k)=\eta < \infty$, (iv) $\mu_j(k)$ is increasing and concave in k for all j, then $Q_t^{(1)} \leq_{ic} Q_t^{(2)}$ for all t.

Clearly, a queue with C servers in parallel and each server operating according to an exponential distribution with rate μ satisfies (iii) and (iv). Therefore, Theorem 2 holds for $DSP/M/C \rightarrow /M/C \rightarrow \ldots \rightarrow /M/C$ tandem systems.

In our third theorem, we consider two $DSP/M(k)/1/N$ loss systems with identical service rate $\mu(k)$ and buffer size N. We consider the number of customers blocked (lost) by time t, i.e., $B_t^{(i)}$.

Theorem 3. If (i) $\lambda_t^{(1)}$ and $\lambda_t^{(2)}$ satisfy the Comparability Condition, (ii) both systems start out at time zero with the same number of customers in the system, (iii) $\mu(k)$ is increasing and concave in k, (iv) $\sup_k \mu(k) =\eta < \infty$, then $B_t^{(1)} \leq_{ic} B_t^{(2)}$ for all t.

Many comparison theorems for queues with DSP arrival processes have been presented in the literature. Ross (1978) considered a family of $DSP/GI/1$ queues with the intensity function of the arrival process, λ_t, driven by a two state Markov process which alternates between states 1 and 2 , with respective rates $\bar{\lambda}_1$ and $\bar{\lambda}_2$. He assumed that the time the Markov process spends in state r is an exponential r.v. with rate $c\alpha_r$, r=1 and 2, and showed that the expected waiting time is a decreasing function of c when $\bar{\lambda}_2 = 0$. He conjectured that a similar monotonicity result holds for $\bar{\lambda}_2 > 0$. Moreover, he conjectured that the expected delay in an $M/GI/1$ queue serves as a lower bound for the expected delays in all $DSP/GI/1$ queues since the arrival process converges to a Poisson process when $c \rightarrow \infty$. He also conjectured that similar results should hold for the blocking (loss) probability in case the queueing system has a finite buffer. Based on Palm probabilities and Jensen's inequality, Rolski (1981)(1983) proved the lower bounds for the expected delays in $DSP/GI/1$ queues and the blocking probabilities in $DSP/M/C$ loss systems for C=1 and 2. Using Lorentz's inequality, which we adapt in this paper, Rolski (1986) proved an upper bound for the expected delay in a $DSP/GI/1$ queue which corresponds to the case $c \to 0$. Fond and Ross (1978) justified the conjecture for blocking probabilities in single server loss systems with exponential service times. Niu (1980) extends their work to general service times with the restriction $\bar{\lambda}_2 = 0$. Svoronos and Green (1987)(1988) considered $DSP/M/1$ loss systems with intensities $\lambda_t = \lambda + cf(t)$, where $f(t)$ is a bounded periodic function of which the integral over the period is 0 and c is a constant. They showed that the blocking probability is increasing convex in c. Chang and Pinedo (1990) showed that the blocking probabilities in $G/M/1$ loss systems only depend on the one-dimensional distribution of interarrival times. They obtained monotonicity results for the blocking probabilities in $DSP/M/1$ loss systems under time scaling ($\lambda_t^{(c)} = \lambda_{\frac{t}{c}}$) as well as amplitude scaling ($\lambda_t^{(c)} = \lambda + cZ_t$, $EZ_t = 0$). A discrete counterpart of Ross's conjecture was considered by Chang, Chao and Pinedo (1990), which suggested an approximation technique for analyzing voice/data integration using the monotonicity results and bounds. For details on comparison theorems for queues with nonstationary arrivals, we refer to Rolski's (1989) survey paper.

The paper is organized as follows: In Section II, we extend the one-dimensional notion of *Stochastically Increasing* and *Convex* (SICX) processes (Shaked and Shanthikumar (1988a,b,c)) to multidimensional processes. Various results are then obtained for segmented processes. In Section III, we prove that the following three processes are stochastically increasing and convex in the Poisson arrival rate λ: (i) the virtual waiting time process in an $M/GI/1$ queue, (ii) the p-dimensional queue length process in a p-stage tandem queue subject to Poisson arrivals and with exponential service rate $\mu_j(k)$, where $\mu_j(k)$ is increasing and concave in k, and (iii) the process of the number of customers blocked (lost) by time t in a $M/M(k)/1/N$ queue with finite buffer size N and exponential service rate that is again increasing and concave in the number of customers present. Using these results, we prove the three theorems. In Section IV, we present three examples of segmentable processes. Moreover, we show that the model presented by Ross, with a DSP arrival process whose intensity is a two-state Markov process, is a special case of one of our models. Our result therefore proves his conjecture.

2. Preliminary Results

In this section, we start out with the notions of stochastic convexity proposed by Shaked and Shanthikumar and extend these to multidimensional processes. Using these notions in conjunction with Lorentz's inequality, we obtain conditions that are useful for our comparisons.

Definition 2.1. A function $f(x_1, \ldots, x_n)$ is L-superadditive if for any $1 \leq i < j \leq n$ and any nonnegative y_1, y_2 the following inequality holds:

$$f(\ldots, x_i + y_1, \ldots, x_j + y_2, \ldots) + f(\ldots, x_i, \ldots, x_j, \ldots)$$
$$\geq f(\ldots, x_i + y_1, \ldots, x_j, \ldots) + f(\ldots, x_i, \ldots, x_j + y_2, \ldots).$$

According to Marshall and Olkin (1979) (Chapter 6), the condition above is equivalent to $\frac{\partial^2 f}{\partial x_i \partial x_j} \geq 0$ for all i, j $(i \neq j)$ if f has second partial derivatives.

Definition 2.2. A function $f(x_1, \ldots, x_n)$ is directionally convex if f is L-superadditive and convex in each variable.

Remark 2.3. It is shown in Shaked and Shanthikumar (1988c) that a function f is directionally convex if and only if for every combination of four vectors $\underline{x}^{(i)}=(x_1^{(i)}, \ldots, x_n^{(i)})$, i=1,2,3,4, with

$\underline{x}^{(i)} \leq \underline{x}^{(4)}$, i=1,2,3,

$\underline{x}^{(2)} + \underline{x}^{(3)} = \underline{x}^{(1)} + \underline{x}^{(4)}$,

the inequality

$f(\underline{x}^{(2)}) + f(\underline{x}^{(3)}) \leq f(\underline{x}^{(1)}) + f(\underline{x}^{(4)})$

holds.

Definition 2.4. (Partial Sum Ordering) Consider two vectors $\underline{x}^{(i)}=(x_1^{(i)},\ldots,x_n^{(i)})$. If $\sum_{l=1}^{j} x_l^{(1)} \le \sum_{l=1}^{j} x_l^{(2)}$ for all $j=1,\ldots,n$, it is said that $\underline{x}^{(1)}$ is smaller than $\underline{x}^{(2)}$ for each partial sum. We write $\underline{x}^{(1)} \le_{ps} \underline{x}^{(2)}$.

Note that the partial sum ordering is weaker than the weak majorization ordering which requires $\underline{x}^{(i)}$ to be descending (Marshall and Olkin (1979)). Let $\underline{\lambda} =(\lambda_1,\ \lambda_2,\ \ldots)$. Consider a family of random vectors $\underline{X}(\underline{\lambda})=(X_1(\underline{\lambda}),\ X_2(\underline{\lambda}),\ \ldots)$ whose distributions are parameterized by vector $\underline{\lambda}$. Throughout this paper, we are only interested in the *distribution* of $\underline{X}(\underline{\lambda})$. As remarked in Shaked and Shanthikumar (1988c), we do not assume any dependence (or independence) properties among the $\underline{X}(\underline{\lambda})$'s. Analogous to their definition of directional convexity for families of r.v.'s, we now define directional convexity for random vectors.

Definition 2.5. $\underline{X}(\underline{\lambda})$ is *Stochastically Increasing* and *Directionally Convex* for each *Partial Sum* in *Sample Path* sense on $\underline{\lambda}$ if for every choice of $\underline{\lambda}^{(i)}$, i=1,2,3,4, with

(c1) $\underline{\lambda}^{(i)} \le \underline{\lambda}^{(4)}$, i=1,2,3,

(c2) $\underline{\lambda}^{(2)} + \underline{\lambda}^{(3)} = \underline{\lambda}^{(1)} + \underline{\lambda}^{(4)}$,

there exist four random vectors $\underline{\hat{X}}^{(i)}$, i=1,2,3,4, defined on a common probability space such that

(c3) $\underline{\hat{X}}^{(i)} \overset{\text{st}}{=} \underline{X}(\underline{\lambda}^{(i)})$, i=1,2,3,4,

(c4) $\underline{\hat{X}}^{(i)} \le \underline{\hat{X}}^{(4)}$ a.s., i=1,2,3,

(c5) $\underline{\hat{X}}^{(2)} + \underline{\hat{X}}^{(3)} \le_{ps} \underline{\hat{X}}^{(1)} + \underline{\hat{X}}^{(4)}$ a.s.

We write $\underline{X}(\underline{\lambda}) \in$ SI-DCX-PS(sp). It should be understood that it is the *distribution* of $\underline{X}(\underline{\lambda})$ that is SI-DCX-PS(sp). If $\underline{\lambda}$=λ is a scalar, we write $\underline{X}(\lambda) \in$ SICX-PS(sp). If $\underline{X}$ is a random variable, we write $X(\underline{\lambda}) \in$ SI-DCX(sp). Moreover, if $\underline{\lambda} = \lambda$ is a scalar and $\underline{X}$ is a random variable, we write $X(\lambda) \in$ SICX(sp).

