Large Deviations, Moderate Deviations, and Queues with Long-Range Dependent Input

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Abstract

Long-range dependence has been recently asserted an important characteristic in modeling telecommunications traffic. Inspired by the integral relationship between the fractional Brownian motion and the standard Brownian motion, we model a process with long-range dependence, Y, as a fractional integral of Riemann-Liouville type applied to a more standard process X — one that does not have long-range dependence. When X takes the form of a sample path process with bounded stationary increments, we provide a criterion for X to satisfy a moderate deviations principle (MDP). Based on the MDP of X, we then establish the MDP for Y. Furthermore, we characterize, in terms of the MDP, the transient behavior of queues when fed with the long-range dependent input process Y. In particular, we identify the most likely path that leads to a large queue, and demonstrate that unlike the case when the input has short-range dependence, the path here is nonlinear.

Key words: sample path large deviations and moderate deviations, long-range dependence, weak convergence, queues.

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1 Introduction

In traditional queueing models, the input (arrival) processes either have independent increments (e.g., Poisson process), or have a "short memory" (e.g., renewal and Markov arrival processes), and tend to "forget" their initial distributions as time increases. On the other hand, via careful experiments, researchers have recently shown that data from telecommunications traffic often exhibit long-range dependence, typically accompanied by self-similarity; see, e.g., Leland *et al* [22], and Willinger [34]. (For preliminaries regarding processes with long-range dependence, as well as short-range dependence, refer to, e.g., Cox [8] and Beran [3].)

Furthermore, it has been demonstrated that queues with long-range dependent input have subexponential tails (Norros [26] and Duffield and O'Connell [12]). This is quite in contrast with the exponential tail distribution in queues with renewal or Markov arrival processes (e.g., Chang [7], Glynn and Whitt [16, 17], Whitt [33], and the references there).

The exponential tail distribution for standard queues — with input that has short-range dependence — is obtained roughly as follows: Suppose the "netput" — input minus (potential) output — is a process with stationary increments. Then, it is known from elementary queueing theory that the steady-state limit of the workload or inventory process is equal in distribution to the limit of the maximum of a random walk, with the netput as its displacement. Assuming that the netput satisfies a *large deviations principle* (LDP), with speed n, one can show that the limiting maximum random walk also satisfies a LDP with speed n, and hence the exponential tail distribution of the steady-state inventory process. Furthermore, it can be shown that most input processes (with short-range dependence) to standard queues do satisfy a LDP. Indeed, in [17], it is shown that an embedded regenerative structure is usually sufficient for the arrival process to satisfy the LDP.

The setting in Duffield and O'Connell [12], however, assumes the netput satisfies a LDP with a speed that is slower than n [cf. (3) below]; and this is usually referred to as a *moderate deviations principle* (MDP); refer to, e.g., Mogulskii [25] and Wu [36]. With this assumption, Duffield and O'Connell show that the maximum random walk also satisfies the MDP. In particular, they showed when the netput is a fractional Brownian motion (FBM) with Hurst parameter $H \in (\frac{1}{2}, 1)$, the MDP is satisfied with a speed $n^{2(1-H)}$. (See Section 3 and Section 4 for more details.) It is known that the FBM is a self-similar process that exhibits long-range dependence (e.g., Mandelbrot and van Ness [24]). Therefore, with the FBM as the netput, the steady-state distribution has a subexponential tail, or, more precisely, the exponential decay rate is $n^{2(1-H)}$ rather than n.

Motivated by the need to characterize queueing behavior under long-range dependent input on the one hand and the more general, MDP based technique in [12] on the other hand, our study is aimed at the following two-fold objective:

(A) To develop a systematic way to generate an input process that has long-range dependence, *and* also satisfies the MDP;

(B) To develop the MDP for queues fed with the long-range dependent input in (A), not just in terms of the steady-state behavior, but more importantly in terms of the *transient* behavior, in particular, to characterize the most likely path that leads to a large queueing build-up.

To accomplish the goal in (A), our starting point is the following linear filter, very much inspired by the FBM model:

$$Y(t) = \int_0^t (t-s)^{H-\frac{1}{2}} dX(s) = (H-\frac{1}{2}) \int_0^t (t-s)^{H-\frac{3}{2}} X(s) ds,$$

[cf. (16) and (17)], where $H \in (\frac{1}{2}, 1)$, and the second inequality follows from integration by parts (assuming X has sample paths of bounded variation and $X(0) = 0$), or from Ito's formula when $X(t) = B(t)$, the standard Brownian motion, in which case, $Y(t) = B_H(t)$, recovering the FBM model of Lévy [23], with H being the Hurst parameter. The second integral is also known as a fractional integral of Riemann-Liouville type (cf. [30]; also, [27]). More generally, if the input to the filter, $X(t)$, is a process with short-range dependence, then clearly, the output process $Y(t)$ will have long-range dependence. (This can be checked by examining the autocovariance function; refer to Section 3.) Moreover, a moderate excursion of $X(t)$ incurs a large excursion of $Y(t)$ through the amplifying effect of the integral.

With this filter as a basic model, our first step is to develop a criterion, in Section 2, by which the MDP of the X process can be verified. In particular, we consider sample path processes with bounded stationary increments. This setting is more general than that of Mogulskii [25] for processes with i.i.d. increments, and hence allows us to consider a wider class of processes as candidates for X. Examples of this class include Markov chains, ϕ-mixing processes, and moving average processes. Next, in Section 3, making use of the contraction principle, we derive the MDP for Y, based on the MDP of X.

