# Moderate Deviations for Queues with Long-Range Dependent Input

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ABSTRACT Based on the integral relationship between Brownian motion and fractional Brownian motion, we model a process with long-range dependence, $Y$, as the output of a time-invariant linear filter applied to a more familiar process $X$ — one that does not have long-range dependence. We develop a general criterion for a broad class of $X$ to satisfy a moderate deviations principle (MDP). Based on the MDP of $X$, we then establish the MDP for $Y$. Using this framework, we develop both transient and steady-state results, in terms of the MDP, regarding the asymptotic behavior of queues fed with the long-range dependent input process $Y$. We study two types of filter models, with bounded and unbounded time intervals, respectively.

## 1 Introduction

Recently, it has been demonstrated that queues with long-range dependent input have subexponential tails (Norros [20], and Duffield and O'Connell [11]). This phenomenon is qualitatively different from the exponential tail distribution in queues with renewal or Markov arrival processes — input that has short-range dependence. The exponential tail behavior is usually established via large deviations techniques; refer to, e.g., Chang [5], Glynn and Whitt [12, 13], Whitt [23], and the references there. (Refer to, e.g., Cox [8] and Beran [2], for preliminaries regarding processes with long-range dependence, as well as short-range dependence. Refer to Leland *et al* [16] and Willinger [24], among others, for studies of data from telecommunications traffic that exhibit long-range dependence, typically accompanied by self-similarity.)

It will be insightful to briefly review here the approach in Duffield and O'Connell [11]. Assuming that the "netput" — input minus (potential) output — satisfies a *large deviations principle* (LDP) with a speed that is slower than $n$ or, more precisely, a *moderate deviations principle* (MDP) (refer to Section 2), Duffield and O'Connell showed that the maximum of the corresponding random walk also satisfies the MDP; in particular, when the netput is a fractional Brownian motion (FBM) with Hurst parameter

$H \in (\frac{1}{2}, 1)$ the MDP holds with speed $n^{2(1-H)}$. (See Section 5 for more details.) Hence, with the FBM as the netput, the steady-state queueing distribution has a subexponential tail; i.e., the exponential decay rate is $n^{2(1-H)}$ rather than $n$.

It is known that the FBM is a *self-similar* process, which exhibits long-range dependence (e.g., Mandelbrot and van Ness [18]). Recall that a stochastic process $\{X(t)\}$ is termed self-similar if the increment process, $\{X(t+\tau) - X(t)\}$, is equal in distribution to $\{a^{-H}[X(t+a\tau) - X(t)]\}$, for any $t$ and any $a > 0$, where $H \geq 0$ is the Hurst parameter. A well-know example of a self-similar process, FBM, denoted $B_H$, admits the representation ([18]):

$$B_H(t) = b_0 + \frac{1}{\Gamma(H+\frac{1}{2})} \cdot \{ \int_0^t (t-s)^{H-1/2} dB(s) + \int_{-\infty}^0 [(t-s)^{H-1/2} - (-s)^{H-1/2}] dB(s) \}, \quad (1.1)$$

where $b_0 = B_H(0)$ is the initial state, $B(t)$ denotes the standard Brownian motion, and $0 < H < 1$ is the Hurst parameter. Also following [18], there is another FBM model, due to Lévy ([17]):

$$B_H(t) = \frac{1}{\Gamma(H+\frac{1}{2})} \cdot \int_0^t (t-s)^{H-1/2} dB(s). \quad (1.2)$$

Note that when $H = 1/2$, the FBM is reduced to the standard Brownian motion. On the other hand, when $H \in (\frac{1}{2}, 1)$, the FBM is known to have long-range dependence (refer to [8] and [18]). Also note from the above representations, (1.2) in particular, that the FBM can be viewed as the output of a time-invariant linear filter with the impulse response $h(t) = t^{H-1/2}$ subject to an input that is Brownian motion. (Similar types of linear filter are widely used in modeling stochastic systems; see, e.g., Wong and Hajek [25].)

In view of the above, our idea here is to use the type of filters in (1.1) and (1.2) as basic models to generate processes with long-range dependence. Specifically, we will replace $B$ above by a stochastic process with short-range dependence, denoted $X$, and denote the resulting output of the filter as $Y$ (which replaces $B_H$ above). The output process $Y$ will then possess long-range dependence. (This can be checked by making use of the autocovariance function; refer to Section 3.) Moreover, a moderate excursion of $X$ incurs a large excursion of $Y$ through the amplifying effect of the integral. This way, we will have a systematic way to generate a broad class of processes with long-range dependence.

We will show that when $X$ satisfies a MDP, so does $Y$; and that a wide class of processes qualify as candidates for $X$ (i.e., satisfying a MDP). Examples include sample path processes corresponding to Markov chains,

$m$-dependent processes, and moving average processes. Indeed, we give a general criterion by which the MDP of $X$, when $X$ belongs to a certain class of processes, may be verified.

Next, we study queues, via conditional limit theorems, that are fed with the long-range dependent input process $Y$. We characterize the transient queueing behavior; in particular, how large queues build up. In contrast to the large queue behavior in the case of queues with short-range dependent input (refer to the conditional limit theorems in Anantharam [1], Dembo and Zajic [9] and Chang [4]), the most likely path to a large queue build-up here is *nonlinear*. Finally, we discuss steady-state results, in terms of the maximum of the associated random walk, and compare them with the results of [11].

We now briefly comment on the individual merits of the two models based on (1.1) and (1.2). The principal advantage of a filter based on (1.2), referred to below as the first model, is the bounded interval $[0, t]$, which leads to a direct application of moderate deviations techniques. The disadvantage is that it does not preserve the stationary increment property, i.e., when $X$ has stationary increments, $Y$ need not have stationary increments. On the other hand, a filter based on (1.1), the second model, does preserve stationary increments, but the unbounded interval $(-\infty, t]$ requires necessary extensions of the MDP theory.

The rest of the article is organized as follows. In Section 2 we develop a general criterion for $X$ to satisfy a MDP, and then derive the MDP of $Y$ based on the MDP of $X$ in Section 3. Queueing applications are presented in Section 4 and Section 5 for transient and steady-state results, respectively. In each of these sections, we start with and focus on the first model, and then present the analogous results for the second model.

This article is intended as a highlight of our recent work in [6, 7]; as such, the emphasis is on conveying the key ideas and presenting the main results. Results of a secondary or intermediate nature and their proofs are omitted. The reader is advised to consult [6, 7] for full details.

## 2 Moderate Deviations: A General Criterion

First recall the definition of the large deviations principle (LDP) on a metric space $\mathcal{S}$ equipped with the topology generated by the metric and the corresponding Borel $\sigma$-field. (Dembo and Zeitouni [10] should be consulted for more details.)

**Definition 2.1** A sequence of probability measures $\mu^{(n)}$ on a metric space $\mathcal{S}$ satisfies the LDP with speed $\delta_n \uparrow \infty$ and a good rate function $I(\cdot) : \mathcal{S} \to [0, \infty]$ if

**(i)** the level sets $\{x \in \mathcal{S} : I(x) \leq \alpha\}$, $\alpha \in \mathbb{R}$, are compact,

**(ii)** for any closed set $F$,

$$\limsup_{n\to\infty} \frac{1}{\delta_n} \log \mu^{(n)}(F) \leq - \inf_{x\in F} I(x),$$

**(iii)** for any open set $G$,

$$\liminf_{n\to\infty} \frac{1}{\delta_n} \log \mu^{(n)}(G) \geq - \inf_{x\in G} I(x).$$

A sequence of random variables is said to satisfy the LDP if the corresponding sequence of distributions does. Furthermore, when the space $\mathcal{S}$ is realized as a space of functions, such as the space of continuous functions on $[0,1]$, we refer to the LDP as a *sample path* LDP, or LDP(sp).