Let $\underline{X}_t(\underline{\lambda})=(X_{1,t}(\underline{\lambda}),\ X_{2,t}(\underline{\lambda}),\ \ldots)$ be a family of multidimensional stochastic processes parameterized by vector $\underline{\lambda}$.

Definition 2.6. $\underline{X}_t(\underline{\lambda})$ is said to be SI-DCX-PS(sp) on $\underline{\lambda}$ for all t, if there exist $\hat{\underline{X}}_t(\underline{\lambda}^{(i)})$, i=1,2,3,4, satisfying (c3), (c4) and (c5) for all $t > 0$ for any choice of $\underline{\lambda}^{(i)}$, i=1,2,3,4, which satisfy (c1) and (c2) and any set of initial conditions $\underline{X}_0(\underline{\lambda})$ that is SI-DCX-PS(sp).

Remark 2.7. If $\underline{X}_t(\underline{\lambda})$ is SI-DCX-PS(sp) on $\underline{\lambda}$ for all t, then $Ef(\sum_{l=1}^{j} X_{l,t}(\underline{\lambda}))$, j=1,2,. . ., is increasing and directionally convex in $\underline{\lambda}$ for all f increasing convex. Note that (c4) can be replaced by the weaker condition, $\hat{\underline{X}}^{(i)} \leq_{ps} \hat{\underline{X}}^{(4)}$, i=1,2,3, to conclude this remark. However, this condition is too weak to be applied for the systems we consider in this paper.

Let $\underline{s}$=(s_0, s_1, s_2, . . .) be an increasing sequence with s_0=0.

Definition 2.8. (Segmented Processes) A process $\underline{X}_t(\underline{s}, \underline{\lambda})$ with $\underline{X}_0(\underline{s}, \underline{\lambda})\overset{\text{st}}{=}\underline{X}_0$, is called a segmented process of $\underline{Z}_t(\lambda)$ with segmentation times $\underline{s}$ and segment parameters $\underline{\lambda}$, if

$$\underline{X}_t(\underline{s}, \underline{\lambda})\overset{\text{st}}{=}\underline{Z}_{t-s_{n-1}}(\lambda_n) \text{ with } \underline{Z}_0(\lambda_n)\overset{\text{st}}{=}\underline{X}_{s_{n-1}}(\underline{s}, \underline{\lambda}) \text{ for } s_{n-1} < t \leq s_n,$$

where $\underline{Z}_t(\lambda)$ is a family of stochastic processes whose finite-dimensional distributions can be determined once the initial distribution of $\underline{Z}_0(\lambda)$ is given, for every λ. For example, if $\underline{Z}_t(\lambda)$ is a family of Markov processes, then $\underline{X}_t(\underline{s}, \underline{\lambda})$ has the same distribution as that of a piecewise Markov process whose transition rates change at the epochs s_{n-1} according to the parameters λ_n. A generalization of this definition is to replace $\underline{Z}_t(\lambda)$ by a set of stochastic processes $\{\underline{Z}_t^{(n)}(\lambda), n = 1, 2, \ldots\}$ which may not have the same finite-dimensional distributions even when the initial distribution and λ are the same. However, the notations are messy and we will stick to the original definition throughout the paper.

Example 2.9. Consider the queue length process of a $G/M/1 \to /M/1$ queueing system. The arrival process is a nonstationary Poisson process with periodic intensity function

$$\lambda_t = \begin{cases} \lambda, & \text{if } 2n < t \leq 2n+1; \\ 3\lambda, & \text{if } 2n+1 < t \leq 2n+2, \end{cases} \qquad n = 0, 1, 2, \ldots.$$

Let $\underline{\lambda}$=(λ_1, λ_2, . . .) with λ_n=(2+ $(-1)^n$)λ and let $\underline{s}$=(0, 1, 2, . . .). Let $\underline{Z}_t(\lambda)$=($Z_{1,t}(\lambda)$, $Z_{2,t}(\lambda)$) be the queue length process of an $M/M/1 \to /M/1$ system with arrival rate λ, where $Z_{1,t}(\lambda)$ ($Z_{2,t}(\lambda)$) denotes the number of customers in the first (second) queue. Now the segmented process $\underline{X}_t(\underline{s},\underline{\lambda})$ of $\underline{Z}_t(\lambda)$ has the same distribution as that of the queue length process of the $G/M/1 \to /M/1$ system.

Lemma 2.10. If $\underline{X}_t(\underline{s},\underline{\lambda})$ is a segmented process of $\underline{Z}_t(\lambda)$ and $\underline{Z}_t(\lambda)$ is SICX-PS(sp) on λ for all t, then $\underline{X}_t(\underline{s},\underline{\lambda})$ is SI-DCX-PS(sp) on $\underline{\lambda}$ for all t.

Proof: As the induction hypothesis, assume $\underline{X}_t(\underline{s},\underline{\lambda})$ is SI-DCX-PS(sp) for all $t \le s_n$. From (c1) and (c2), we have $\lambda_n^{(i)} \le \lambda_n^{(4)}$, i=1,2,3, and $\lambda_n^{(2)}+\ \lambda_n^{(3)} = \lambda_n^{(1)}\ +\lambda_n^{(4)}$ for all n. Since $\underline{Z}_t(\lambda)$ is SICX-PS(sp) on λ for all t, we can construct $\underline{X}_t(\underline{s},\underline{\lambda}^{(i)})\stackrel{\text{st}}{=}\underline{Z}_{t-s_n}(\lambda^{(i)})$, $i = 1,2,3,4$, to satisfy (c3), (c4) and (c5), so that $\underline{X}_t(\underline{s},\underline{\lambda})$ is SI-DCX-PS(sp) for all $s_n < t \le s_{n+1}$. Induction on s_n completes the proof. ∎

Now let $\underline{\Lambda}^{(i)}$=($\Lambda_1^{(i)}$, $\Lambda_2^{(i)}$, ...) be two sequences of r.v.'s and $\underline{S}^{(i)}$=($S_0^{(i)}$, $S_1^{(i)}$, ...) be two increasing sequences of r.v.'s that are independent of the two $\underline{\Lambda}^{(i)}$. Moreover, $S_0^{(i)}$=0.

Lemma 2.11. (Variability Condition) If

(v1) the $\underline{\Lambda}^{(i)}$ are two sequences of *independent* r.v.'s with $\underline{\Lambda}^{(1)} \le_{ic} \underline{\Lambda}^{(2)}$,

(v2) $\underline{S}^{(1)}\stackrel{\text{st}}{=}\underline{S}^{(2)}$,

(v3) the two processes $\underline{X}_t(\underline{S}^{(i)},\underline{\Lambda}^{(i)})$ with identical initial conditions $\underline{X}_0(\underline{S}^{(1)},\underline{\Lambda}^{(1)})$ $= \underline{X}_0(\underline{S}^{(2)},\underline{\Lambda}^{(2)})$ are segmented processes of $\underline{Z}_t(\lambda)$, where $\underline{Z}_t(\lambda)$ is SICX-PS(sp) on λ for all t,

then

(v4) $\sum_{l=1}^{j} X_{l,t}(\underline{S}^{(1)},\underline{\Lambda}^{(1)}) \le_{ic} \sum_{l=1}^{j} X_{l,t}(\underline{S}^{(2)},\underline{\Lambda}^{(2)})$ for $j \ge 1$ and all t.

Proof: Since the $\underline{S}^{(i)}$ are independent of the $\underline{\Lambda}^{(i)}$ and $\underline{S}^{(1)}\stackrel{\text{st}}{=}\underline{S}^{(2)}$, we condition on $\underline{S}^{(i)}$=$\underline{s}$, i=1 and 2. By Lemma 2.10., $\underline{X}_t(\underline{s},\underline{\lambda})$ is SI-DCX-PS(sp) on $\underline{\lambda}$ for all t. It follows from Remark 2.7. that $Ef(\sum_{l=1}^{j} X_{l,t}(\underline{s},\underline{\lambda}))$, j=1,2,..., is increasing and directionally convex in $\underline{\lambda}$ for all f increasing convex. Since $\underline{\Lambda}^{(i)}$ consists of independent r.v.'s and $\underline{\Lambda}^{(1)} \le_{ic} \underline{\Lambda}^{(2)}$, we apply the preservation property of variability ordering (Ross (1983), Proposition 8.5.4, although Ross seems to (implicitly) assume *joint* convexity of f, his proof is valid under the weaker assumption of convexity in *each* variable) to complete the proof. ∎

Lemma 2.12. (Subsequence Condition) If

(s1) $\underline{S}^{(2)} \stackrel{\text{st}}{=} (S^{(1)}_{n_0},\, S^{(1)}_{n_1},\, S^{(1)}_{n_2},\, \ldots)$, $0 = n_0 < n_1 < n_2 < \ldots$, is a subsequence of $\underline{S}^{(1)}$.

(s2) $\underline{\Lambda}^{(2)}$ is a sequence of independent r.v.'s and the random vectors $(\Lambda^{(1)}_{n_{m-1}+1}, \ldots, \Lambda^{(1)}_{n_m})$, $m \geq 0$, are independent with $\Lambda^{(1)}_n \stackrel{\text{st}}{=} \Lambda^{(2)}_m$, $n_{m-1}+1 \leq n \leq n_m$.