Having accomplished the goal in (A), we move on to the tasks in (B). In Section 4, we study queues that are fed with the long-range dependent input process Y via conditional limit theorems. We characterize the transient queueing behavior; in particular, how large queues build up. In contrast to queues with short-range dependent input (refer to the conditional limit theorems in Anantharam [1], Dembo and Zajic [10] and Chang [6]), the most likely path to a large queue build-up here is *nonlinear*. Finally, in Section 5, we discuss a steady-state result, in terms of the maximum random walk, and compare it with the results in [12].

2 Sample Path Large and Moderate Deviations

2.1 Preliminaries

We first recall the definition of the large deviations principle (LDP) on a metric space $\mathcal{S}$ equipped with the topology generated by the metric and the corresponding Borel σ-field. (Dembo and Zeitouni [10] should be consulted for more details.)

Definition 2.1 A sequence of probability measures $\mu^{(n)}$ on a metric space $\mathcal{S}$ satisfies the LDP with speed $\delta_n \uparrow \infty$ and good rate function $I(\cdot) : \mathcal{S} \to [0, \infty]$ if

(i) the level sets $\{x \in \mathcal{S} : I(x) \le \alpha\}$, $\alpha \in \mathbb{R}$, are compact,

(ii) for any closed set F,

$$\limsup_{n\to\infty} \frac{1}{\delta_n} \log \mu^{(n)}(F) \le - \inf_{x\in F} I(x), \tag{1}$$

(iii) for any open set G,

$$\liminf_{n\to\infty} \frac{1}{\delta_n} \log \mu^{(n)}(G) \ge - \inf_{x\in G} I(x). \tag{2}$$

A sequence of random variables is said to satisfy the LDP if the corresponding sequence of distributions does.

Let $\mathcal{C}$ and $\mathcal{D}$ be the spaces of $\mathbb{R}$-valued functions on the interval $[0, 1]$ that are, respectively, continuous and right continuous with left limits. Each of these spaces is equipped with the supremum norm topology generated by the norm

$$\|\phi\|_\infty = \sup_{0\le t\le 1} |\phi(t)|$$

and the corresponding Borel σ-field.

In addition to the LDP in the space $\mathbb{R}^d$ we shall also consider the LDP in function spaces, such as $\mathcal{D}$ and $\mathcal{C}$, in which case we shall refer to the LDP as a *sample path* LDP or LDP(sp).

We say that a sequence $\mu^{(n)}$ satisfies a moderate deviation principle (MDP) if $\mu^{(n)}$ satisfies the LDP with speed γ_n^2, where γ_n satisfies

$$\lim_{n\to\infty} \frac{\gamma_n}{\sqrt{n}} = 0 \quad \text{and} \quad \lim_{n\to\infty} \gamma_n = \infty. \tag{3}$$

Analogous to LDP(sp), we shall use MDP(sp) to denote a sample path moderate deviation principle in a function space. Allowing that the definition of $\mu^{(n)}$ may depend on γ_n, we say $\mu^{(n)}$ satisfies the *complete* MDP (resp. MDP(sp)) if $\mu^{(n)}$ satisfies the MDP (resp. MDP(sp)) for every sequence that satisfies (3).

Consider a sequence of random variables in $I\!R$, $\{a(t), t = 0, 1, 2, \ldots\}$. Let $A(t_1, t_2) = \sum_{t=t_1+1}^{t_2} a(t)$ and define a sequence of stochastic processes as follows:

$$A^{(n)}(t) = \frac{1}{n} A(0, [nt]), \qquad t \in [0, 1]. \tag{4}$$

Denote by $\nu_A^{(n)}$ the distribution of the process $\{\frac{\sqrt{n}}{\gamma_n} A^{(n)}(t), 0 \le t \le 1\}$.

When $\{a(t), t = 0, 1, 2, \ldots\}$ is a sequence of *bounded* i.i.d. random variables with $Ea(0) = 0$ and $Ea^2(0) = \sigma^2$, it is known from Mogulskii [25], Theorem 1, that the sequence $\nu_A^{(n)}$ satisfies the complete MDP(sp) with rate function

$$I(\phi) = \begin{cases} \int_0^1 \frac{1}{2\sigma^2} (\phi'(t))^2 dt & \text{if } \phi \in \mathcal{AC},\ \phi(0) = 0 \\ \infty & \text{otherwise}, \end{cases} \tag{5}$$

where $\mathcal{AC}$ denotes those functions in $\mathcal{C}$ which are absolutely continuous.

2.2 A General Criterion for MDP

The following theorem provides a useful criterion for verifying that the sample path process, (4), of a given sequence $a(t)$ satisfies the complete MDP(sp) when $a(t)$ is not necessarily i.i.d.

Theorem 2.2 Let $\{a(t), t = 0, 1, 2, \ldots\}$ be a stationary sequence of bounded, mean zero, real-valued random variables. For a fixed integer m, and $0 = t_0 < t_1 < \cdots < t_m \le 1$, let

$$Z_n = \left(\frac{\sqrt{n}}{\gamma_n} A^{(n)}(t_1), \frac{\sqrt{n}}{\gamma_n}(A^{(n)}(t_2) - A^{(n)}(t_1)), \ldots, \frac{\sqrt{n}}{\gamma_n}(A^{(n)}(t_m) - A^{(n)}(t_{m-1}))\right).$$

Suppose that for each positive integer m and partition $\{t_i\}_{i=1}^m$, the sequence $\{Z_n\}$ satisfies the complete MDP in $I\!R^m$ with good rate function

$$I_m(z) = \sum_{i=1}^m \frac{1}{2\sigma^2} \frac{z_i^2}{(t_i - t_{i-1})},$$

for some $\sigma^2 > 0$, where $z = (z_1, \ldots, z_m)$. In addition, assume that

$$\limsup_{n\to\infty} \frac{1}{\gamma_n^2} \log \mathsf{E}[\exp(\frac{\gamma_n \sum_{i=1}^n a(i)}{\sqrt{n}}) + \exp(\frac{-\gamma_n \sum_{i=1}^n a(i)}{\sqrt{n}})] < \infty \tag{6}$$

for every sequence γ_n satisfying (3). Then the sequence $\nu_A^{(n)}$ satisfies the complete MDP(sp) in $\mathcal{D}[0,1]$ with the good rate function in (5).