We say that a sequence $\mu^{(n)}$ satisfies a moderate deviations principle (MDP), if $\mu^{(n)}$ satisfies the LDP with speed $\gamma_n^2$, where $\gamma_n$ satisfies

$$\lim_{n\to\infty} \frac{\gamma_n}{\sqrt{n}} = 0 \quad \text{and} \quad \lim_{n\to\infty} \gamma_n = \infty. \tag{1.3}$$

Analogous to LDP(sp), we shall use MDP(sp) when the space $\mathcal{S}$ is realized as a space of functions. Allowing that the definition of $\mu^{(n)}$ may depend on $\gamma_n$, in case $\mu^{(n)}$ satisfies the MDP (resp. MDP(sp)) for every sequence satisfying (1.3), we say $\mu^{(n)}$ satisfies the *complete* MDP (resp. MDP(sp)).

Throughout, we shall refer to the following function spaces. Denote by $\mathcal{C}[a,b]$, $\mathcal{D}[a,b]$ and $\mathcal{AC}[a,b]$, the spaces of real-valued functions defined on the interval $[a,b]$ which are continuous, right continuous with left limits and absolutely continuous, respectively. Each of these spaces is equipped with the supremum norm

$$\|\phi\|_\infty = \sup_{a\leq t\leq b} |\phi(t)|$$

and the topology and Borel $\sigma$-field generated by this norm. In addition, we shall consider the spaces $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{AC}$ of real-valued functions defined on $(-\infty,1]$ which are continuous, right continuous with left limits, and absolutely continuous, respectively. These latter spaces are equipped with the metric of uniform convergence on compact sets

$$d(\phi,\psi)_\infty = \sum_{n=1}^{\infty} \frac{2^{-n} \sup_{t\in[-n,1]} |\phi(t)-\psi(t)|}{1+\sup_{t\in[-n,1]} |\phi(t)-\psi(t)|} .$$

and the topology and Borel $\sigma$-field generated by this metric.

A starting point for the main results in the following sections is that a certain process — specifically, a scaled version of the input process to a filter — satisfies a MDP(sp). We now describe a class of processes for which a useful criterion may be applied to verify whether a MDP(sp) is satisfied.

We do so for the setting of both filters mentioned in the Introduction, beginning with the first one in mind.

Consider a sequence of random variables in $I\!R$, $\{a(t), t = 0, 1, 2, \ldots\}$. Let $A(t_1, t_2) = \sum_{t=t_1+1}^{t_2} a(t)$ and define a sequence of stochastic processes as follows:

$$A^{(n)}(t) = \frac{1}{n} A(0, [nt]), \qquad t \in [0, 1]. \tag{1.4}$$

Denote by $\nu_A^{(n)}$ the distribution of the process $\{\frac{\sqrt{n}}{\gamma_n} A^{(n)}(t), 0 \le t \le 1\}$.

When $\{a(t), t = 0, 1, 2, \ldots\}$ is a sequence of *bounded* i.i.d. random variables with $Ea(0) = 0$ and $Ea^2(0) = \sigma^2 > 0$, it is known from Mogulskii [19], Theorem 1, that the sequence $\nu_A^{(n)}$ satisfies the complete MDP(sp) in $\mathcal{D}[0, 1]$ with good rate function

$$I(\phi) = \begin{cases} \int_0^1 \frac{1}{2\sigma^2} (\phi'(t))^2 dt & \text{if } \phi \in \mathcal{AC}[0, 1],\ \phi(0) = 0 \\ \infty & \text{otherwise .} \end{cases} \tag{1.5}$$

The following theorem provides a useful criterion for verifying that the *sample path process*, (1.4), of a given sequence $a(t)$ satisfies the complete MDP(sp) when $a(t)$ is not necessarily i.i.d.

**Theorem 2.2** Let $\{a(t), t = 0, 1, 2, \ldots\}$ be a stationary sequence of bounded, mean zero, real-valued random variables. For a fixed integer $m$, and $0 = t_0 < t_1 < \cdots < t_m \le 1$, let

$$\begin{aligned} Z_n \quad = \quad & (\frac{\sqrt{n}}{\gamma_n} A^{(n)}(t_1), \frac{\sqrt{n}}{\gamma_n}(A^{(n)}(t_2) - A^{(n)}(t_1)), \\ & \ldots, \frac{\sqrt{n}}{\gamma_n}(A^{(n)}(t_m) - A^{(n)}(t_{m-1}))). \end{aligned}$$

Suppose that for each positive integer $m$ and partition $\{t_i\}_{i=1}^m$, the sequence $\{Z_n\}$ satisfies the complete MDP in $I\!R^m$ with good rate function

$$I_m(z) = \sum_{i=1}^m \frac{1}{2\sigma^2} \frac{z_i^2}{(t_i - t_{i-1})}\ , \tag{1.6}$$

for some $\sigma^2 > 0$, where $z = (z_1, \ldots, z_m)$. In addition, assume that

$$\limsup_{n \to \infty} \frac{1}{\gamma_n^2} \log \mathsf{E}[\exp(\frac{\gamma_n \sum_{i=1}^n a(i)}{\sqrt{n}})] < \infty \tag{1.7}$$

for every sequence $\gamma_n$ satisfying (1.3). Then the sequence $\nu_A^{(n)}$ satisfies the complete MDP(sp) in $\mathcal{D}[0, 1]$ with the good rate function in (1.5).

The above theorem can be proved by mimicking the proof of Theorem 5.1.2 of Dembo and Zeitouni [10].

In the setting of the second model, we proceed as follows. Consider a sequence of random variables in $I\!R$, $\{a(t), t = \ldots, -2, -1, 0, 1, 2, \ldots\}$. Let $A(t_1, t_2) = \sum_{t=t_1+1}^{t_2} a(t)$ and define a sequence of stochastic processes as follows:

$$A^{(n)}(t) = \begin{cases} \frac{1}{n}A(0,[nt]), & t \in [0,1] \\ \frac{1}{n}A([nt],0), & t \in (-\infty, 0] \end{cases} .$$

Denote by $\nu_A^{(n)}$ the distribution of $\{\frac{\sqrt{n}}{\gamma_n}A^{(n)}(t),\ -\infty < t \leq 1\}$.

The following is then the counterpart of Theorem 2.2:

**Theorem 2.3** Let $\{a(t), t = \ldots, -2, -1, 0, 1, 2, \ldots\}$ be a stationary sequence of bounded, mean zero, real-valued random variables. For a fixed integer $m$ and $t_0 < t_1 < \cdots < t_m \leq 1$, let

$$\begin{aligned} Z_n \quad &= \quad \Big(\frac{\sqrt{n}}{\gamma_n}[A^{(n)}(t_1) - A^{(n)}(t_0)], \frac{\sqrt{n}}{\gamma_n}[A^{(n)}(t_2) - A^{(n)}(t_1)], \\ &\qquad \ldots, \frac{\sqrt{n}}{\gamma_n}[A^{(n)}(t_m) - A^{(n)}(t_{m-1})]\Big). \end{aligned}$$

Suppose that for each positive integer $m$ and partition $\{t_i\}_{i=1}^m$, $\{Z_n\}$ satisfies the complete MDP in $I\!R^m$ with rate function given by (1.6) for some $\sigma^2 > 0$ and that (1.7) holds. Then the sequence $\nu_A^{(n)}$ satisfies the complete MDP(sp) in $\mathcal{D}$ with good rate function

$$I(\phi) = \begin{cases} \int_{-\infty}^1 \frac{1}{2\sigma^2}(\phi'(t))^2 dt & \text{if } \phi \in \mathcal{AC},\ \phi(0) = 0 \\ \infty & \text{otherwise .} \end{cases}$$

The proof of Theorem 2.3 is analogous to that of Theorem 2.2.