(s3) the two processes $\underline{X}_t(\underline{S}^{(i)}, \underline{\Lambda}^{(i)})$ with identical initial conditions $\underline{X}_0(\underline{S}^{(1)}, \underline{\Lambda}^{(1)}) = \underline{X}_0(\underline{S}^{(2)}, \underline{\Lambda}^{(2)})$ are segmented processes of $\underline{Z}_t(\lambda)$, where $\underline{Z}_t(\lambda)$ is SICX-PS(sp) on λ for all t,

then

(s4) $\sum_{l=1}^{j} X_{l,t}(\underline{S}^{(1)}, \underline{\Lambda}^{(1)}) \leq_{ic} \sum_{l=1}^{j} X_{l,t}(\underline{S}^{(2)}, \underline{\Lambda}^{(2)})$ for $j \geq 1$ and all t.

Proof: We need Lorentz's inequality in the proof. It is stated in Rolski (1986), Lemma 5 that if $X_1, \ldots, X_n$ have a common distribution, then $Ef(X_1, \ldots, X_n) \leq Ef(X_1, \ldots, X_1)$ for any L-superadditive function f. This inequality can be easily extended to multidimensional random vectors. Let $\underline{X}^{(i)}=(X^{(i)}_1, \ldots, X^{(i)}_{n_i})$, $i = 1, \ldots m$, be m independent random vectors. If the r.v.'s $X^{(i)}_j$, $j = 1, \ldots, n_i$, have a common distribution $F^{(i)}(x) = \mathrm{P}(\, X^{(i)}_j \leq x)$, then for any L-superadditive function f,

$$\begin{aligned}
&Ef(X^{(1)}_1, \ldots, X^{(1)}_{n_1}, X^{(2)}_1, \ldots, X^{(2)}_{n_2}, \ldots, X^{(m)}_1, \ldots, X^{(m)}_{n_m}) \\
&\leq Ef(X^{(1)}_1, \ldots, X^{(1)}_1, X^{(2)}_1, \ldots, X^{(2)}_1, \ldots, X^{(m)}_1, \ldots, X^{(m)}_1).
\end{aligned}$$

For details, see Chang (1989), Lemma 4.2.8.

Now consider the special case $\underline{S}^{(1)}=(S_0,\, S_1,\, S_2, \ldots)$ and $\underline{S}^{(2)}=(S_0,\, S_{n_1},\, S_{n_1+1}, \ldots)$ for some $n_1 \geq 1$. Let $\tilde{\Lambda}^{(2)}_n=\Lambda^{(2)}_{n-n_1+1}$ for $n > n_1$ and $\tilde{\Lambda}^{(2)}_n = \Lambda^{(2)}_1$ for $n \leq n_1$. Then $\underline{X}_t(\underline{S}^{(2)}, \underline{\Lambda}^{(2)}) \stackrel{\text{st}}{=} \underline{X}_t(\underline{S}^{(1)}, \underline{\tilde{\Lambda}}^{(2)})$. Applying Lorentz's inequality yields

$$Ef(\sum_{l=1}^{j} X_{l,t}(\underline{S}^{(1)}, \underline{\Lambda}^{(1)})) \leq Ef(\sum_{l=1}^{j} X_{l,t}(\underline{S}^{(1)}, \underline{\tilde{\Lambda}}^{(2)})), \;\; j = 1, 2, \ldots.$$

due to the fact $Ef(\sum_{l=1}^{j} X_{l,t}(\underline{s}, \underline{\lambda}))$, j=1,2,..., is increasing and directionally convex in $\underline{\lambda}$ for all f increasing convex. Using the same arguments, the result follows for arbitrary subsequences. ■

Remark 2.13. (i) If the $\underline{\Lambda}^{(i)}$ are two sequences of i.i.d. r.v.'s and $\underline{\Lambda}^{(1)} \stackrel{\text{st}}{=} \underline{\Lambda}^{(2)}$, then (s2) is satisfied. (ii) If $\underline{S}^{(1)} = (\ S_0,\ \ldots,\ S_m)$, $\underline{S}^{(2)} \stackrel{\text{st}}{=} (S_0,\ S_m)$ and $\Lambda_n^{(1)} \stackrel{\text{st}}{=} \Lambda_1^{(2)}$, $1 \le n \le m$, then (s2) is satisfied. We will use (i) to prove the monotonicity result of Ross's conjecture and (ii) to prove a similar upper bound obtained by Rolski (1986).

3. Main Results

In this section, we first prove that certain processes embedded in the three queueing systems are SICX-PS(sp) for all t. Then, we prove the three theorems. The proofs of our results are based on standard coupling and uniformization techniques (see Keilson (1979) and Shaked and Shanthikumar (1988a,b,c)).

Proposition 3.1. (i) If $max(x_1, x_2, x_3) \le x_4$, $x_2 + x_3 \le x_1 + x_4$ and $max(y_1, y_2, y_3) \le y_4$, $y_2 + y_3 \le y_1 + y_4$, then $max(x_1 + y_1, x_2 + y_2, x_3 + y_3) \le x_4 + y_4$ and $x_2 + y_2 + x_3 + y_3 \le x_1 + y_1 + x_4 + y_4$.

(ii) If $max(x_1, x_2, x_3) \le x_4$, and $x_2 + x_3 \le x_1 + x_4$, then $max(f(x_1), f(x_2), f(x_3)) \le f(x_4)$ and $f(x_2) + f(x_3) \le f(x_1) + f(x_4)$ for any f increasing convex.

Proof: The proof of this proposition is straightforward. For details, see Chang (1989).

Lemma 3.2. The family of virtual waiting time processes $V_t(\lambda)$ of $M/GI/1$ queues with Poisson arrival rate λ is SICX(sp) on λ for all t.

Proof: Consider the virtual waiting time processes $V_t(\lambda^{(i)})$, i=1,2,3,4, of four $M/GI/1$ queues with Poisson arrival rate $\lambda^{(i)}$ which satisfy (c1) and (c2), i.e.,

(c1) $\lambda^{(i)} \le \lambda^{(4)}$, i=1,2,3, (c2) $\lambda^{(2)} + \lambda^{(3)} = \lambda^{(1)} + \lambda^{(4)}$.

Assume that (c3), (c4) and (c5) hold for initial conditions $V_0(\lambda^{(i)})$, i=1,2,3,4, i.e., there exist four r.v.'s $\hat{V}_0^{(i)}$, i=1,2,3,4, defined on a common probability space such that

(c3) $\hat{V}_0^{(i)} \overset{\text{st}}{=} V_0(\lambda^{(i)})$, i=1,2,3,4,

(c4) $\hat{V}_0^{(i)} \leq \hat{V}_0^{(4)}$ a.s., i=1,2,3,

(c5) $\hat{V}_0^{(2)} + \hat{V}_0^{(3)} \leq \hat{V}_0^{(1)} + \hat{V}_0^{(4)}$ a.s.

According to Definition 2.6., we have to show that $V_t(\lambda^{(i)})$, i=1,2,3,4, satisfy (c3), (c4) and (c5) for all t, i.e., there exist four stochastic processes $\hat{V}_t^{(i)}$, i=1,2,3,4, defined on a common probability space such that

(c3) $\hat{V}_t^{(i)} \overset{\text{st}}{=} V_t(\lambda^{(i)})$, i=1,2,3,4,

(c4) $\hat{V}_t^{(i)} \leq \hat{V}_t^{(4)}$ a.s., i=1,2,3,

(c5) $\hat{V}_t^{(2)} + \hat{V}_t^{(3)} \leq \hat{V}_t^{(1)} + \hat{V}_t^{(4)}$ a.s.

The idea of the construction is the following: first, we construct the arrivals such that queue 4 always has the largest amount of arriving work and we show that $\hat{V}_t^{(4)}$ is the largest. Second, we distribute the arrivals in such a way that the total amount of work that arrives at queue 1 and queue 4 is the same as that of queue 2 and queue 3. Based on the internal convexity of queues, we then show that $\hat{V}_t^{(2)} + \hat{V}_t^{(3)} \leq \hat{V}_t^{(1)} + \hat{V}_t^{(4)}$. Assume $\lambda^{(1)} \leq \lambda^{(2)} \leq \lambda^{(3)} \leq \lambda^{(4)}$. Construct a Poisson process with rate $\lambda^{(4)}$ and let $\{\tau_n, n \geq 1\}$ be the arrival epochs. Generate a sequence of i.i.d. r.v.'s $\{U_n, n \geq 1\}$ uniformly distributed over [0,1] and a sequence of i.i.d. r.v.'s $\{\alpha_n, n \geq 1\}$ from the service distribution. For $i = 1, 2, 3, 4$, construct $\hat{V}_t^{(i)}$ now as follows:

$$\hat{V}_t^{(i)} = \begin{cases} (\hat{V}_0^{(i)} - t)^+, & \text{if } 0 \leq t < \tau_1, \\ (\hat{V}_{\tau_n}^{(i)} + \alpha_n \beta_n - (t - \tau_n))^+, & \text{if } \tau_n \leq t < \tau_{n+1}, n \geq 1, \end{cases}$$

where $(x)^+ = max(0, x)$ and

$\beta_n^{(1)} = 1\{\frac{\lambda^{(4)} - \lambda^{(3)}}{\lambda^{(4)}} \leq U_n < \frac{\lambda^{(2)}}{\lambda^{(4)}}\}$,

$\beta_n^{(2)} = 1\{0 \leq U_n < \frac{\lambda^{(2)}}{\lambda^{(4)}}\}$,

$\beta_n^{(3)} = 1\{\frac{\lambda^{(4)}-\lambda^{(3)}}{\lambda^{(4)}} \leq U_n < 1\}$,
$\beta_n^{(4)} = 1\{0 \leq U_n < 1\}$.