In proving Theorem 2.2 we follow the approach in the proof of Theorem 5.1.2 of Dembo and Zeitouni [11]. Consider the polygonal approximation

$$\tilde{A}^{(n)}(t) = A^{(n)}(t) + (t - \frac{[nt]}{n})a([nt]+1). \tag{7}$$

Let $\tilde{\nu}_A^{(n)}$ be the corresponding distribution of $\frac{\sqrt{n}}{\gamma_n}\tilde{A}^{(n)}(t)$. We begin with the following three lemmas whose proofs are postponed until after the proof of Theorem 2.2. In each of these lemmas we assume the conditions in Theorem 2.2 are in force.

Lemma 2.3 The distributions $\nu_A^{(n)}$ and $\tilde{\nu}_A^{(n)}$ are exponentially equivalent in $\mathcal{D}$ with respect to any speed satisfying (3).

Lemma 2.4 Let S denote the set of all maps from [0,1] to $I\!R$ such that $t = 0$ is mapped to 0, and suppose that S is equipped with the pointwise convergence topology on [0,1]. Then, $\tilde{\nu}_A^{(n)}$ satisfies the MDP(sp) in S with the rate function (5) for any speed satisfying (3).

Lemma 2.5 The sequence of measures $\tilde{\nu}_A^{(n)}$ is exponentially tight in the space $\mathcal{C}$ with respect to any speed satisfying (3).

Proof. [Theorem 2.2]

As in the proof of Theorem 5.1.2 of Dembo and Zeitouni [11], Lemmas 2.4 and 2.5 allow us to conclude the MDP(sp) for $\tilde{\nu}_A^{(n)}$ in the space $\mathcal{C}$ with the rate function (5). As $\mathcal{C}$ is a closed subset of $\mathcal{D}$, by Lemma 4.1.5 of Dembo and Zeitouni [11], we have the MDP(sp) for $\tilde{\nu}_A^{(n)}$ in $\mathcal{D}$. Lemma 2.3 allows us to conclude the same MDP(sp) for $\nu_A^{(n)}$. ∎

Proof. [Lemma 2.3]

Note that

$$|\frac{\sqrt{n}}{\gamma_n}\tilde{A}^{(n)}(t) - \frac{\sqrt{n}}{\gamma_n}A^{(n)}(t)| \le \frac{|a([nt]+1)|}{\sqrt{n}\gamma_n}.$$

Thus, for any $\eta > 0$,

$$\mathsf{P}(\sup_{0\le t\le 1} |\frac{\sqrt{n}}{\gamma_n}\tilde{A}^{(n)}(t) - \frac{\sqrt{n}}{\gamma_n}A^{(n)}(t)| > \eta) \le n \max_{1\le k\le n+1} \mathsf{P}(|a(k)| > \sqrt{n}\gamma_n\eta) \ . \tag{8}$$

As $a(k)$ is bounded, the right hand side equals zero for n large enough and therefore the distributions $\nu_A^{(n)}$ and $\tilde{\nu}_A^{(n)}$ are exponentially equivalent in $\mathcal{D}$ (with respect to any speed satisfying (3)). (See Dembo and Zeitouni [11] Definition 4.2.10.) ■

Proof. [Lemma 2.4]

The proof follows the proof of Lemma 5.1.6 of Dembo and Zeitouni [11], mutatis mutandis.

■

Proof. [Lemma 2.5]

By [29, Theorem 3], it suffices to show that

$$\limsup_{n\to\infty} \frac{1}{\gamma_n^2} \log \sup_{1\ge t>s\ge 0} \mathsf{E}[\exp(\frac{\sqrt{n}\gamma_n|\tilde{A}^{(n)}(t) - \tilde{A}^{(n)}(s)|}{\sqrt{(t-s)}})] < \infty \ . \tag{9}$$

But this last quantity is bounded above by

$$\limsup_{n\to\infty} \frac{1}{\gamma_n^2} \log \sup_{\substack{\theta_1,\theta_2\in[0,1) \\ m+\theta_1+\theta_2\in(0,n]}} \mathsf{E}[\exp(\frac{\gamma_n|\theta_1 a(0) + \sum_{i=1}^m a(i) + \theta_2 a(m+1)|}{\sqrt{(m+\theta_1+\theta_2)}})] \ . \tag{10}$$

Now

$$\begin{aligned} & \mathsf{E}[\exp(\frac{\gamma_n|\theta_1 a(0) + \sum_{i=1}^m a(i) + \theta_2 a(m+1)|}{\sqrt{(m+\theta_1+\theta_2)}})] \\ \le \ & \mathsf{E}[\exp(2\frac{\gamma_n|\theta_1 a(0) + \theta_2 a(m+1)|}{\sqrt{(m+\theta_1+\theta_2)}})]^{1/2} \mathsf{E}[\exp(2\frac{\gamma_n|\sum_{i=1}^m a(i)|}{\sqrt{(m+\theta_1+\theta_2)}})]^{1/2} \ , \end{aligned}$$

and

$$\begin{aligned} & \mathsf{E}[\exp(2\frac{\gamma_n|\sum_{i=1}^m a(i)|}{\sqrt{(m+\theta_1+\theta_2)}})]^{1/2} \\ \le \ & \mathsf{E}[\exp(2\frac{\gamma_n\sum_{i=1}^m a(i)}{\sqrt{m}})]^{1/2} + \mathsf{E}[\exp(-2\frac{\gamma_n\sum_{i=1}^m a(i)}{\sqrt{m}})]^{1/2} \ . \end{aligned}$$