Below we present three classes of random processes that satisfy the hypotheses of both Theorem 2.2 and Theorem 2.3. We remark that the complete MDP(sp) in the space $\mathcal{D}[0,1]$ also holds for the sample path process of the process considered in the first example when the Markov chain there has arbitrary initial distribution (and hence is not stationary).

**Example 1.** Let $\{b(t), t = 0, \pm 1, \pm 2, \ldots\}$ be a stationary Markov chain taking values in a Polish space and satisfying the conditions of Theorem 2.1 of Wu [26]. Let, for $t = 0, \pm 1, \pm 2, \ldots$

$$a(t) = f(b(t)) - E_\mu[f(b(t))]$$

for any bounded real-valued function $f$, where $\mu$ is the stationary distribution of the chain. Here $\sigma^2 = E_\mu[a^2(0)] + 2\sum_{k=1}^\infty E_\mu[a(k)a(0)]$, which we require to be positive.

**Example 2.** Let $\{\xi(t), t = 0, \pm 1, \pm 2, \ldots\}$ be a sequence of bounded, mean zero, i.i.d. random variables and $\{b(t), t = 0, \pm 1, \pm 2, \ldots\}$ a sequence of real numbers which is absolutely summable. Let the sequence $\{a(t), t = 0, \pm 1, \pm 2, \ldots\}$ be the infinite moving average process defined by

$$a(t) = \sum_{i=-\infty}^{\infty} b(i+t)\xi(i).$$

Here $\sigma^2 = (\sum_{i=-\infty}^{\infty} b_i)^2 E[\xi^2(0)]$, which we require to be positive.

**Example 3.** Suppose $\{a(t), t = 0, \pm 1, \pm 2, \ldots\}$ is a stationary sequence of mean zero, bounded random variables which are *m-dependent*. By this we mean that the $\sigma$-fields $\sigma\{a(j); j \leq k\}$ and $\sigma\{a(j); j \geq k+m\}$ are independent for any integer $k$. Examples of such sequences include sequences $\{f(b(t)) - E[f(b(t))]\}$, where $f$ is a bounded real-valued function and $\{b(t)\}$ is a stationary Gaussian sequence that is either $\phi$-mixing or $\psi$-mixing (see, for example, Chapters 4 and 5 of [15]). Here $\sigma^2 = E[a^2(0)] + 2\sum_{k=1}^{m-1} E[a(k)a(0)]$, which we require to be positive.

## 3 MDP for the Filtered Processes

We first study the effect of feeding a process with short-range dependence into the following linear filter, our first model, which is based on (1.2):

$$Y(t) = \int_0^t (t-s)^{H-\frac{1}{2}} dX(s). \tag{1.8}$$

Specifically, given that $X$ satisfies a MDP, we want to show that $Y$, a process with long-range dependence, also satisfies a MDP.

As we mentioned in the Introduction, the advantage of the filter in (1.8) is the boundedness of the interval $[0, t]$, which allows the direct application of the MDP results of the last section. The disadvantage, however, is that the filter does not preserve the stationary increment property (of $X$). Neither does the filter preserve self-similarity: when $X$ is a self-similar process, $Y$ is only self-similar with respect to $t = 0$.

Since $Y$ does not have stationary increments, the autocovariance function (of its increments)

$$\mathsf{Cov}[Y(\tau+1) - Y(\tau),\, Y(t+\tau+1) - Y(t+\tau)]$$

depends on $\tau$ (as well as on $t$). However, we can still use the standard definition for long-range dependence (e.g., Cox [8]), namely, that the autocovariance function is not summable over $t$ (for any $\tau$). (For instance, when $X$ is the standard Brownian motion, one can verify that the above autocovariance is of order $t^{-(\frac{3}{2}-H)}$.)

When considering the first model we restrict ourselves to processes $X$ which are either Brownian motion or possess sample paths of bounded variation on finite intervals. When $X$ is a process with sample paths of bounded variation on finite intervals, the integral in (1.8) can be viewed as a sample path Lebesgue-Stieltjes integral (see e.g., Wong and Hajek [25]). Integration by parts on (1.8) yields

$$Y(t)=\int_0^t (H-\frac{1}{2})(t-s)^{H-\frac{3}{2}}X(s)ds, \tag{1.9}$$

since $X(0)=0$ and $H>\frac{1}{2}$. On the other hand, if $X=B$, the standard Brownian motion, applying Ito's formula (see, e.g., [25]) to (1.8) yields the same expression as above.

With this in mind, we may consider the transformation of $X$ into $Y$ given by either (1.8) or (1.9). Letting $Y^{(n)}(t)=Y(nt)/n$ and $X^{(n)}(t)=X(nt)/n$, a simple change of variable ($s=n\tau$) yields

$$Y^{(n)}(t)=\int_0^t (H-\frac{1}{2})(t-\tau)^{H-\frac{3}{2}}(n^{H-\frac{1}{2}}X^{(n)}(\tau))d\tau. \tag{1.10}$$

Using the relation (1.10), we derive below a MDP for $Y$ and related conditional limit theorems, assuming that $X$ satisfies the MDP.

**Theorem 3.1** Suppose, for some $\frac{1}{2}<H<1$, the probability law of $\{n^{H-\frac{1}{2}}X^{(n)}(t), t\in[0,1]\}$, denoted $\mu_X^{(n)}$, satisfies the LDP(sp) on $\mathcal{D}[0,1]$ with speed $n^{2(1-H)}$ and a good rate function $I_X(\cdot):\mathcal{D}[0,1]\mapsto \mathbb{R}$,

$$I_X(\phi)=\begin{cases}\int_0^1 \frac{1}{2\sigma^2}(\phi'(t))^2dt & \text{if } \phi\in\mathcal{AC}[0,1], \phi(0)=0\\ \infty & \text{otherwise}\end{cases}.$$

Then, $Y(n)/n$ satisfies the LDP with speed $n^{2(1-H)}$ and a good rate function $\Lambda_Y^*(\gamma):\mathbb{R}\mapsto\mathbb{R}$,

$$\Lambda_Y^*(\gamma)=\frac{H}{\sigma^2}\gamma^2.$$

**Theorem 3.2** Suppose the conditions in Theorem 3.1 are in force.

**(i)** Let $\tilde{\mu}_X^{(n)}$ denote the law of $\{n^{H-\frac{1}{2}}X^{(n)}(\cdot)|Y^{(n)}(1)\geq\gamma\}$ for some $\gamma\geq 0$. Then, $\tilde{\mu}_X^{(n)}$ converges in distribution to $\delta_{\phi_\gamma(\cdot)}$, where

$$\phi_\gamma(t)=\gamma\frac{2H}{H+\frac{1}{2}}(1-(1-t)^{H+\frac{1}{2}}),\ \ t\in[0,1].$$

**(ii)** Let $\tilde{\mu}_Y^{(n)}$ denote the law of $\{Y^{(n)}(\cdot)|Y^{(n)}(1)\geq\gamma\}$. Then $\tilde{\mu}_Y^{(n)}$ converges in distribution to $\delta_{\psi_\gamma(\cdot)}$, where, for $t\in[0,1]$,

$$\psi_\gamma(t)=2H\gamma\int_0^t (t-s)^{H-\frac{1}{2}}(1-s)^{H-\frac{1}{2}}ds.$$

In proving Theorem 3.1, we make use of Lemma 3.3 below, which is readily proved via the contraction principle (Dembo and Zeitouni [10], Theorem 4.2.1). The proof of Theorem 3.1 then yields the most likely path for the input process in Theorem 3.2(i). The most likely path for $Y$ is then simply the output of the filter with the most likely input path.