Observe that $\mathrm{P}(\beta_n^{(i)}{=}1){=}\frac{\lambda^{(i)}}{\lambda^{(4)}}$, i=1,2,3,4. It is clear that these four processes, $\hat{V}_t^{(i)}$, i=1,2,3,4, are stochastically identical to the virtual waiting time processes, $V_t(\lambda^{(i)})$, i=1,2,3,4, of the $M/GI/1$ queues with Poison arrival rate $\lambda^{(i)}$, i=1,2,3,4. Notice that $\beta_n^{(i)} \leq \beta_n^{(4)}$, i=1,2,3, and $\beta_n^{(2)} + \beta_n^{(3)} = \beta_n^{(1)} + \beta_n^{(4)}$. Under the coupling, the total amount of work that arrives at queue 1 and queue 4 is the same as that at queue 2 and queue 3. Furthermore, queue 4 always has the largest amount of arriving work. Since $max(0,x)$ is an increasing convex function of x, conditions (c4) and (c5) for the work load in the four systems are preserved for all $t > 0$ by the initial conditions for the work load in the four systems at time 0 and by Proposition 3.1. ∎

The idea of the coupled input has been presented before by Shaked and Shanthikumar (1988a,b,c), among many others. In the following theorem for tandem systems, we need to couple both inputs and outputs because the input of a intermediate queue is the output of its previous one. Consider a family of p-stage tandem queueing systems at which customers arrive according to a Poisson process with rate λ. The service rate at the j-th stage is $\mu_j(k)$ when k customers are present. Let $Q_{j,t}(\lambda)$, $j = 1, \ldots, p$, be the number of customers in the j-th queue at time t and let $\underline{Q}_t(\lambda) = (Q_{1,t}(\lambda), \ldots, Q_{p,t}(\lambda))$.

Lemma 3.3. If $\mu_j(k)$ is increasing and concave in k for each j and $\sup_{j,k} \mu_j(k) = \eta < \infty$, then $\underline{Q}_t(\lambda) \in$ SICX-PS(sp) on λ for all t.

Proof: Consider the queue length processes $\underline{Q}_t^{(i)}$, i=1,2,3,4, of the four tandem queues with Poisson rates $\lambda^{(i)}$ which satisfy (c1) and (c2). Assume also that (c3), (c4) and (c5) hold for initial conditions $\underline{Q}_0^{(i)}$, i=1,2,3,4. According to Definition 2.6., we have to show $\underline{Q}_t^{(i)}$ satisfy (c3), (c4) and (c5) for all t.

Again, we use standard coupling and uniformization techniques. Construct a Poisson process with rate $\Delta = \lambda^{(4)} + p\eta$ and let $\{\tau_n, n \geq 1\}$ be the arrival epochs. Generate a sequence of i.i.d. r.v.'s $\{U_n, n \geq 1\}$ uniformly distributed over [0,1]. Construct the uniformized Markov chains $\underline{\hat{Q}}^{(i)}_{\tau_n}$, i=1,2,3,4, as follows:

(i) Arrivals at the first stage ($0 \leq U_n < \frac{\lambda^{(4)}}{\Delta}$):

The idea and the construction is the same as in the proof of Lemma 3.2. in the arrival case.

$$\hat{Q}^{(i)}_{1,\tau_{n+1}} = \hat{Q}^{(i)}_{1,\tau_n} + \beta^{(i)}_n,\ i\text{=1,2,3,4},$$
$$\hat{Q}^{(i)}_{l,\tau_{n+1}} = \hat{Q}^{(i)}_{l,\tau_n},\quad l \neq 1,\ i\text{=1,2,3,4},$$

where

$$\beta^{(1)}_n = 1\{\tfrac{\lambda^{(4)}-\lambda^{(3)}}{\Delta} \leq U_n < \tfrac{\lambda^{(2)}}{\Delta}\},$$
$$\beta^{(2)}_n = 1\{0 \leq U_n < \tfrac{\lambda^{(2)}}{\Delta}\},$$
$$\beta^{(3)}_n = 1\{\tfrac{\lambda^{(4)}-\lambda^{(3)}}{\Delta} \leq U_n < \tfrac{\lambda^{(4)}}{\Delta}\},$$
$$\beta^{(4)}_n = 1\{0 \leq U_n < \tfrac{\lambda^{(4)}}{\Delta}\}.$$

Observe that only the first stages are affected by arrivals. Since $\beta^{(i)}_n \leq \beta^{(4)}_n$, i=1,2,3, and $\beta^{(2)}_n + \beta^{(3)}_n = \beta^{(1)}_n + \beta^{(4)}_n$, it follows from Proposition 3.1. and the induction hypothesis that

$$\hat{Q}^{(i)}_{1,\tau_{n+1}} \leq \hat{Q}^{(4)}_{1,\tau_{n+1}},\ i\text{=1,2,3},$$

and that

$$\textstyle\sum_{l=1}^{j} (\hat{Q}^{(2)}_{l,\tau_{n+1}} + \hat{Q}^{(3)}_{l,\tau_{n+1}}) \leq \sum_{l=1}^{j} (\hat{Q}^{(1)}_{l,\tau_{n+1}} + \hat{Q}^{(4)}_{l,\tau_{n+1}}),\ j = 1, \ldots, p.$$

(ii) Departures from stage j ($\frac{\lambda_4+(j-1)\eta}{\Delta} \leq U_n < \frac{\lambda^{(4)}+j\eta}{\Delta}$), $j = 1, \ldots, p$

This construction is more complicated than the previous one because of the state dependency of the service rates. We have to consider two subcases,

(iia) $\hat{Q}^{(1)}_{j,\tau_n} + \hat{Q}^{(4)}_{j,\tau_n} \leq \hat{Q}^{(2)}_{j,\tau_n} + \hat{Q}^{(3)}_{j,\tau_n}$ and (iib) $\hat{Q}^{(1)}_{j,\tau_n} + \hat{Q}^{(4)}_{j,\tau_n} > \hat{Q}^{(2)}_{j,\tau_n} + \hat{Q}^{(3)}_{j,\tau_n}$. The idea is to construct the departures in such a way that in (iia), the total service rate at the j-th stages of systems 1 and 4 is smaller than that of systems 2 and 3 and thus the total number of customers in the first j stages is larger. Note that there is at most one departure from the j-th queue for each system. In (iib), the number of customers at the j-th stages in systems 1 and 4 is larger than that in systems 2 and 3. We then control the departures in such a way that the difference between the departures from the j-th stages of systems 1 and 4 and those from the j-th stages of systems 2 and 3 is smaller than the difference between the number of customers at the j-th stages of systems 1 and 4 and those at the j-th stages of systems 2 and 3. Thus, we still have (iib) satisfied after the departures.

(iia) $\hat{Q}^{(1)}_{j,\tau_n} + \hat{Q}^{(4)}_{j,\tau_n} \leq \hat{Q}^{(2)}_{j,\tau_n} + \hat{Q}^{(3)}_{j,\tau_n}$

By the induction hypothesis (c4), $\hat{Q}^{(i)}_{j,\tau_n} \leq \hat{Q}^{(4)}_{j,\tau_n}$, i=1,2,3. In conjunction with $\hat{Q}^{(1)}_{j,\tau_n} + \hat{Q}^{(4)}_{j,\tau_n} \leq \hat{Q}^{(2)}_{j,\tau_n} + \hat{Q}^{(3)}_{j,\tau_n}$, $\hat{Q}^{(1)}_{j,\tau_n} \leq \hat{Q}^{(i)}_{j,\tau_n}$, i=2,3,4. Let $a^{(i)} \stackrel{\text{def}}{=} \mu_j(\hat{Q}^{(i)}_{j,\tau_n})$ be the service rate at the j-th stage of the i-th system at time τ_n. Analogous to Proposition 3.1, we have $a^{(1)} + a^{(4)} \leq a^{(2)} + a^{(3)}$ and $a^{(i)} \leq a^{(4)}$, i=1,2,3, from the assumption that $\mu_j(k)$ is increasing and concave in k.