Suppose

$$\limsup_{n\to\infty} \frac{1}{\gamma_n^2} \log \mathsf{E}[\exp(2\frac{\gamma_n\sum_{i=1}^{m_n} a(i)}{\sqrt{m_n}})]^{1/2} = \infty, \tag{11}$$

where m_n is the m that achieves the supremum in (10). Then, we must have $\limsup \sqrt{m_n}/\gamma_n = \infty$. For otherwise, with M a bound on $a(i)$, we would have

$$\limsup_{n\to\infty} \frac{1}{\gamma_n^2} \log \mathsf{E}[\exp(2\frac{\gamma_n \sum_{i=1}^{m_n} a(i)}{\sqrt{m_n}})] \leq 2M \limsup \sqrt{m_n}/\gamma_n < \infty .$$

Hence we may choose a subsequence, again indexed by n, such that m_n/γ_n^2 increases to ∞. We then have, by Varadhan's lemma,

$$\limsup_{n\to\infty} \frac{1}{\gamma_n^2} \log \mathsf{E}[\exp(2\frac{\gamma_n \sum_{i=1}^{m_n} a(i)}{\sqrt{m_n}})] = 2\sigma^2 . \tag{12}$$

But this contradicts (11) and so (9) holds. ■

2.3 Examples

Here we present three examples of sequences that satisfy the hypotheses of Theorem 2.2.

Example 1. Let $\{b(t), t = 0, 1, 2, \ldots\}$ be a Markov chain taking values in a Polish space and satisfying the conditions of Theorem 2.1 of Wu [36]. Let $a(t) = f(b(t)) - \mathsf{E}_\mu[f(b(t))]$ for any bounded real valued function f, where μ is the stationary distribution of $b(t)$. Note that here the sequence $\{a(t)\}$ is not necessarily stationary, and hence we can not directly apply Theorem 2.2. However, we note that Theorem 2.2 remains valid in the non-stationary setting once the statement of Lemma 2.5 is verified. So we first indicate the necessary modifications to the proof of Lemma 2.5. It suffices to show, in place of (10), that

$$\limsup_{n\to\infty} \frac{1}{\gamma_n^2} \log \sup_{\substack{\theta_1,\theta_2\in[0,1) \\ m+\theta_1+\theta_2\in(0,n]}} \mathsf{E}_x[\exp(\frac{\gamma_n|\theta_1 a(0) + \sum_{i=1}^{m} a(i) + \theta_2 a(m+1)|}{\sqrt{(m+\theta_1+\theta_2)}})] < \infty , \tag{13}$$

uniformly in x. However, by Theorem 2.1 of [36], we have, uniformly in x,

$$\limsup_{n\to\infty} \frac{1}{\gamma_n^2} \log \mathsf{E}_x[\exp(2\frac{\gamma_n \sum_{i=1}^{m_n} a(i)}{\sqrt{m_n}})] = 2\sigma^2 .$$

Hence (13) can be proved in the same fashion as (9). To verify that (6) holds uniformly in x then follows from Theorem 2.1 of [36], while the complete MDP of the sequence $\{Z_n\}$ can be established following the proof of Theorem 3 in [10]. Note that the MDP(sp) is uniform in the initial condition.

Example 2. Let $\{a(t), t = 0, \pm 1, \pm 2, \ldots\}$ be a stationary sequence of bounded, mean zero random variables. Let, for n a positive integer,

$$\phi(n) = \sup(|P(A|B) - P(B)|;\ A \in \mathcal{F}_k \text{ with } P(A) > 0, B \in \mathcal{F}^{k+n}, k \in \mathbb{N}) ,$$

where, for $k \in \mathbb{N}$, $\mathcal{F}_k = \sigma\{a(i) : i \leq k\}$ and $\mathcal{F}^k = \sigma\{a(i) : i \geq k\}$. Recall that when $\lim_{n\to\infty} \phi(n) = 0$, the sequence is said to be ϕ- or uniform-mixing. We shall further assume that $\phi(n) = 0(e^{-\alpha n})$ for some $\alpha > 0$.

To verify the hypotheses of Theorem 2.2, we consider only $m = 2$, the treatment for $m > 2$ then being clear. Let

$$\sigma^2 = \lim_{n\to\infty} \frac{1}{n} \mathsf{E}[\sum_{i=1}^{n} a(i)]^2 .$$

If we can show that, for $\beta_1,\ \beta_2 \in I\!R$,

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{\gamma_n^2} \log \mathsf{E}[\exp\{\beta_1 \frac{\gamma_n}{\sqrt{n}} \sum_{i=1}^{[nt_1]} a(i) + \beta_2 \frac{\gamma_n}{\sqrt{n}} \sum_{i=[nt_1]+1}^{[nt_2]} a(i)\}] \\ = \ & \frac{\beta_1^2}{2}\sigma^2 t_1 + \frac{\beta_2^2}{2}\sigma^2 (t_2 - t_1) \ , \end{aligned} \tag{14}$$