**Lemma 3.3** Suppose the conditions in Theorem 3.1 hold. Then, the distribution of $\{Y^{(n)}(t), t \in [0,1]\}$, denoted $\mu_Y^{(n)}$, satisfies the LDP(sp) with speed $n^{2(1-H)}$ and a good rate function $I_Y(\psi) : \mathcal{C}[0,1] \mapsto I\!R$,

$$I_Y(\psi) = \inf\{I_X(\phi) : \psi(t) = \int_0^t (t-s)^{H-\frac{1}{2}}\phi'(s)ds,\ t \in [0,1]\}.$$

**Proof.** [Theorem 3.1]
Note that $Y^{(n)}(1) = Y(n)/n$ and the continuity of the mapping $\psi \mapsto \psi(1)$ from $\mathcal{C}[0,1]$ to $I\!R$. It then follows from the contraction principle and Lemma 3.3 that the distribution of $Y^{(n)}(1)$ satisfies the LDP with speed $n^{2(1-H)}$ and a good rate function

$$\begin{aligned}\Lambda_Y^*(\gamma) &= \inf\{I_Y(\psi) : \psi(1) = \gamma\} \\ &= \inf\{I_X(\phi) : \int_0^1 (1-s)^{H-\frac{1}{2}}\phi'(s)ds = \gamma\}.\end{aligned}$$

¿From the definition of $I_X(\phi)$ (in Theorem 3.1), the infimum above is tantamount to solving the following minimization problem:

$$\begin{aligned}\min_\phi \quad & \int_0^1 \frac{1}{2\sigma^2}(\phi'(s))^2 ds \\ \text{s.t.} \quad & \int_0^1 (1-s)^{H-\frac{1}{2}}\phi'(s)ds = \gamma, \quad \phi(0) = 0.\end{aligned}$$

Using that $\int_0^1 (1-s)^{2(H-\frac{1}{2})}ds = 1/(2H)$, it may be shown that

$$\Lambda_Y^*(\gamma) = \frac{H}{\sigma^2}\gamma^2.$$

Moreover, the infimum is uniquely achieved by

$$\phi_\gamma'(s) = \gamma(2H)(1-s)^{H-\frac{1}{2}},$$

the derivative of the function appearing in the statement of Theorem 3.2. ■

**Proof.** [Theorem 3.2]

(i) Denote
$$B_\epsilon = \{\phi \in \mathcal{D}[0,1] : ||\phi - \phi_\gamma||_\infty \geq \epsilon\},$$
for some $\epsilon > 0$ (with $\phi_\gamma$ following the definition in Theorem 3.2), and
$$\Gamma_D = \{\phi \in \mathcal{D}[0,1] : \int_0^1 (H-1/2)(1-s)^{H-\frac{3}{2}}\phi(s)ds \geq \gamma\}.$$
It is enough to show that for any given $\epsilon > 0$ we have
$$\lim_{n\to\infty} \tilde{\mu}_X^{(n)}(B_\epsilon) = \lim_{n\to\infty} \frac{\mu_X^{(n)}(B_\epsilon \cap \Gamma_D)}{\mu_X^{(n)}(\Gamma_D)} = 0. \tag{1.11}$$
Since $\mu_X^{(n)}(\Gamma_D) = \mathsf{P}[Y(n)/n \geq \gamma]$, it follows from Theorem 3.1 and $\gamma \geq 0$ that
$$\lim_{n\to\infty} \frac{1}{n^{2(1-H)}} \log \mu_X^{(n)}(\Gamma_D) = -\Lambda_Y^*(\gamma). \tag{1.12}$$
On the other hand, from the LDP(sp) upper bound for $\mu_X^{(n)}$, we have
$$\lim_{n\to\infty} \frac{1}{n^{2(1-H)}} \log \mu_X^{(n)}(B_\epsilon \cap \Gamma_D) \leq -\inf_{\phi \in B_\epsilon \cap \Gamma_D} I_X(\phi)\ . \tag{1.13}$$
Combining (1.12) and (1.13), and making use of Lemma 2.8 of Shwartz and Weiss [21], we have, for some $\delta > 0$,
$$\limsup_{n\to\infty} \frac{1}{n^{2(1-H)}} \log \Big(\frac{\mu_X^{(n)}(B_\epsilon \cap \Gamma_D)}{\mu_X^{(n)}(\Gamma_D)}\Big) \leq -\delta^2,$$
which clearly implies (1.11).
(ii) Using the continuity of the mapping appearing in (1.9), the desired weak convergence follows from (i). ∎

We now consider the second model, the filter based on (1.1):
$$\begin{aligned} Y(t) &= \int_0^t (t-s)^{H-1/2} dX(s) \\ &\quad + \int_{-\infty}^0 [(t-s)^{H-1/2} - (-s)^{H-1/2}] dX(s)\ . \end{aligned} \tag{1.14}$$

Before going further, we make precise the processes we shall consider as possible inputs, $X$, to the filter (1.14). In order to allow our results to apply to FBM we allow $X$ to be a standard Brownian motion. Aside from this, we restrict ourselves to sample path processes of the type considered in Section 2.

Similar to the first model, given the process $X(t)$, letting $X^{(n)}(t) = X(nt)/n$ and $Y^{(n)}(t) = Y(nt)/n$, where $Y$ relates to $X$ through (1.14), we have

$$\begin{aligned} Y^{(n)}(t) &= \int_0^t (t-s)^{H-1/2} d(n^{H-1/2} X^{(n)}(s)) \\ &\quad + \int_{-\infty}^0 [(t-s)^{H-1/2} - (-s)^{H-1/2}] d(n^{H-1/2} X^{(n)}(s)) \; . \end{aligned}$$

Given this, we see that, in case $X$ is taken to be a sample path process, meaning needs to be given to the integrals appearing in the definition of each $Y^{(n)}$. We do so as follows. Let $Y_a^{(n)}$ denote the process $Y^{(n)}$ modified so that the second integral begins at $a$ instead of $-\infty$ and note the this process takes value in $C[0,1]$ and, moreover, belongs to $L^2[C[0,1]]$, i.e., $E[(\sup_{t\in[0,1]} Y_a^{(n)}(t))^2] < \infty$. We require of the process $X$ that, for each $n = 1, 2, \ldots$ fixed, the sequence $\{Y_{-j/n}^{(n)}\}_{j=1}^{\infty}$ be a Cauchy sequence in $L^2[C[0,1]]$. We then take $Y^{(n)}$ to be the limit of this sequence.