Observe that $a^{(1)}$ is the smallest among the four terms. Now construct $\hat{Q}^{(i)}_{j,\tau_{n+1}}$, i=1,2,3,4, as follows:

$$\begin{aligned}
&\hat{Q}^{(i)}_{j,\tau_{n+1}} = \hat{Q}^{(i)}_{j,\tau_n} - \beta^{(i)}_n,\\
&\hat{Q}^{(i)}_{j+1,\tau_{n+1}} = \hat{Q}^{(i)}_{j+1,\tau_n} + \beta^{(i)}_n,\\
&\hat{Q}^{(i)}_{l,\tau_{n+1}} = \hat{Q}^{(i)}_{l,\tau_n},\ l \neq j, j+1,
\end{aligned}$$

where

$$\begin{aligned}
&\beta^{(1)}_n = 1\{\tfrac{\lambda_4+(j-1)\eta+a^{(2)}-a^{(1)}}{\Delta} \leq U_n < \tfrac{\lambda_4+(j-1)\eta+a^{(2)}}{\Delta}\},\\
&\beta^{(2)}_n = 1\{\tfrac{\lambda_4+(j-1)\eta}{\Delta} \leq U_n < \tfrac{\lambda_4+(j-1)\eta+a^{(2)}}{\Delta}\},\\
&\beta^{(3)}_n = 1\{\tfrac{\lambda_4+(j-1)\eta+a^{(4)}-a^{(3)}}{\Delta} \leq U_n < \tfrac{\lambda_4+(j-1)\eta+a^{(4)}}{\Delta}\},\\
&\beta^{(4)}_n = 1\{\tfrac{\lambda_4+(j-1)\eta}{\Delta} \leq U_n < \tfrac{\lambda_4+(j-1)\eta+a^{(4)}}{\Delta}\}.
\end{aligned}$$

From the construction, it follows that $\hat{Q}^{(i)}_{l,\tau_{n+1}} = \hat{Q}^{(i)}_{l,\tau_n}$, $l \neq j, j+1$ and that $\sum_{l=1}^{m} \hat{Q}^{(i)}_{l,\tau_{n+1}} = \sum_{l=1}^{m} \hat{Q}^{(i)}_{l,\tau_n}$, $m \neq j$. Therefore, we only have to verify that

$\hat{Q}^{(i)}_{l,\tau_{n+1}} \leq \hat{Q}^{(4)}_{l,\tau_{n+1}}$, i=1,2,3, for $l = j, j+1$ and

$\sum_{l=1}^{j}(\hat{Q}^{(2)}_{l,\tau_{n+1}}+\hat{Q}^{(3)}_{l,\tau_{n+1}}) \le \sum_{l=1}^{j}(\hat{Q}^{(1)}_{l,\tau_{n+1}}+\hat{Q}^{(4)}_{l,\tau_{n+1}})$.

Observe that $\beta_n^{(i)} \le \beta_n^{(4)}$, i=1,2,3. Hence $\hat{Q}^{(i)}_{j+1,\tau_{n+1}} \le \hat{Q}^{(4)}_{j+1,\tau_{n+1}}$, i=1,2,3, by induction on n. Also notice that $\beta_n^{(1)} + \beta_n^{(4)} \le \beta_n^{(2)} + \beta_n^{(3)}$. Hence

$$\begin{aligned}\sum_{l=1}^{j}(\hat{Q}^{(2)}_{l,\tau_{n+1}}+\hat{Q}^{(3)}_{l,\tau_{n+1}}) &= \sum_{l=1}^{j}(\hat{Q}^{(2)}_{l,\tau_n}+\hat{Q}^{(3)}_{l,\tau_n}) - \beta_n^{(2)} - \beta_n^{(3)} \\ &\le \sum_{l=1}^{j}(\hat{Q}^{(1)}_{l,\tau_n}+\hat{Q}^{(4)}_{l,\tau_n}) - \beta_n^{(1)} - \beta_n^{(4)} = \sum_{l=1}^{j}(\hat{Q}^{(1)}_{l,\tau_{n+1}}+\hat{Q}^{(4)}_{l,\tau_{n+1}}).\end{aligned}$$

If $\hat{Q}^{(4)}_{j,\tau_n} > \hat{Q}^{(i)}_{j,\tau_n}$ for some i, then

$$\hat{Q}^{(4)}_{j,\tau_{n+1}} = \hat{Q}^{(4)}_{j,\tau_n} - \beta_n^{(4)} \ge \hat{Q}^{(4)}_{j,\tau_n} - 1 \ge \hat{Q}^{(i)}_{j,\tau_n} \ge \hat{Q}^{(i)}_{j,\tau_n} - \beta_n^{(i)} = \hat{Q}^{(i)}_{j,\tau_{n+1}}.$$

If $\hat{Q}^{(4)}_{j,\tau_n} = \hat{Q}^{(i)}_{j,\tau_n}$ for some i, then $\beta_n^{(4)} = \beta_n^{(i)}$ and $\hat{Q}^{(4)}_{j,\tau_{n+1}} = \hat{Q}^{(i)}_{j,\tau_{n+1}}$. So we still have $\hat{Q}^{(i)}_{j,\tau_{n+1}} \le \hat{Q}^{(4)}_{j,\tau_{n+1}}$, i=1,2,3.

(iib) $\hat{Q}^{(1)}_{j,\tau_n} + \hat{Q}^{(4)}_{j,\tau_n} > \hat{Q}^{(2)}_{j,\tau_n} + \hat{Q}^{(3)}_{j,\tau_n}$

Similarly, construct $\hat{Q}^{(i)}_{j,\tau_{n+1}}$, i=1,2,3,4, as follows:

$\hat{Q}^{(i)}_{j,\tau_{n+1}} = \hat{Q}^{(i)}_{j,\tau_n} - \beta_n^{(i)}$,
$\hat{Q}^{(i)}_{j+1,\tau_{n+1}} = \hat{Q}^{(i)}_{j+1,\tau_n} + \beta_n^{(i)}$,
$\hat{Q}^{(i)}_{l,\tau_{n+1}} = \hat{Q}^{(i)}_{l,\tau_n}$, $l \ne j, j+1$.

Now construct $\beta_n^{(i)}$, i=1,2,3,4, according to the following rules:

$\beta_n^{(i)} = 1\{\frac{\lambda_4+(j-1)\eta}{\Delta} \le U_n < \frac{\lambda_4+(j-1)\eta+a^{(i)}}{\Delta}\}$, i=1,2,3,4,

where $a^{(i)} \stackrel{\text{def}}{=} \mu_j(\hat{Q}^{(i)}_{j,\tau_n})$ is again the service rate at the j-th stage of the i-th system at time τ_n. Again, we only have to verify that

$\hat{Q}^{(i)}_{l,\tau_{n+1}} \le \hat{Q}^{(4)}_{l,\tau_{n+1}}$, i=1,2,3, for $l = j, j+1$ and
$\sum_{l=1}^{j}(\hat{Q}^{(2)}_{l,\tau_{n+1}}+\hat{Q}^{(3)}_{l,\tau_{n+1}}) \le \sum_{l=1}^{j}(\hat{Q}^{(1)}_{l,\tau_{n+1}}+\hat{Q}^{(4)}_{l,\tau_{n+1}})$.

Observe that $\beta_n^{(i)} \le \beta_n^{(4)}$ as $\mu_j(k)$ is increasing in k. Because of the induction hypothesis (c4), i.e., $\hat{Q}^{(i)}_{j,\tau_n} \le \hat{Q}^{(4)}_{j,\tau_n}$, i=1,2,3, we have $\hat{Q}^{(i)}_{j+1,\tau_{n+1}} \le \hat{Q}^{(4)}_{j+1,\tau_{n+1}}$, i=1,2,3. From an argument similar to the one used in the previous subcase, it follows that $\hat{Q}^{(i)}_{j,\tau_{n+1}} \le \hat{Q}^{(4)}_{j,\tau_{n+1}}$,

i=1,2,3. Observe that $\beta_n^{(1)} + \beta_n^{(4)} \le 1 + \beta_n^{(2)} + \beta_n^{(3)}$ if $\hat{Q}^{(1)}_{j,\tau_n} \le \max(\hat{Q}^{(2)}_{j,\tau_n}, \hat{Q}^{(3)}_{j,\tau_n})$. Hence

$$\hat{Q}^{(1)}_{j,\tau_{n+1}} + \hat{Q}^{(4)}_{j,\tau_{n+1}} = \hat{Q}^{(1)}_{j,\tau_n} + \hat{Q}^{(4)}_{j,\tau_n} - \beta_n^{(1)} - \beta_n^{(4)}$$
$$\ge \hat{Q}^{(2)}_{j,\tau_n} + \hat{Q}^{(3)}_{j,\tau_n} - \beta_n^{(2)} - \beta_n^{(3)} = \hat{Q}^{(2)}_{j,\tau_{n+1}} + \hat{Q}^{(3)}_{j,\tau_{n+1}}.$$

However, if $\hat{Q}^{(1)}_{j,\tau_n} > \max(\hat{Q}^{(2)}_{j,\tau_n}, \hat{Q}^{(3)}_{j,\tau_n})$, then

$$\hat{Q}^{(4)}_{j,\tau_n} \ge \hat{Q}^{(1)}_{j,\tau_n} \ge 1 + \hat{Q}^{(i)}_{j,\tau_n}, \quad i = 2, 3,$$

and therefore

$$\hat{Q}^{(4)}_{j,\tau_n} + \hat{Q}^{(1)}_{j,\tau_n} \ge 2 + \hat{Q}^{(2)}_{j,\tau_n} + \hat{Q}^{(3)}_{j,\tau_n}.$$

So

$$\hat{Q}^{(1)}_{j,\tau_{n+1}} + \hat{Q}^{(4)}_{j,\tau_{n+1}} \ge \hat{Q}^{(2)}_{j,\tau_{n+1}} + \hat{Q}^{(3)}_{j,\tau_{n+1}}.$$

From the induction hypothesis (c5), i.e.,

$$\sum_{l=1}^{j-1} (\hat{Q}^{(2)}_{l,\tau_n} + \hat{Q}^{(3)}_{l,\tau_n}) \le \sum_{l=1}^{j-1} (\hat{Q}^{(1)}_{l,\tau_n} + \hat{Q}^{(4)}_{l,\tau_n}),$$

it now follows that

$$\sum_{l=1}^{j} (\hat{Q}^{(2)}_{l,\tau_{n+1}} + \hat{Q}^{(3)}_{l,\tau_{n+1}}) \le \sum_{l=1}^{j} (\hat{Q}^{(1)}_{l,\tau_{n+1}} + \hat{Q}^{(4)}_{l,\tau_{n+1}}). \ \blacksquare$$

Remark 3.4. (i) Shaked and Shanthikumar (1988b) obtained the result for a single queue. (ii) The result can be extended to the case with bulk arrivals in the same way as was done by Shaked and Shanthikumar (1988b). (iii) If the buffer size between two stages is finite, the result still holds for a communication type of blocking (i.e. if a service completion occurs when the buffer downstream is full, the customer initiates a new service time at the same server).