the hypotheses of Theorem 2.2, for $m = 2$, then follow by the Gärtner-Ellis theorem, assuming $\sigma^2 > 0$. We verify (14) with "$=$" replaced by "$\leq$", the opposite inequality can be established in a similar way. For β_1, β_2 real numbers and n, l positive integers, we have, by Hölder's inequality,

$$\begin{aligned} &\frac{1}{\gamma_n^2} \log \mathsf{E}[\exp(\beta_1 \frac{\gamma_n}{\sqrt{n}} \sum_{1}^{[nt_1]} a(i) + \beta_2 \frac{\gamma_n}{\sqrt{n}} \sum_{[nt_1]+1}^{[nt_2]} a(i))] \\ \leq \ & \frac{1}{p\gamma_n^2} \log \mathsf{E}[\exp(p\beta_1 \frac{\gamma_n}{\sqrt{n}} \sum_{1}^{[nt_1]} a(i) + p\beta_2 \frac{\gamma_n}{\sqrt{n}} \sum_{[nt_1]+l}^{[nt_2]} a(i))] \\ & + \frac{1}{q\gamma_n^2} \log \mathsf{E}[\exp(q\beta_2 \frac{\gamma_n}{\sqrt{n}} \sum_{1}^{l-1} a(i))], \end{aligned}$$

for $p, q > 0$ such that $1/p + 1/q = 1$. By Lemma 20.1 of [4], the right hand side above is dominated by

$$\begin{aligned} &\frac{1}{p\gamma_n^2} \log\{\mathsf{E}[\exp(p\beta_1 \frac{\gamma_n}{\sqrt{n}} \sum_{1}^{[nt_1]} a(i))]\mathsf{E}[\exp(p\beta_2 \frac{\gamma_n}{\sqrt{n}} \sum_{[nt_1]+l}^{[nt_2]} a(i))] \\ & + 2\sqrt{\phi(l)\mathsf{E}[\exp(2p\beta_1 \frac{\gamma_n}{\sqrt{n}} \sum_{1}^{[nt_1]} a(i))]\mathsf{E}[\exp(2p\beta_2 \frac{\gamma_n}{\sqrt{n}} \sum_{[nt_1]+l}^{[nt_2]} a(i))]}\} \\ & + \frac{1}{q\gamma_n^2} \log \mathsf{E}[\exp(q\beta_2 \frac{\gamma_n}{\sqrt{n}} \sum_{1}^{l-1} a(i))], \end{aligned}$$

which, in turn, is dominated by

$$\frac{1}{p\gamma_n^2} \log\{\mathsf{E}[\exp(p\beta_1 \frac{\gamma_n}{\sqrt{n}} \sum_{1}^{[nt_1]} a(i))](\mathsf{E}[\exp(rp\beta_2 \frac{\gamma_n}{\sqrt{n}} \sum_{[nt_1]}^{[nt_2]} a(i))])^{1/r}(\mathsf{E}[\exp(-sp\beta_2 \frac{\gamma_n}{\sqrt{n}} \sum_{1}^{l-1} a(i))])^{1/s}$$

$$+2\sqrt{\phi(l)\mathsf{E}[\exp(2p\beta_1\frac{\gamma_n}{\sqrt{n}}\sum_1^{[nt_1]}a(i))]\mathsf{E}[\exp(2p\beta_2\frac{\gamma_n}{\sqrt{n}}\sum_{[nt_1]}^{[nt_2]}a(i))]])\}}$$
$$+\frac{1}{q\gamma_n^2}\log\mathsf{E}[\exp(q\beta_2\frac{\gamma_n}{\sqrt{n}}\sum_1^{l-1}a(i))],$$

with $r, s > 0$ such that $1/r + 1/s = 1$, following, again, Hölder's inequality.

Choosing now l so that $\gamma_n^2 << l << \gamma_n\sqrt{n}$ and using Theorem 1.2 of [14], we have, letting first $n \to \infty$ and then $p, r \to 1$,

$$\begin{aligned}&\limsup_{n\to\infty}\frac{1}{\gamma_n^2}\log\mathsf{E}[\exp(\beta_1\frac{\gamma_n}{\sqrt{n}}\sum_1^{[nt_1]}a(i)+\beta_2\frac{\gamma_n}{\sqrt{n}}\sum_{[nt_1]+1}^{[nt_2]}a(i))]\\ \leq\ &\frac{\beta_1^2}{2}\sigma^2t_1+\frac{\beta_2^2}{2}\sigma^2(t_2-t_1)\ .\end{aligned}$$

Example 3. Let $\{\xi(t), t = 0, \pm1, \pm2, \ldots\}$ be a sequence of bounded, mean zero, i.i.d. random variables and $\{b(t), t = 0, \pm1, \pm2, \ldots\}$ a sequence of real numbers which is absolutely summable. Let the sequence $\{a(t), t = 0, 1, 2, \ldots\}$ be the infinite moving average sequence defined by $a(t) = \sum_{i=-\infty}^{\infty} b(i + t)\xi(i)$. Jiang, Wang and Rao [20] have demonstrated a MDP for the sequence $A^{(n)}(1)$.