We shall also require the assumption that, for any $\eta > 0$

$$\begin{aligned} &\lim_{m\to\infty} \limsup_{n\to\infty} \frac{1}{n^{2(1-H)}} \log \mathsf{P}\{ \sup_{t\in[0,1]} | \int_{-\infty}^{-m} [(t-s)^{H-1/2} - (-s)^{H-1/2}] \\ &\qquad d(n^{H-1/2} X^{(n)}(s))| \geq \eta \} = -\infty \; . \end{aligned} \tag{1.15}$$

We remark that this assumption holds when $X$ is Brownian motion. In addition, it can be shown that the processes $Y^{(n)}$ exist in the sense specified in the previous paragraph and that (1.15) holds when $X$ is a sample path process of the type considered in the examples of Section 2, with the following additional conditions being satisfied for the first two examples. For Example 1, it suffices to ask that the Markov chain be *reversible* and that there exists a bounded solution, $h$ say, of the equation $Ph - h = f$, i.e. Poisson's equation, where $P$ is the transition kernel of the chain. For Example 2, we ask that

$$\sum_{k=1}^{\infty} ( \sum_{j=-\infty}^{-k} |b(j)| + \sum_{j=k}^{\infty} |b(j)|) < \infty \; .$$

The following two theorems are the analogues of Theorem 3.1 and Theorem 3.2, respectively:

**Theorem 3.4** Suppose, for some $\frac{1}{2} < H < 1$, the probability law of $\{n^{H-\frac{1}{2}} X^{(n)}(t), t \in (-\infty, 1]\}$, denoted $\mu_X^{(n)}$, satisfies the LDP(sp) on $\mathcal{D}$ with speed $n^{2(1-H)}$ and a good rate function $I_X(\cdot) : \mathcal{D} \mapsto I\!R$,

$$I_X(\phi) = \begin{cases} \int_{-\infty}^1 \frac{1}{2\sigma^2} (\phi'(t))^2 dt & \text{if } \phi \in \mathcal{AC}, \phi(0) = 0 \\ \infty & \text{otherwise} \end{cases} .$$

Also, assume that (1.15) holds for any $\eta > 0$. Then, $Y(n)/n$ satisfies the LDP with speed $n^{2(1-H)}$ and a good rate function $\Lambda^*{}_Y(\gamma) : I\!R \mapsto I\!R$,

$$\Lambda^*{}_Y(\gamma) = \frac{\tilde{H}}{\sigma^2}\gamma^2,$$

where

$$\tilde{H} = \frac{1}{2}\Big[\int_0^1 (1-s)^{2(H-\frac{1}{2})}ds + \int_{-\infty}^0 [(1-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}]^2 ds\Big]^{-1}. \tag{1.16}$$

**Theorem 3.5** Suppose the conditions in Theorem 3.4 are in force.

**(i)** Let $\tilde{\mu}_X^{(n)}$ denote the law of $\{n^{H-\frac{1}{2}}X^{(n)}(\cdot)|Y^{(n)}(1) \geq \gamma\}$ for some $\gamma \geq 0$. Then, $\tilde{\mu}_X^{(n)}$ converges in distribution to $\delta_{\phi_\gamma(\cdot)}$, where

$$\phi_\gamma(t) = \begin{cases} \gamma\frac{2\tilde{H}}{H+\frac{1}{2}}(1-(1-t)^{H+\frac{1}{2}}) & t \in [0,1] \\ \gamma\frac{2\tilde{H}}{H+\frac{1}{2}}(1-(1-t)^{H+\frac{1}{2}} + (-t)^{H+1/2}) & t \in (-\infty, 0] \end{cases}.$$

**(ii)** Let $\tilde{\mu}_Y^{(n)}$ denote the law of $\{Y^{(n)}(\cdot)|Y^{(n)}(1) \geq \gamma\}$. Then $\tilde{\mu}_Y^{(n)}$ converges in distribution to $\delta_{\psi_\gamma(\cdot)}$, where, for $t \in [0,1]$,

$$\psi_\gamma(t) = \int_0^t (t-s)^{H-1/2}d\phi_\gamma(s) + \int_{-\infty}^0 [(t-s)^{H-1/2} - (-s)^{H-1/2}]d\phi_\gamma(s).$$

To facilitate the proof of Theorem 3.4, we require a lemma analogous to Lemma 3.3. In proving such a lemma we make use of condition (1.15). More precisely, for each fixed integer $m$, we apply the contraction principle to yield the LDP(sp) in $\mathcal{C}[0,1]$ for the sequence $Y_m^{(n)}$, where

$$\begin{aligned} Y_m^{(n)}(t) &= \int_0^t (t-s)^{H-1/2}d(n^{H-1/2}X^{(n)}(s)) \\ &+ \int_{-m}^0 [(t-s)^{H-1/2} - (-s)^{H-1/2}]d(n^{H-1/2}X^{(n)}(s)), \quad t \in [0,1], \end{aligned}$$

with a good rate function given by

$$\begin{aligned} I_Y^m(\psi) = \inf\{I_X(\phi) : \psi(t) &= \int_0^t (t-s)^{H-1/2}\phi'(s)ds \\ &+ \int_{-m}^0 [(t-s)^{H-1/2} - (-s)^{H-1/2}]\phi'(s)ds,\ t \in [0,1]\}. \end{aligned}$$

Furthermore, by (1.15), the $Y_m^{(n)}$ are exponentially good approximations of $Y^{(n)}$ (cf. [10, Definition 4.2.14]), and therefore [10, Theorem 4.2.23] allows to conclude the following lemma.

**Lemma 3.6** Suppose the conditions in Theorem 3.4 hold. Then, the distribution of $\{Y^{(n)}(t), t \in [0,1]\}$, denoted $\mu_Y^{(n)}$, satisfies the LDP(sp) with speed $n^{2(1-H)}$ and good rate function $I_Y(\psi) : \mathcal{C}[0,1] \mapsto I\!R$,

$$I_Y(\psi) = \inf\{I_X(\phi) : \psi(t) = \int_0^t (t-s)^{H-1/2}\phi'(s)ds + \int_{-\infty}^0 [(t-s)^{H-1/2} - (-s)^{H-1/2}]\phi'(s)ds,\ t \in [0,1]\}.$$

**Proof.** [Theorem 3.4]
Given Lemma 3.6, the proof may be carried out as was that of Theorem 3.1. ■

**Proof.** [Theorem 3.5]
(i) As in the proof of Theorem 3.2 (i), it suffices to show

$$\inf_{\phi \in B_\epsilon \cap \Gamma} \int_{-\infty}^1 \frac{1}{2\sigma^2}(\phi'(s))^2 ds > \Lambda^*{}_Y(\gamma)\ .$$

for all $\epsilon > 0$, where

$$B_\epsilon = \{\phi \in \mathcal{D} : d(\phi, \phi_\gamma)_\infty \geq \epsilon\},$$

and

$$\begin{aligned}\Gamma &= \{\phi \in \mathcal{AC} : \int_0^1 (1-s)^{H-\frac{1}{2}}\phi'(s)ds \\ &\quad + \int_{-\infty}^0 [(1-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}]\phi'(s)ds \geq \gamma\}.\end{aligned}$$

That this is the case follows as in the proof of Theorem 3.2.
(ii) Note that the map $\tilde{F}^m : \mathcal{D} \to \mathcal{C}[0,1]$, defined by

$$\begin{aligned}\tilde{F}^m(\phi)(t) &= \int_0^t (H-1/2)(t-s)^{H-3/2}\phi(s)ds \\ &\quad + \int_{-m}^0 (H-1/2)[(t-s)^{H-3/2} - (-s)^{H-3/2}]\phi(s)ds \\ &\quad -((t+m)^{(H-1/2)} - m^{(H-1/2)})\phi(m)\ , \qquad (1.17)\end{aligned}$$

is continuous. Therefore the law of $\{\tilde{F}^m(n^{H-1/2}X^{(n)}(\cdot))|Y^{(n)}(1) \geq \gamma\}$ converges in distribution to $\tilde{F}^m(\phi_\gamma(\cdot))$. Recalling that $\{X(t)\}$ has sample paths of bounded variation on finite intervals, we conclude that the law of $\{F^m(n^{H-1/2}X^{(n)}(\cdot))|Y^{(n)}(1) \geq \gamma\}$ converges in distribution to $F^m(\phi_\gamma(\cdot))$, where

$$\begin{aligned} F^m(\phi)(t) &= \int_0^t (t-s)^{H-1/2} d\phi(s) \\ &\quad + \int_{-m}^0 [(t-s)^{H-1/2} - (-s)^{H-1/2}] d\phi(s) . \qquad (1.18) \end{aligned}$$

To see this note that (1.17) and (1.18) are equivalent, via integration by parts, when $\phi$ is of bounded variation. Using (1.15) then allows us to conclude the desired result. ■

# 4 Queueing Applications: Transient Results

Consider a queue with an input process $\{Y(t)\}$, which, in turn, is the output of a linear filter subject to the input process $\{X(t)\}$, following (1.8) or (1.14). Hence in general, the input to the queue (the $Y$ process) has long-range dependence.