Consider now a family of $M/M(k)/1/N$ queues with Poisson arrival rates λ. The service rate is $\mu(k)$ when there are k customers in system. An arrival is blocked (lost) if it finds N customers in system. Let $B_t(\lambda)$ be the number of customers lost by time t and $Q_t(\lambda)$ the number of customers in system at time t. In order to form a Markov process, we have to enlarge our state space and consider the bivariate process, $\underline{X}_t(\lambda) \stackrel{\text{def}}{=} (\ B_t(\lambda),\ Q_t(\lambda)\ )$.

Lemma 3.5. If $\mu(k)$ is increasing and concave in k and $\sup_k \mu(k) = \eta < \infty$, then $\underline{X}_t(\lambda) \in$ SICX-PS (sp) on λ for all t.

Proof: Consider four processes $\underline{X}_t(\lambda^{(i)})$, i=1,2,3,4, with Poisson rates $\lambda^{(i)}$ which satisfy (c1) and (c2). Assume also that (c3), (c4) and (c5) hold for initial conditions $\underline{X}_0(\lambda^{(i)})$, i.e., there exist four random vectors $\hat{X}_0^{(i)} = (\hat{B}_0^{(i)},\ \hat{Q}_0^{(i)})$ such that

(c3B) $B_0(\lambda^{(i)}) \stackrel{\text{st}}{=} \hat{B}_0^{(i)}$,

(c3Q) $Q_0(\lambda^{(i)}) \stackrel{\text{st}}{=} \hat{Q}_0^{(i)}$,

(c4B) $\hat{B}_0^{(i)} \leq \hat{B}_0^{(4)}$, a.s., i=1,2,3,

(c4Q) $\hat{Q}_0^{(i)} \leq \hat{Q}_0^{(4)}$, a.s., i=1,2,3,

(c5B) $\hat{B}_0^{(2)} + \hat{B}_0^{(3)} \leq \hat{B}_0^{(1)} + \hat{B}_0^{(4)}$, a.s.,

(c5Q) $\hat{B}_0^{(2)} + \hat{Q}_0^{(2)} + \hat{B}_0^{(3)} + \hat{Q}_0^{(3)} \leq \hat{B}_0^{(1)} + \hat{Q}_0^{(1)} + \hat{B}_0^{(4)} + \hat{Q}_0^{(4)}$, a.s.

We now have to show $\underline{X}_t(\lambda^{(i)})$, i=1,2,3,4, satisfy (c3), (c4) and (c5) for all t, i.e.,

(c3B) $B_t(\lambda^{(i)}) \stackrel{\text{st}}{=} \hat{B}_t^{(i)}$,

(c3Q) $Q_t(\lambda^{(i)}) \stackrel{\text{st}}{=} \hat{Q}_t^{(i)}$,

(c4B) $\hat{B}_t^{(i)} \leq \hat{B}_t^{(4)}$, a.s., i=1,2,3,

(c4Q) $\hat{Q}_t^{(i)} \leq \hat{Q}_t^{(4)}$, a.s., i=1,2,3,

(c5B) $\hat{B}_t^{(2)} + \hat{B}_t^{(3)} \leq \hat{B}_t^{(1)} + \hat{B}_t^{(4)}$, a.s.,

(c5Q) $\hat{B}_t^{(2)} + \hat{Q}_t^{(2)} + \hat{B}_t^{(3)} + \hat{Q}_t^{(3)} \leq \hat{B}_t^{(1)} + \hat{Q}_t^{(1)} + \hat{B}_t^{(4)} + \hat{Q}_t^{(4)}$, a.s.

The key ideas are similar to those in the proofs of Lemma 3.2. and Lemma 3.3. The arrivals are distributed evenly and system 4 has the largest number of arrivals. If the number of customers blocked in systems 1 and 4 at an arrival epoch is *the same* as that in systems 2 and 3, we show that the number of arriving-and-blocked customers in systems

1 and 4 is larger than that in systems 2 and 3. If the number of customers blocked in systems 1 and 4 at an arrival epoch is *larger* than that in systems 2 and 3, we control the difference of the arriving-and-blocked customers and show that the number of customers blocked in systems 1 and 4 after arrivals is still larger than that in systems 2 and 3. As before, generate a Poisson process with rate $\Delta = \lambda^{(4)} + \mu(N)$ and let $\{\tau_n, n \geq 1\}$ be its arrival epochs. Generate a sequence of i.i.d. r.v.'s $\{U_n, n \geq 1\}$ uniformly distributed over [0,1]. Construct the uniformized Markov chains $\underline{X}^{(i)}_{\tau_n}$, i=1,2,3,4, as follows:

(i) Arrivals ($0 \leq U_n < \frac{\lambda^{(4)}}{\Delta}$):

$$\hat{B}^{(i)}_{\tau_{n+1}} = \hat{B}^{(i)}_{\tau_n} + \beta^{(i)}_n 1\{\hat{Q}^{(i)}_{\tau_n} = N\},$$
$$\hat{Q}^{(i)}_{\tau_{n+1}} = \hat{Q}^{(i)}_{\tau_n} + \beta^{(i)}_n 1\{\hat{Q}^{(i)}_{\tau_n} < N\},$$

where

$$\beta^{(1)}_n = 1\{\tfrac{\lambda^{(4)}-\lambda^{(3)}}{\Delta} \leq U_n < \tfrac{\lambda^{(2)}}{\Delta}\},$$
$$\beta^{(2)}_n = 1\{0 \leq U_n < \tfrac{\lambda^{(2)}}{\Delta}\},$$
$$\beta^{(3)}_n = 1\{\tfrac{\lambda^{(4)}-\lambda^{(3)}}{\Delta} \leq U_n < \tfrac{\lambda^{(4)}}{\Delta}\},$$
$$\beta^{(4)}_n = 1\{0 \leq U_n < \tfrac{\lambda^{(4)}}{\Delta}\}.$$

From the fact that $\beta^{(i)}_n \leq \beta^{(4)}_n$ and that $1\{\hat{Q}^{(i)}_{\tau_n} = N\} \leq 1\{\hat{Q}^{(4)}_{\tau_n} = N\}$, i=1,2,3, it follows that (c4B) is satisfied. If $1\{\hat{Q}^{(4)}_{\tau_n} < N\} = 1$, then (c4Q) is satisfied. If $1\{\hat{Q}^{(4)}_{\tau_n} < N\}$ $= 0$, then $\hat{Q}^{(4)}_{\tau_n} = N$ and (c4Q) is satisfied as well. Since $\beta^{(2)}_n + \beta^{(3)}_n = \beta^{(1)}_n + \beta^{(4)}_n$, (c5Q) is satisfied. Observe that $\beta^{(4)}_n 1\{\hat{Q}^{(4)}_{\tau_n} = N\} + 1 \geq \beta^{(2)}_n\ 1\{\hat{Q}^{(2)}_{\tau_n} = N\} + \beta^{(3)}_n\ 1\{\hat{Q}^{(3)}_{\tau_n} = N\}$. Thus, if $\hat{B}^{(1)}_{\tau_n} + \hat{B}^{(4)}_{\tau_n} > \hat{B}^{(2)}_{\tau_n} + \hat{B}^{(3)}_{\tau_n}$, then (c5B) is satisfied. However, if $\hat{B}^{(1)}_{\tau_n} + \hat{B}^{(4)}_{\tau_n} = \hat{B}^{(2)}_{\tau_n} + \hat{B}^{(3)}_{\tau_n}$, then $\hat{Q}^{(2)}_{\tau_n} + \hat{Q}^{(3)}_{\tau_n} \leq \hat{Q}^{(1)}_{\tau_n} + \hat{Q}^{(4)}_{\tau_n}$ from the induction hypothesis (c5Q), i.e., if $\max(\hat{Q}^{(2)}_{\tau_n}, \hat{Q}^{(3)}_{\tau_n}) = N$, then $\hat{Q}^{(4)}_{\tau_n} = N$ and $\hat{Q}^{(1)}_{\tau_n} \geq \min(\hat{Q}^{(2)}_{\tau_n}, \hat{Q}^{(3)}_{\tau_n})$. Thus,

$$\beta^{(2)}_n 1\{\hat{Q}^{(2)}_{\tau_n} = N\} + \beta^{(3)}_n 1\{\hat{Q}^{(3)}_{\tau_n} = N\} \leq \beta^{(1)}_n 1\{\hat{Q}^{(1)}_{\tau_n} = N\} + \beta^{(4)}_n 1\{\hat{Q}^{(4)}_{\tau_n} = N\}$$

when $\hat{B}^{(1)}_{\tau_n} + \hat{B}^{(4)}_{\tau_n} = \hat{B}^{(2)}_{\tau_n} + \hat{B}^{(3)}_{\tau_n}$. Again, from Proposition 3.1., $\hat{B}^{(2)}_{\tau_{n+1}} + \hat{B}^{(3)}_{\tau_{n+1}} \leq \hat{B}^{(1)}_{\tau_{n+1}}$ $+ \hat{B}^{(4)}_{\tau_{n+1}}$ and (c5B) is satisfied.