To verify the hypotheses of Theorem 2.2, we consider only $m = 2$, the treatment for $m > 2$ then being clear. Let $\sigma_\xi = \mathsf{E}[\xi^2(0)]$. If we can show that, for $\beta_1,\ \beta_2 \in I\!R$,

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{\gamma_n^2}\log\mathsf{E}[\exp\{\beta_1\frac{\gamma_n}{\sqrt{n}}\sum_{i=1}^{[nt_1]}a(i)+\beta_2\frac{\gamma_n}{\sqrt{n}}\sum_{i=[nt_1]+1}^{[nt_2]}a(i)\}]\\ =\ &\frac{\beta_1^2}{2}(\sum_{i=-\infty}^{\infty}b_i)^2\sigma_\xi^2t_1+\frac{\beta_2^2}{2}(\sum_{i=-\infty}^{\infty}b_i)^2\sigma_\xi^2(t_2-t_1)\ ,\end{aligned}\tag{15}$$

the hypotheses of Theorem 2.2, for $m = 2$, then follow by the Gärtner-Ellis theorem, with

$$\sigma^2=(\sum_{i=-\infty}^{\infty}b_i)^2\sigma_\xi^2\ ,$$

assuming $\sigma^2 > 0$. To verify (15), we write

$$\begin{aligned}&\log\mathsf{E}[\exp\{\beta_1\frac{\gamma_n}{\sqrt{n}}\sum_{i=1}^{[nt_1]}a(i)+\beta_2\frac{\gamma_n}{\sqrt{n}}\sum_{i=[nt_1]+1}^{[nt_2]}a(i)\}]\\ =\ &\sum_{i=-\infty}^{\infty}\log\mathsf{E}[\exp\{\beta_1\frac{\gamma_n}{\sqrt{n}}b_{[nt_1]i}\xi_1+\beta_2\frac{\gamma_n}{\sqrt{n}}(b_{[nt_2]i}-b_{[nt_1]i})\xi_1\}]\ ,\end{aligned}$$

where $b_{mi} = \sum_{j=1}^{m} b(i+j)$ for $m \geq 1$, $-\infty < i < \infty$. Expanding terms as in [20] and using the fact that, for any $t \geq s \geq 0$ and positive integer k,

$$\frac{1}{n} \sum_{i=-\infty}^{\infty} \Big(\sum_{j=[ns]+1}^{[nt]} b(i+j) \Big)^k \to (t-s) \Big(\sum_{i=-\infty}^{\infty} b(i) \Big)^k ,$$

we obtain (15).

3 MDP for Processes with Long-Range Dependence

As mentioned in the Introduction, our idea of creating input processes to queues with long-range dependence is to let a standard X process — one with short-range dependence — go through the following linear filter:

$$Y(t) = \int_0^t (t-s)^{H-\frac{1}{2}} dX(s), \qquad (16)$$

so that Y is a process with long-range dependence, and then let Y be the input process to a queue.

This is mainly motivated by Lévy's model of fractional Brownian motion, where X is the standard Brownian motion and Y is the fractional Brownian motion (FBM) with the Hurst parameter H; refer to Lévy [23], also to Mandelbrot and van Ness [24]. Obviously, when $H = \frac{1}{2}$, the FBM reduces to the standard Brownian motion. On the other hand, when $H \in (\frac{1}{2}, 1)$, the FBM is known to have long-range dependence (refer to [8] and [24]; also, [22] and [34]). Hence, below we will assume that $H \in (\frac{1}{2}, 1)$ throughout.

Some remarks are in order. The advantage of the filter in (16) lies in the fact that it is a familiar object widely used in many engineering applications (e.g., [35]). Even more importantly, the boundedness of the interval $[0, t]$ allows the direct application of the MDP results of the last section. The disadvantage, however, is that the filter does not preserve the stationary increment property (of X). In addition , when X is a self-similar process, the process Y is only self-similar with respect to $t = 0$ according to the definition of [24]. (Following [24], Definition 3.2, a stochastic process $\{X(t)\}$ is termed *self-similar* if the increment process, $\{X(t+\tau) - X(t)\}$, is equal in distribution to $\{a^{-H}[X(t+a\tau) - X(t)]\}$, for any t and any $a > 0$, where $H \geq 0$ is the Hurst parameter. Note, however, that Y *is* self-similar following the definition in [3].) Expanding the range of the integration to $(-\infty, t)$ (and suitably modifying the integrand) will yield stationary increments and self-similarity. But this then requires extending the MDP to unbounded intervals, which involves a level of technicality that is substantially more complex than what is required in the proofs below.

Since Y does not have stationary increments, the autocovariance function (of the increments)

$$\mathsf{Cov}(dY(\tau), dY(\tau+t))$$

depends on τ (as well as on t). Modulo this twist, however, we can still use the standard definition for long-range dependence (e.g., Cox [8]), namely, that the autocovariance function is not summable over $t \in (0,\infty)$ (for any τ). (For instance, when X is the standard Brownian motion, one can verify that the above autocovariance is of order $t^{-(\frac{3}{2}-H)}$.)

When X is a process with (locally) finite variation (i.e., with probability one the variation over each interval $[0,t]$ is finite), the integral in (16) can be viewed as a sample path Lebesgue-Stieltjes integral (see e.g., Wong and Hajek [35]). Integration by parts yields

$$\begin{aligned} Y(t) &= (t-s)^{H-\frac{1}{2}}X(s)|_0^t + \int_0^t (H-\frac{1}{2})(t-s)^{H-\frac{3}{2}}X(s)ds \\ &= \int_0^t (H-\frac{1}{2})(t-s)^{H-\frac{3}{2}}X(s)ds, \end{aligned}$$

since $X(0)=0$ and $H>\frac{1}{2}$. This last integral is known as a fractional integral of Riemann-Liouville type (cf. [30] and also [27]). On the other hand, if $X(t)\equiv B(t)$, the standard Brownian motion, then applying Ito's formula (see, e.g., [35]) to (16) yields the same expression as above.