We start with the first model and assume that the conditions in Theorem 3.1 are in force. Consider $V(t) = \sup_{0 \leq s \leq t}(Y(t) - Y(s) - c(t-s))$. Note that $V(t)$ is the *state* process of a queue with input $Y$ and a deterministic service mechanism that depletes work at rate $c$. In queueing literature, $V(t)$ is often referred to as the virtual waiting time process, or the workload/inventory process. See, for instance, Chang [3], Duffield and O'Connell [11], Harrison [14], and Sigman and Yao [22], among others.

Let $V^{(n)}(t) = V(nt)/n$; in particular, $V^{(n)}(1) = V(n)/n$.

**Theorem 4.1** The process $\{V(n)/n, n \geq 0\}$ satisfies the LDP with speed $n^{2(1-H)}$ and good rate function $\Lambda_V^*(\gamma) : I\!R \mapsto I\!R$,

$$\Lambda_V^*(\gamma) = \frac{H}{\sigma^2} \inf_{0 \leq s \leq 1} \frac{(c(1-s)+\gamma)^2}{v(s) + (1-s)^{2H}}, \qquad (1.19)$$

where

$$v(s) = 2H \int_0^s [(1-\tau)^{H-\frac{1}{2}} - (s-\tau)^{H-\frac{1}{2}}]^2 d\tau.$$

**Theorem 4.2** (i) Let $\hat{\mu}_X^{(n)}$ denote the law of $\{n^{H-\frac{1}{2}}X^{(n)}(\cdot)|V^{(n)}(1) \geq \gamma\}$ for some $\gamma \geq 0$. Suppose $s^*$ achieves the minimum in (1.19). Then, $\hat{\mu}_X^{(n)}$

converges in distribution to $\delta_{\hat{\phi}_\gamma(\cdot)}$, where

$$\hat{\phi}_\gamma(t) = \begin{cases} 2H\frac{c(1-s^*)+\gamma}{v(s^*)+(1-s^*)^{2H}} \int_0^t [(1-\tau)^{H-\frac{1}{2}} - (s^*-\tau)^{H-\frac{1}{2}}]d\tau & \text{if } 0 \le t \le s^* \\ \hat{\phi}_\gamma(s^*) + 2H\frac{c(1-s^*)+\gamma}{v(s^*)+(1-s^*)^{2H}} \int_{s^*}^t (1-\tau)^{H-\frac{1}{2}} d\tau & \text{if } s^* < t \le 1 \end{cases}.$$

(ii) Let $\hat{\mu}_V^{(n)}$ denote the law of $\{V^{(n)}(\cdot) | V^{(n)}(1) \ge \gamma\}$. Then $\hat{\mu}_V^{(n)}$ converges in distribution to $\delta_{\hat{\psi}_\gamma(\cdot)}$, where, for $t \in [0,1]$,

$$\hat{\psi}_\gamma(t) = \sup_{0 \le s \le t} [\int_0^t (t-\tau)^{H-1/2} d\hat{\phi}_\gamma(\tau) - \int_0^s (s-\tau)^{H-1/2} d\hat{\phi}_\gamma(\tau) - c(t-s)] .$$

Note that the conditional limit in Theorem 4.2 (ii), which illustrates how the queue builds up when the input has long-range dependence, is qualitatively different from the more standard case of input with short-range dependence (e.g., [1, 4, 9]), where the most likely path is linear. Here, the most likely path is *nonlinear*.

**Proof.** [Theorem 4.1]
Using the continuity of the reflection mapping,

$$\psi(t) = \sup_{0 \le s \le t} \{\phi(t) - \phi(s) - c(t-s)\}, \quad t \in [0,1],$$

it follows from the contraction principle that the distribution of $V^{(n)}(1)$ satisfies the LDP with speed $n^{2(1-H)}$ and a good rate function

$$\Lambda_V^*(\gamma) = \inf\{I_X(\phi) : \phi \in \hat{F}\},$$

where

$$\begin{aligned} \hat{F} &= \{\phi \in \mathcal{AC} : \sup_{0 \le s \le 1} [\int_0^1 (1-\tau)^{H-\frac{1}{2}} \phi'(\tau) d\tau \\ &\quad - \int_0^s (s-\tau)^{H-\frac{1}{2}} \phi'(\tau) d\tau - c(1-s)] = \gamma\}. \end{aligned}$$

It remains to further identify the rate function. Let

$$\hat{F} = \cup_{\{0 \le s \le 1\}} \hat{F}_s ,$$

where

$$\begin{aligned} \hat{F}_s &= \{\phi \in \mathcal{AC} : \phi(0) = 0, \int_0^s [(1-\tau)^{H-\frac{1}{2}} - (s-\tau)^{H-\frac{1}{2}}] \phi'(\tau) d\tau \\ &\quad + \int_s^1 (1-\tau)^{H-\frac{1}{2}} \phi'(\tau) d\tau = c(1-s) + \gamma\} . \end{aligned}$$

It may be shown that

$$\inf_{\phi\in\hat{F}_s}\int_0^1 \frac{1}{2\sigma^2}(\phi'(\tau))^2 d\tau = \frac{H}{\sigma^2}\frac{(c(1-s)+\gamma)^2}{v(s)+(1-s)^{2H}},$$

where

$$v(s) = 2H\int_0^s [(1-\tau)^{H-\frac{1}{2}} - (s-\tau)^{H-\frac{1}{2}}]^2 d\tau.$$

Furthermore, the minimum is uniquely achieved by

$$\hat{\phi}'_\gamma(\tau) = \begin{cases} 2H\frac{c(1-s)+\gamma}{v(s)+(1-s)^{2H}}[(1-\tau)^{H-\frac{1}{2}} - (s-\tau)^{H-\frac{1}{2}}] & \text{if } 0\le\tau<s \\ 2H\frac{c(1-s)+\gamma}{v(s)+(1-s)^{2H}}(1-\tau)^{H-\frac{1}{2}} & \text{if } s\le\tau\le 1 \end{cases}.$$

Given the above, the representation (1.19) for the rate function is clear. ■

To prove Theorem 4.2, we need the following lemma (proved in [6]).

**Lemma 4.3** The solution to the minimization problem in (1.19), $s^*$, is unique.

**Proof.** [Theorem 4.2]
As in the proof of Theorem 3.2 and Theorem 3.5, it suffices to show that

$$\inf_{\phi\in B_\epsilon\cap\hat{F}} I_X(\phi) > \Lambda^*_V(\gamma) ,$$

for all $\epsilon > 0$, where

$$B_\epsilon = \{\phi\in\mathcal{D}[0,1] : ||\phi-\hat{\phi}_\gamma||_\infty \ge \epsilon\} ,$$

and $\hat{F}$ is defined as in the proof of Theorem 4.1. The proof may be carried out as were those of these earlier theorems.
(ii) The result follows from the fact that the mappings under consideration are continuous. ■

For the second model, i.e., when $Y$ follows (1.14), the following theorem replaces the transient results in Theorem 4.2, and the proof is analogous. The counterpart of Theorem 4.1 is presented in the next section. The statement of the following theorem makes use of the quantity $s^*$ given by

$$s^* = \begin{cases} -\frac{H\gamma}{c(1-H)} & \text{if } \gamma < c(1-H)/H \\ -1 & \text{otherwise} \end{cases}.$$

It is assumed that the process $X$ satisfies the conditions of Theorem 3.4.