(ii) Departures ($\frac{\lambda^{(4)}}{\Delta} \leq U_n < 1$)

Observe that a departure does not affect $\hat{B}^{(i)}_{\tau_n}$. It is the same as a departure from the last stage of the tandem system in Lemma 3.3. The proof now follows an argument similar to the one in case (ii) of Lemma 3.3. ■

Remark 3.6. The result of Lemma 3.5. can be extended to tandem systems with either infinite buffers between two stages or finite buffers with a communication type of blocking.

We are now ready for the proofs of Theorems 1, 2 and 3.

Proof of Theorem 1: Consider two single-server queues with i.i.d. service times from an arbitrary distribution. The arrival processes are DSP processes with segmentable intensity functions $\lambda_t^{(i)}$. Observe that the virtual waiting processes $V_t^{(i)}$, i=1,2, are segmented processes of the virtual waiting process considered in Lemma 3.2 with segmentation times $\underline{S}^{(i)}$ and segment parameters $\underline{\Lambda}^{(i)}$, i=1 and 2. From Lemmas 2.11 (Variability Condition), 2.12. (Subsequence Condition) and 3.2., it then follows that $V_t^{(1)}(\underline{S}^{(1)}, \underline{\Lambda}^{(1)}) \leq_{ic} V_t^{(2)}(\underline{S}^{(2)}, \underline{\Lambda}^{(2)})$ for all t if the intensity functions $\lambda_t^{(i)}$, i=1 and 2, satisfy the Comparability Condition and the initial conditions of these two systems are the same, i.e., $V_0^{(1)}(\underline{S}^{(1)}, \underline{\Lambda}^{(1)}) = V_0^{(2)}(\underline{S}^{(2)}, \underline{\Lambda}^{(2)})$.

In case of identical input, but different initial conditions, it follows from Lemma 3.2. that $Ef(V_t|V_0)$ is increasing convex in V_0 for all f increasing convex. Thus, $V_t^{(1)} \leq_{ic} V_t^{(2)}$ if $V_0^{(1)} \leq_{ic} V_0^{(2)}$. ■

Proof of Theorem 2: Analogous to the proof of Theorem 1, observe that the p-dimensional queue length processes are segmented processes of the process considered in Lemma 3.3. Application of Lemmas 2.11., 2.12. and 3.3. proves the theorem. ■

Proof of Theorem 3: Observe that $\underline{X}_t^{(i)}(\underline{S}^{(i)}, \underline{\Lambda}^{(i)}) = (\ B_t(\underline{S}^{(i)}, \underline{\Lambda}^{(i)}),\ Q_t(\underline{S}^{(i)}, \underline{\Lambda}^{(i)}))$ are segmented processes of the process considered in Lemma 3.5. From Lemmas 2.11, 2.12. and 3.5., the result follows. ■

Remark 3.7. Observe that Lemmas 3.2., 3.3. and 3.5. can be extended to systems that are subject to a superposition of two arrival processes: one being a Poisson and the second being an arbitrary arrival process that is independent of the first and the parameterization is on λ, the rate of the Poisson process. These proofs can be argued as follows: Let $\{\hat{\tau}_n, n \geq 1\}$ be the arrival epochs of the arbitrary process. At each $\hat{\tau}_n$, all these four systems considered in the proofs have an arrival that is a special case of τ_n. By induction on n, the proofs then go through. Thus, these three theorems can be extended to systems subject to a superposition of independent DSP arrivals if each pair of the intensity functions satisfies the comparability condition. To show this, we apply the well-known pairwise interchange argument. Consider auxiliary systems that differ from one another by a DSP process. Consider the other DSP processes as an arbitrary arrival process and apply the superposition result concerning a Poisson process and an arbitrary process to the proofs of these three theorems. A partial ordering can then be shown among these systems. This completes the argument (a rigorous proof, which requires extending the definition of the segmented process as remarked in Definition 2.8., is omitted since it is notationally messy).

4. Discussion

One may raise questions about the characteristics of stochastic processes that are segmentable. In order to provide some more insight into these processes, we present three examples.

Example 4.1. (Simple Processes) A stochastic process X_t is called simple if there exists an increasing sequence $\underline{s} = (s_0, s_1, \dots)$ with $s_0 = 0$ and a sequence of r.v.'s $\underline{\Lambda} = (\Lambda_1, \Lambda_2, \dots)$ such that $X_t = X_0 + \sum_{n=1}^{\infty} \Lambda_n 1\{s_{n-1} < t \leq s_n\}$ (see Karatzas and Shreve (1988)). Clearly, a simple process is a segmentable process with segmentation times $\underline{s}$ and segment variables $\underline{\Lambda}$.

Remark 4.2. Rolski (1986) obtained upper bounds for the expected waiting times in

$DSP/GI/1$ queues. He showed that the expected stationary virtual waiting time is not greater than $Ev(\lambda_0)$, where λ_0 is λ_t at time 0 and $v(a)$ denotes the expected stationary waiting time in the $M/GI/1$ queue with arrival rate a. Analogous to his proof, we may approximate an intensity function by a sequence of simple processes. Using Subsequence Condition (Remark 2.13.(ii)), similar upper bounds can be obtained for tandem queues and queues with finite buffers (cases (ii) and (iii)) subject to DSP arrivals with *stationary* intensity functions, i.e., the expected total number of customers in the tandem system considered in Theorem 2 is not greater than $E[\bar{Q}(\lambda_0)]$, where $\bar{Q}(a)$ denotes the expected total number of customers in the tandem system subject to Poisson arrivals with rate a, and the blocking probability P_B in the system considered in Theorem 3 is not greater than $E[P_B(\lambda_0)]$, where $P_B(a)$ denotes the blocking probability in the $M/M(k)/1/N$ queue with arrival rate a (the blocking probability P_B is defined as the limit of the number of customers blocked by time t divided by the number of arrivals by time t, i.e.,

$$P_B = \lim_{t\to\infty} \frac{B_t}{N_t} = \frac{\lim_{t\to\infty} \frac{B_t}{t}}{\lim_{t\to\infty} \frac{N_t}{t}} = \frac{1}{\lambda} \lim_{t\to\infty} \frac{EB_t}{t},$$

where N_t is the number of arrivals by time t).

Example 4.3. Consider two stationary Markov processes $\lambda_t^{(i)}$, i=1 and 2, with M-states $\{\bar{\lambda}_1^{(i)}, \ldots, \bar{\lambda}_M^{(i)}\}$, i=1 and 2, and transition rates q_{rs} which satisfy $q_{rs} = \alpha_s$, $r \neq s$ and q_{rr} $= -\sum_{s\neq r} \alpha_s$ for some $\alpha_s > 0$, s=1, ..., M. Then $\lambda_t^{(i)} \in \mathcal{G}$. If, moreover, $\lambda_0^{(1)} \leq_{ic} \lambda_0^{(2)}$ ($\lambda_t^{(i)}$ at time 0), then $\lambda_t^{(i)}$ satisfy the Variability Condition. In order to verify this, we proceed as follows:

Observe that the stationary distributions of $\lambda_t^{(i)}$, i=1 and 2, are P($\lambda_t^{(i)} = \bar{\lambda}_r^{(i)}$) $=$ $\alpha_r / \sum_{s=1}^M \alpha_s \stackrel{\text{def}}{=} \pi_r$, $r = 1, \ldots, M$. Let $\underline{T} = (\, T_1,\, T_2,\, \ldots\,)$ be a sequence of independent and exponentially distributed r.v.'s with mean $1/\sum_{s=1}^M \alpha_s$ and let $\underline{\Lambda}^{(i)} = (\, \Lambda_1^{(i)},\, \Lambda_2^{(i)},\, \ldots\,)$ be two sequences of i.i.d. r.v.'s which are independent of $\underline{T}$ with distributions P($\Lambda_n^{(i)} =$ $\bar{\lambda}_r^{(i)}$) $= \pi_r$. Let $\underline{S} = (\, S_0,\, S_1,\, \ldots\,)$ with $S_0 = 0$ and $S_n = \sum_{j=1}^n T_j$ and let $\lambda_t^{(i)} = \Lambda_n^{(i)}$ for $S_{n-1} < t \leq S_n$.

In order to show that these $\lambda_t^{(i)}$, i=1 and 2, are stationary M-state Markov processes with transition rates q_{rs}, it has to be verified that the transition time $T_{rs}^{(i)}$, i=1 and 2, from state r to state s is an exponential r.v. with mean $\frac{1}{\alpha_s}$ and the transition probability $P_{rs}^{(i)}$, i=1 and 2, from state r to state s is $\alpha_s/\sum_{r\neq s}\alpha_r$.