Therefore, more generally, we will consider the transformation of X into Y as through the following mapping:

$$Y(t) = \int_0^t (H-\frac{1}{2})(t-s)^{H-\frac{3}{2}}X(s)ds. \tag{17}$$

Let $Y^{(n)}(t) = Y(nt)/n$ and $X^{(n)}(t) = X(nt)/n$. A simple change of variable ($s = n\tau$) yields

$$Y^{(n)}(t) = \int_0^t (H-\frac{1}{2})(t-\tau)^{H-\frac{3}{2}}(n^{H-\frac{1}{2}}X^{(n)}(\tau))d\tau. \tag{18}$$

Through the mapping in (18), we derive below a LDP(sp) for Y and related conditional limit theorems, assuming that X satisfies the MDP in Theorem 3.1.

Theorem 3.1 Suppose, for some $\frac{1}{2} < H < 1$, the probability law of $\{n^{H-\frac{1}{2}}X^{(n)}(t), t\in[0,1]\}$, denoted $\mu_X^{(n)}$, satisfies LDP(sp) with speed $n^{2(1-H)}$ and a good rate function $I_X(\phi): \mathcal{D} \mapsto I\!R$,

$$I_X(\phi) = \begin{cases} \int_0^1 \frac{1}{2\sigma^2}(\phi'(t))^2 dt & \text{if } \phi\in\mathcal{AC}, \phi(0)=0 \\ \infty & \text{otherwise} \end{cases}.$$

Then, the distribution of $\{Y^{(n)}(t), t \in [0,1]\}$, denoted $\mu_Y^{(n)}$, satisfies LDP(sp) with speed $n^{2(1-H)}$ and good rate function $I_Y(\psi) : \mathcal{D} \mapsto I\!R$,

$$I_Y(\psi) = \inf\{I_X(\phi) : \psi(t) = \int_0^t (t-s)^{H-\frac{1}{2}} \phi'(s) ds, \, t \in [0,1]\}.$$

It is known that, for a given function ψ, if there exists a function $\phi' \in L^2([0,1])$ satisfying the integral equation appearing in the definition of $I_Y(\psi)$, then this ϕ' is unique (cf. [21]). In addition, the rate function $I_Y(\cdot)$ also appears in the LDP(sp) of certain fractional ARIMA processes with normally distributed noises; refer to Barbe and Broniatowski [2]. Also note that in Section 2 we have provided examples of processes for which the corresponding $\mu_X^{(n)}$ satisfies the required LDP(sp) in Theorem 3.1.

Theorem 3.2 Suppose the conditions in Theorem 3.1 are in force.

(i) Let $\tilde{\mu}_X^{(n)}$ denote the law of $\{n^{H-\frac{1}{2}} X^{(n)}(\cdot) | Y^{(n)}(1) \geq \gamma\}$ for some $\gamma \geq 0$. Then, $\tilde{\mu}_X^{(n)}$ converges in distribution to $\delta_{\phi_\gamma(\cdot)}$, where

$$\phi_\gamma(t) = \gamma \frac{2H}{H+\frac{1}{2}} (1-(1-t)^{H+\frac{1}{2}}), \quad t \in [0,1].$$

(ii) Let $\tilde{\mu}_Y^{(n)}$ denote the law of $\{Y^{(n)}(\cdot) | Y^{(n)}(1) \geq \gamma\}$. Then $\tilde{\mu}_Y^{(n)}$ converges in distribution to $\delta_{\psi_\gamma(\cdot)}$, where

$$\psi_\gamma(t) = \gamma(2H) \int_0^t (t-s)^{H-\frac{1}{2}} (1-s)^{H-\frac{1}{2}} ds, \quad t \in [0,1].$$

In the rest of this section, we prove the above two theorems. We first prove Theorem 3.1. We next present an elementary Lemma 3.3, where we summarize the solutions to the type of variational problems that will appear repeatedly later, followed by a corollary of Theorem 3.1. The proof of the corollary also yields the most likely path for the input process in Theorem 3.2(i). The most likely path for the filtered process is then simply the output of the filter with the most likely input path. To prove the conditional limit in Theorem 3.2(i), we show the probability outside the ϵ-tube around the most likely path is exponentially smaller than the probability of the most likely path.

Proof. [Theorem 3.1]

We first show that

$$\psi(t) = \int_0^t (H-\frac{1}{2})(t-s)^{H-\frac{3}{2}} \phi(s) ds, \quad t \in [0,1]$$

is a continuous mapping from $\mathcal{D}$ to $\mathcal{D}$ [cf. the mapping in (17)]. Consider $\phi_1, \phi_2 \in \mathcal{D}$ satisfying $||\phi_1 - \phi_2||_\infty < \epsilon$ for some $\epsilon > 0$. Then, the corresponding ψ_1 and ψ_2 satisfy:

$$\begin{aligned} ||\psi_1 - \psi_2||_\infty &= \sup_{t\in[0,1]} |\int_0^t (H-\frac{1}{2})(t-s)^{H-\frac{3}{2}}(\phi_1(s)-\phi_2(s))ds| \\ &\leq \epsilon \sup_{t\in[0,1]} \int_0^t (H-\frac{1}{2})(t-s)^{H-\frac{3}{2}} ds \\ &= \epsilon. \end{aligned}$$