**Theorem 4.4** (i) Let $\hat{\mu}_X^{(n)}$ denote the law of $\{n^{H-\frac{1}{2}}X^{(n)}(\cdot)|V^{(n)}(1) \geq \gamma\}$ for some $\gamma \geq 0$. Then, $\hat{\mu}_X^{(n)}$ converges in distribution to $\delta_{\hat{\phi}_\gamma}$, where $\hat{\phi}_\gamma(t) = \phi_\gamma(t-1) - \phi_\gamma(-1)$ and

$$\phi_\gamma(t) =$$
$$\begin{cases} -\frac{2\tilde{H}(\gamma-cs^*)}{|s^*|^{2H}}\int_t^0(-\tau)^{H-1/2}d\tau & \text{if } s^* \leq t \leq 0 \\ \phi_\gamma(s^*) + \frac{2\tilde{H}(\gamma-cs^*)}{|s^*|^{2H}}\int_t^{s^*}[(s-\tau)^{H-1/2}-(-\tau)^{H-1/2}]d\tau & \text{if } -\infty < t \leq s^* \end{cases}.$$

(ii) Let $\hat{\mu}_V^{(n)}$ denote the law of $\{V^{(n)}(\cdot)|V^{(n)}(1) \geq \gamma\}$. Then $\hat{\mu}_V^{(n)}$ converges in distribution to $\delta_{\hat{\psi}_\gamma(\cdot)}$, where, for $t \in [0,1]$,

$$\hat{\psi}_\gamma(t) = \sup_{0\leq s\leq t}[\int_0^t(t-\tau)^{H-1/2}d\hat{\phi}_\gamma(\tau)+\int_{-\infty}^0[(t-\tau)^{H-1/2}-(-\tau)^{H-1/2}]d\hat{\phi}_\gamma(\tau)$$

$$-\int_0^s(s-\tau)^{H-1/2}d\hat{\phi}_\gamma(\tau)-\int_{-\infty}^0[(s-\tau)^{H-1/2}-(-\tau)^{H-1/2}]d\hat{\phi}_\gamma(\tau)-c(t-s)].$$

# 5 Queueing Applications: Steady-State Results

In queueing theory, the steady-state limit, $V(\infty)$, is typically obtained through the limit of another process, $\tilde{M}(t) = \sup_{0\leq s\leq t}(-Y(-s) - cs)$, the maximum of a random walk. The key linkage between the two is the stationary increment property of the process $Y$, which guarantees that for each $t$, $V(t)$ and $\tilde{M}(t)$ are equal in distribution, and hence so are $V(\infty)$ and $\tilde{M}(\infty)$. This fact is crucial in characterizing the steady-state limit, since whereas $\tilde{M}(t)$ is an increasing process (in $t$), $V(t)$ is not; and short of the linkage to $\tilde{M}(\infty)$, there is no easy way to even ensure the existence of $V(\infty)$.

The filter in our first model unfortunately does not yield a process, $Y$, with stationary increments even when $X$ has stationary increments. Hence, in this context, we cannot claim that $\tilde{M}(\infty)$ gives a characterization of the steady-state limit of the queueing process. We do, however, in Theorem 5.1 below, consider a process, $M(t)$, in the setting of the first model whose definition is analogous to that of $\tilde{M}(t)$. Like the process $\tilde{M}(t)$, considered in Theorem 5.4 below, associated to the process $M(t)$ is a limiting random variable, $M(\infty)$. As seen from the statements of Theorem 5.1 and Theorem 5.4, the random variables $M(\infty)$ and $\tilde{M}(\infty)$ both exhibit subexponential tail behavior. Moreover, the proofs of the two theorems are carried out in a similar fashion.

Throughout this section we assume that the process $X$ satisfies the conditions of Theorem 3.1 (resp. Theorem 3.4) when considering the first (resp. second) model.

**Theorem 5.1** With $M(t) = \sup_{0 \le s \le t}(Y(s) - cs)$ for some $c > 0$, $M(t)$ converges almost surely to a finite random variable $M(\infty)$, which has a subexponential tail distribution. Specifically,

$$\lim_{x \to \infty} \frac{1}{x^{2(1-H)}} \log \mathsf{P}[M(\infty) > x] = -\theta^*, \tag{1.20}$$

where

$$\theta^* = \frac{H}{\sigma^2} \frac{c^{2H}}{H^{2H}(1-H)^{2(1-H)}}. \tag{1.21}$$

To prove Theorem 5.1, we need two lemmas (the proofs are quite similar to the proof of Theorem 3.1).

**Lemma 5.2** For all $\gamma \ge 0$

$$\lim_{n \to \infty} \frac{1}{n^{2(1-H)}} \log \mathsf{P}[M(n)/n > \gamma] = -g(\gamma),$$

where

$$g(\gamma) = \begin{cases} \frac{H}{\sigma^2}(c+\gamma)^2 & \text{if } \gamma \ge \frac{c(1-H)}{H} \\ \frac{H}{\sigma^2} \frac{c^{2H}}{H^{2H}(1-H)^{2(1-H)}} \gamma^{2(1-H)} & \text{if } \gamma < \frac{c(1-H)}{H} \end{cases}.$$

**Lemma 5.3** Let $Z(n) = \sup_{n-1 < s \le n}(Y(s) - cs)$, $n = 1, 2, \ldots$. For all $\gamma \ge 0$,

$$\lim_{n \to \infty} \frac{1}{n^{2(1-H)}} \log \mathsf{P}[Z(n)/n > \gamma] = -\frac{H}{\sigma^2}(c+\gamma)^2.$$

**Proof.** [Theorem 5.1]
Since $M(t)$ is an increasing sequence, it converges almost surely to a random variable $M(\infty)$.

We prove the limit (1.20) by arguing that the liminf and the limsup are dominated from below and from above by the same bound.

To establish the lower bound, we follow the approach in Chang [3]. Note that for all $\gamma > 0$

$$\begin{aligned} & \liminf_{x \to \infty} \frac{1}{x^{2(1-H)}} \log \mathsf{P}[M(\infty) > x] \\ \ge \; & \liminf_{n \to \infty} \frac{1}{(\gamma n)^{2(1-H)}} \log \mathsf{P}[Y(n) - cn > \gamma n] \\ \ge \; & -\frac{H}{\sigma^2} \frac{(\gamma + c)^2}{\gamma^{2(1-H)}}, \end{aligned}$$

where we have used Theorem 3.1. Thus,

$$\begin{aligned}
&\liminf_{x\to\infty} \frac{1}{x^{2(1-H)}} \log \mathsf{P}[M(\infty) > x] \\
\geq \quad & -\inf_{\gamma>0} \frac{H}{\sigma^2} \frac{(\gamma+c)^2}{\gamma^{2(1-H)}} \\
= \quad & -\theta^*.
\end{aligned}$$

To establish the upper bound, note that for all $\gamma > 0$

$$\begin{aligned}
&\limsup_{x\to\infty} \frac{1}{x^{2(1-H)}} \log \mathsf{P}[M(\infty) \geq x] \\
= \quad & \limsup_{n\to\infty} \frac{1}{(\gamma n)^{2(1-H)}} \log \mathsf{P}[M(\infty) \geq \gamma n].
\end{aligned}$$

Write

$$\begin{aligned}
&\mathsf{P}[M(\infty) \geq \gamma n] \\
= \quad & \mathsf{P}[M(n) \geq \gamma n] + \mathsf{P}[M(n) < \gamma n, \ \sup_{s>n}(Y(s) - cs) \geq \gamma n] \\
\leq \quad & \mathsf{P}[M(n) \geq \gamma n] + \mathsf{P}[\sup_{m\geq n+1} Z(m) \geq 0], \qquad (1.22)
\end{aligned}$$

where $Z(n)$ follows the definition in Lemma 5.3.