Let $Z^{(i)} = \inf\{n > 0 : \Lambda_n^{(i)} = \bar{\lambda}_s^{(i)}\}$, i=1 and 2. Then $Z^{(i)}$ is a geometric r.v. with $\mathrm{P}(Z^{(i)} = n) = \pi_s(1-\pi_s)^{n-1}$, i=1 and 2. Observe that $T_{rs}^{(i)} \stackrel{\text{st}}{=} \sum_{n=1}^{Z^{(i)}} T_n$. From the fact that a geometric sum of exponential r.v.'s is an exponential r.v. and from Wald's equation, it follows that

$$ET_{rs}^{(i)} = EZ^{(i)}ET_1 = \frac{1}{\pi_s}\frac{1}{\sum_{r=1}^{M}\alpha_r} = \frac{1}{\alpha_s}$$

and that

$$P_{rs}^{(i)} = P(\Lambda_1^{(i)} = \bar{\lambda}_s^{(i)} | \Lambda_1^{(i)} \neq \bar{\lambda}_r^{(i)}) = \frac{\pi_s}{1-\pi_r} = \frac{\alpha_s}{\sum_{r\neq s}\alpha_r}.$$

Therefore, $\lambda_t^{(i)} \in \mathcal{G}$.

If, moreover, $\lambda_0^{(1)} \leq_{ic} \lambda_0^{(2)}$, then $\Lambda_n^{(1)} \leq_{ic} \Lambda_n^{(2)}$ for all n and $\lambda_t^{(i)}$, i=1 and 2, satisfy the Variability Condition. ■

Example 4.4. Consider two stationary Markov processes $\lambda_t^{(i)}$, i=1 and 2, with M states $\{\bar{\lambda}_1, \ldots, \bar{\lambda}_M\}$ and transition rates $q_{rs}^{(i)} = c^{(i)}\alpha_s$, $r \neq s$ and $q_{rr}^{(i)} = -c^{(i)}\sum_{s\neq r}\alpha_s$ for some $c^{(1)} > c^{(2)}$ and $\alpha_s > 0$, $s = 1, \ldots, M$. Then $\lambda_t^{(i)}$, i=1 and 2, satisfy the Subsequence Condition (Remark 2.13 (i)). This can be shown as follows:

Assume $c^{(1)}$=1 and construct $\underline{S}$ and $\underline{T}$ as in the previous example. Let $\underline{\Lambda} = (\Lambda_1, \Lambda_2, \ldots)$ be a sequence of i.i.d. r.v.'s which are independent of $\underline{S}$ with $\mathrm{P}(\Lambda_n = \bar{\lambda}_r) = \alpha_r/\sum_{s=1}^{M}\alpha_s \stackrel{\text{def}}{=} \pi_r$, $r = 1, \ldots, M$. Let $\underline{B} = (B_1, B_2, \ldots)$ be a sequence of i.i.d. Bernoulli r.v.'s which are independent of $\underline{S}$ and $\underline{\Lambda}$ with $\mathrm{P}(B_n = 0) = c^{(2)}$, $\mathrm{P}(B_n = 1) = 1 - c^{(2)}$. Choose $\underline{S}^{(2)}$ from $\underline{S}$ when B_n=0. Then $\underline{S}^{(2)}$ is clearly a subsequence of $\underline{S}$. Now construct $\lambda_t^{(i)}$, i=1 and 2, as follows:

$\lambda_t^{(1)} = \Lambda_n$, $S_{n-1} < t \leq S_n$,

$\lambda_t^{(2)} = \Lambda_n,\, S_{n-1}^{(2)} < t \leq S_n^{(2)}.$

As in the previous example, $\lambda_t^{(1)} \in \mathcal{G}$. It remains to be shown that the transition time $T_{rs}^{(2)}$ from state r to s is exponential with mean $\frac{1}{c^{(2)}\alpha_s}$.

If $Z = \inf\{n \geq 0 : \Lambda_n = \bar{\lambda}_s, B_n = 0\}$, then P(Z=n) $= \pi_s c^{(2)}$ (1- $\pi_s c^{(2)}$ $)^{n-1}$. Observe that $T_{rs}^{(2)} \stackrel{\text{st}}{=} \sum_{j=1}^{Z} T_n$ and

$$ET_{rs}^{(2)} = EZET_1 = \frac{1}{\pi_s c^{(2)}} \frac{1}{\sum_{r=1}^{M} \alpha_s} = \frac{1}{c^{(2)}\alpha_s}.$$

Clearly, the Subsequence Condition (Remark. 2.13.(i)) is satisfied. ∎

Remark 4.5. Ross (1978) considered a $DSP/GI/1$ queue with an arrival process driven by a two-state Markov process which alternates between states 1 and 2 with respective rates $\bar{\lambda}_1$ and $\bar{\lambda}_2$. The time that the two-state Markov process spends in state r is exponential with rate $c\alpha_r$, r=1 and 2. Ross conjectured that the expected waiting time in the queue is a decreasing function of c. The conjecture holds, due to the fact that the two-state Markov processes considered by Ross are segmentable processes that satisfy our Subsequence Condition (Example 4.4.). From the property of work conservation for $G/GI/1$ queues and Little's formula (see Rolski (1986), (3)),

$$EV = EWES + \frac{\lambda ES^2}{2},$$

where V, W and S are stationary virtual waiting time, stationary waiting time and service time, and

$$\lambda = \frac{\alpha_2}{\alpha_1 + \alpha_2}\bar{\lambda}_1 + \frac{\alpha_1}{\alpha_1 + \alpha_2}\bar{\lambda}_2.$$

The conjecture then follows from Theorem 1. Moreover, we have shown that the total number of customers in a tandem system is a decreasing function of c as well. From Little's formula, it follows that the expected sojourn time in the tandem system is a decreasing function of c. Moreover, in a $DSP/M/C/N$ queue, the expected number of customers blocked (lost) by time t is a decreasing function of c and thus the blocking probability is a decreasing function of c.

Acknowledgments

We would like to thank Prof. J.G. Shanthikumar for his valuable advice and discussion. A general theory on regularity of stochastic processes, which generalizes our variability condition and subsequence condition, will appear in a paper of his (Shanthikumar (1988)). Special thanks are due to the referee for their very careful review of this paper. Their insightful comments have led to a greatly improved version of the paper. The work was partially done while C.S. Chang and X.L. Chao were with the Center for Telecommunications Research, Columbia University.

References

[1] Chang, C.S. (1989) Comparison Theorems for Queueing Systems and Their Applications to ISDN, Ph.D. dissertation, Dept. of Electrical Engineering, Columbia University, New York.

[2] Chang, C.S. & Pinedo, M.(1990) Bounds and Inequalities for Single Server Loss Systems. To appear in *Queueing Systems*.

[3] Chang, C.S., Chao, X.L. & Pinedo, M.(1990) Integration of Discrete-Time Correlated Markov Processes in a TDM System: Structural Results. *Probability in the Engineering and Informational Sciences* 4, 29-56.

[4] Fond, S. and Ross. S.M.(1978) A Heterogeneous Arrival and Service Queueing Loss Model. *Naval Research Logistics Quarterly* 25, 483-488.

[5] Karatzas, I. & Shreve, S.E. (1988) *Brownian Motion and Stochastic Calculus*, Springer -Verlag, New York.

[6] Keilson, T.(1979) *Markov Chain Models-Rarity and Exponentiality*, Springer-Verlag, New York.

[7] Niu,S.-C.(1980) A Single Server Queueing Loss Model with Heterogeneous arrival and Service. *Oper. Res.* 28, 584-593.

[8] Rolski, T.(1981) Queues with Non-stationary Input Stream: Ross's Conjecture. *Adv. Appl. Prob.* 13, 603-618.

[9] Rolski, T.(1983) Comparison Theorems for Queues with Dependent Inter-Arrival Times, Modelling and Performance Evaluation Proceeding of the International Seminar, Paris, France, Jan 24-26, 42-67.

[10] Rolski, T.(1986) Upper Bounds for Single Server Queues with Doubly Stochastic Poisson Arrivals. *Math. of Oper. Res.* 11, 442-450.

[11] Rolski, T.(1989) Queues with Nonstationary Arrivals. *Queueing Systems* 5, 113-130.

[12] Ross, S.M.(1978) Average Delay in Queues with Non-stationary Poisson Arrivals. *J. Appl. Prob.* 15, 602-609.

[13] Ross, S.M.(1983) *Stochastic Processes.* J. Wiley & Sons, New York.

[14] Shaked, M. & Shanthikumar, J.G.(1988a) Temporal Stochastic Convexity and Concavity, *Stochastic Processes and Their Applications* 27, 1-20.

[15] Shaked, M. & Shanthikumar, J.G.(1988b) Stochastic Convexity and Its Applications, *Adv. Appl. Prob.* 20, 427-446.

[16] Shaked, M. & Shanthikumar, J.G.(1988c) Parametric Stochastic Convexity and Concavity of Stochastic Processes, Technical Report, Univ. of California, Berkeley.

[17] Shanthikumar, J.G.(1988) Regularity of Stochastic Processes: A Theory Based on Directional Convexity, Technical Report, Univ. of California, Berkeley.

[18] Stoyan,D.(1983) *Comparison Methods for Queues and Other Stochastic Models.* J. Wiley & Sons, Berlin.

[19] Svoronos, A. & Green, L.(1987) The N-Seasons S-Servers Loss System, *Naval Research Logistics Quarterly* 34, 579-591.

[20] Svoronos, A. & Green, L.(1988) A Convexity Result for Single Server Exponential Loss Systems with Nonstationary Arrivals, *J. Appl. Prob.* 25, 224-227.