The key now is to argue (cf. [11]) that $\gamma$ can be made small enough so that "large excursions" occur at least as likely in the first term on the right side of (1.22) as in the second term. Regarding the first term, if $\gamma < \frac{c(1-H)}{H}$, then from Lemma 5.2,

$$\limsup_{n\to\infty} \frac{1}{(\gamma n)^{2(1-H)}} \log \mathsf{P}[M(n) \geq \gamma n] = -\theta^*. \qquad (1.23)$$

It remains to show that $\gamma$ can also be chosen small enough such that the second term grows no faster than the first. In view of the upper bound for $\{Z(n)/n, n \geq 0\}$ in Lemma 5.3, for every $\epsilon > 0$ there exists an $n_0$ such that for all $n \geq n_0$ we have

$$\mathsf{P}(Z(n)/n \geq 0) \leq \exp\Big( -n^{2(1-H)}(\frac{H}{\sigma^2}c^2 - \epsilon)\Big).$$

Now choose $\epsilon$ such that $0 < \epsilon < \frac{H}{2\sigma^2}c^2$, and then choose $\gamma$ small enough such that

$$\gamma^{2(1-H)}\theta^* + \epsilon < \frac{H}{\sigma^2}c^2 - \epsilon \quad \text{and} \quad \gamma < \frac{c(1-H)}{H}.$$

This way, we have, for $n \geq n_0$,

$$\mathsf{P}[\sup_{m\geq n+1} Z(m) \geq 0]$$

$$\leq \sum_{m=n+1}^{\infty} \mathsf{P}[Z(m)/m \geq 0]$$

$$\leq \sum_{m=n+1}^{\infty} \exp\Big( - n^{2(1-H)}(\gamma^{2(1-H)}\theta^* + \epsilon)\Big). \tag{1.24}$$

Simple algebra yields

$$\limsup_{n\to\infty} \frac{1}{(\gamma n)^{2(1-H)}} \log \sum_{m=n+1}^{\infty} \exp\Big( - n^{2(1-H)}(\gamma^{2(1-H)}\theta^* + \epsilon)\Big) \leq -\theta^*. \tag{1.25}$$

(Note that for $\alpha > 0, z > 0$, $n > (\frac{2}{\alpha z})^{1/\alpha}$

$$ne^{-n^\alpha z} = \int_n^\infty (\alpha z s^\alpha - 1)e^{-s^\alpha z} ds \geq \int_n^\infty e^{-s^\alpha z} ds.$$

Choose $\alpha = 2(1-H)$, $z = \gamma^{2(1-H)}\theta^* + \epsilon$, with $\epsilon > 0$ arbitrarily small.) Therefore, combining (1.22), (1.23), (1.24) and (1.25), we have established the desired upper bound,

$$\limsup_{x\to\infty} \frac{1}{x^{2(1-H)}} \log \mathsf{P}(M(\infty) \geq x) \leq -\theta^*.$$

■

Now, consider the second model. The following theorem and lemma replace Theorem 5.1 and Lemma 5.2

**Theorem 5.4** With $\tilde{M}(t) = \sup_{0\leq s\leq t}(-Y(-s) - cs)$ for some $c > 0$, $\tilde{M}(t)$ converges almost surely to a finite random variable $\tilde{M}(\infty)$, which has a subexponential tail distribution. Specifically,

$$\lim_{x\to\infty} \frac{1}{x^{2(1-H)}} \log \mathsf{P}[\tilde{M}(\infty) > x] = -\frac{\tilde{H}}{H}\theta^*,$$

where $\theta^*$ follows (1.21), and $\tilde{H}$ follows (1.16).

**Lemma 5.5** For all $\gamma \geq 0$, under the conditions of Theorem 3.4,

$$\lim_{n\to\infty} \frac{1}{n^{2(1-H)}} \log \mathsf{P}[\tilde{M}(n)/n > \gamma] = -\tilde{g}(\gamma),$$

where

$$\tilde{g}(\gamma) = \begin{cases} \frac{\tilde{H}}{\sigma^2}(c+\gamma)^2 & \text{if } \gamma \geq \frac{c(1-H)}{H} \\ \frac{\tilde{H}}{\sigma^2}\frac{c^{2H}}{H^{2H}(1-H)^{2(1-H)}}\gamma^{2(1-H)} & \text{if } \gamma < \frac{c(1-H)}{H} \end{cases}.$$

Recall that

$$V(t) = \sup_{0 \le s \le t} (Y(t) - Y(s) - c(t-s)), \tag{1.26}$$

and $V^{(n)}(t) = V(nt)/n$. Recall also that when $Y$ has stationary increments, as is the case when $X$ is the standard Brownian motion, the random variable $V(t)$ has the same distribution as $\tilde{M}(t)$. We claim that it still holds that $V(n) =_d \tilde{M}(n)$ for all integer $n$, when $X$ is a sample path process of the type considered in Section 2. To see this claim, note that, for $u \in [0,1]$ arbitrary

$$\begin{aligned} & n^{3/2-H}(Y^{(n)}(1) - Y^{(n)}(u)) \\ = & \sum_{j=-\infty}^{n} (1 - j/n)^{H-1/2} a(j) - \sum_{j=-\infty}^{[nu]} (u - j/n)^{H-1/2} a(j), \end{aligned}$$

which, by stationarity,

$$\begin{aligned} =_d & \sum_{j=-\infty}^{n} (1 - j/n)^{H-1/2} a(j+n) - \sum_{j=-\infty}^{[nu]} (u - j/n)^{H-1/2} a(j+n) \\ = & \sum_{j=-\infty}^{0} (-j/n)^{H-1/2} a(j) - \sum_{j=-\infty}^{[n(u-1)]} ((u-1) - j/n)^{H-1/2} a(j) \\ = & \; n^{3/2-H}(Y^{(n)}(0) - Y^{(n)}(u-1)) \; . \end{aligned}$$

Given this, in part (ii) of the following theorem the values of $t$ should be taken as integers when $X$ is a sample path process. The proof of the theorem is a direct consequence of Theorem 5.4 and Lemma 5.5.

**Theorem 5.6** Suppose that the conditions in Theorem 5.4 hold.

**(i)** The process $\{V(n)/n, n \ge 0\}$ satisfies the LDP with speed $n^{2(1-H)}$ and a good rate function $\Lambda_V^*(\gamma) : I\!R \mapsto I\!R$, where $\Lambda_V^*(\gamma) = \tilde{g}(\gamma)$.

**(ii)** The random variable $V(t)$ converges in distribution to a finite random variable $V(\infty)$, which has a subexponential tail distribution. Specifically,

$$\lim_{x \to \infty} \frac{1}{x^{2(1-H)}} \log \mathsf{P}[V(\infty) > x] = -\frac{\tilde{H}}{H} \theta^* .$$

**Remark.** Note that when $Y$ is the FBM in (1.1), the result in Theorem 5.4 is consistent with the result in Duffield and O'Connell [11] for FBM. To see this, note that when $X(t) \equiv \sqrt{2\tilde{H}} B(t)$, where $B(t)$ is standard Brownian motion, $\mathsf{E}[Y^2(t)] = t^{2H}$, which is consistent with the relation (57) of [11]. In addition, $X$ satisfies the conditions in Theorem 3.1 with $\sigma^2 = 2\tilde{H}$. It then suffices to note that the minimization problem appearing in (60) of [11] has value $\theta^*$.

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