It then follows from the contraction principle (see, for example, Dembo and Zeitouni [11], Theorem 4.2.1) that $\mu_Y^{(n)}$ satisfies LDP(sp) with speed $n^{2(1-H)}$ and the good rate function $I_Y(\cdot)$, taking into account that when $\phi \in \mathcal{AC}$ and $\phi(0) = 0$, integration by parts yields

$$\psi(t) = \int_0^t (t-s)^{H-\frac{1}{2}} \phi'(s) ds.$$

■

Lemma 3.3 (i) With $g \in L^2([0,1])$ such that $G^2 := \int_a^b g^2(t)dt > 0$, the unique optimal solution to the minimization problem:

$$\min_{f\in L^2([0,1])} \int_a^b f^2(t)dt \qquad \text{s.t.} \quad \int_a^b g(t)f(t)dt = c \text{ (or } \geq c, \text{ if } c \geq 0),$$

is $f^*(t) = (c/G^2)g(t)$, $t \in [a,b]$, and the corresponding minimum is c^2/G^2.
(ii) With $g_1, g_2 \in L^2([0,1])$ such that $G_1^2 := \int_a^{b_1} g_1^2(t)dt > 0$ and/or $G_2^2 := \int_{b_1}^b g_2^2(t)dt > 0$, the unique optimal solution to the minimization problem:

$$\min_{f\in L^2([0,1])} \int_a^b f^2(t)dt \qquad \text{s.t.} \quad \int_a^{b_1} g_1(t)f(t)dt + \int_{b_1}^b g_2(t)f(t)dt = c \quad \text{(or } \geq c, \text{ if } c \geq 0),$$

is $f^*(t) = [c/(G_1^2+G_2^2)]g_1(t)$, for $t \in [a,b_1)$, and $f^*(t) = [c/(G_1^2+G_2^2)]g_2(t)$, for $t \in [b_1,b]$, and the corresponding minimum is $c^2/(G_1^2+G_2^2)$,

Proof. (i) Note that $\int_a^b (f^*(t))^2dt = c^2/G^2$. From the Cauchy-Schwartz inequality, we have

$$\int_a^b f^2(t)dt \int_a^b (f^*(t))^2dt \geq (\int_a^b f(t)f^*(t)dt)^2 = (\frac{c}{G^2})^2(\int_a^b g(t)f(t)dt)^2.$$

Following the constraint, we know the right hand side above is equal to (or, when $c \geq 0$, dominates) c^4/G^4. Dividing both sides by $\int_a^b (f^*(t))^2dt = c^2/G^2$ yields the desired solution.
(ii) Let $g(t) = g_1(t)$ for $t \in [a,b_1)$ and $g(t) = g_2(t)$ for $t \in [b_1,b]$ and use (i). ■

Corollary 3.4 Suppose the conditions in Theorem 3.1 are in force. Then the process $\{Y(n)/n, n \geq 0\}$ satisfies LDP with speed $n^{2(1-H)}$ and a good rate function $\Lambda_Y^*(\gamma) : \mathbb{R} \mapsto \mathbb{R}$,

$$\Lambda_Y^*(\gamma) = \frac{H}{\sigma^2}\gamma^2.$$

Proof. Note that $Y^{(n)}(1) = Y(n)/n$. Consider the mapping $f : \psi \mapsto \psi(1)$ from $\mathcal{D}$ to $\mathbb{R}$, which is clearly continuous. Thus, as a direct consequence of the contraction principle and Theorem 3.1, the distribution of $Y^{(n)}(1)$ satisfies LDP with speed $n^{2(1-H)}$ and a good rate function

$$\begin{aligned} \Lambda_Y^*(\gamma) &= \inf\{I_Y(\psi) : \psi(1) = \gamma\} \\ &= \inf\{I_X(\phi) : \int_0^1 (1-s)^{H-\frac{1}{2}}\phi'(s)ds = \gamma\}. \end{aligned}$$

¿From the definition of $I_X(\phi)$ (in Theorem 3.1), the infimum above is tantamount to solving the following minimization problem:

$$\begin{aligned} \min_\phi \quad & \int_0^1 \frac{1}{2\sigma^2}(\phi'(s))^2 ds \\ \text{s.t.} \quad & \int_0^1 (1-s)^{H-\frac{1}{2}}\phi'(s)ds = \gamma, \quad \phi(0) = 0. \end{aligned}$$

Since $\int_0^1 (1-s)^{2(H-\frac{1}{2})}ds = 1/(2H)$, it follows from Lemma 3.3(i) that

$$\Lambda_Y^*(\gamma) = \frac{H}{\sigma^2}\gamma^2.$$

Moreover, the infimum above is uniquely achieved by

$$\phi_\gamma'(s) = \gamma(2H)(1-s)^{H-\frac{1}{2}},$$

which, along with $\phi(0) = 0$, yields

$$\phi_\gamma(t) = \gamma\frac{2H}{H+\frac{1}{2}}(1-(1-t)^{H+\frac{1}{2}}), \quad t \in [0,1].$$

■

Proof. [Theorem 3.2]

(i) Denote

$$B_\epsilon = \{\phi \in \mathcal{D} : ||\phi - \phi_\gamma||_\infty \geq \epsilon\},$$

for some $\epsilon > 0$ (with ϕ_γ following the definition in Theorem 3.2), and

$$\Gamma_D = \{\phi \in \mathcal{D} : \int_0^1 (H-1/2)(1-s)^{H-\frac{3}{2}}\phi(s)ds \geq \gamma\}.$$

It is enough to show that for any given $\epsilon > 0$ we have

$$\lim_{n\to\infty} \tilde{\mu}_X^{(n)}(B_\epsilon) = \lim_{n\to\infty} \frac{\mu_X^{(n)}(B_\epsilon \cap \Gamma_D)}{\mu_X^{(n)}(\Gamma_D)} = 0. \tag{19}$$

Since $\mu_X^{(n)}(\Gamma_D) = \mathsf{P}[Y(n)/n \geq \gamma]$, it follows from Corollary 3.4 and $\gamma \geq 0$ that

$$\begin{aligned}
-\frac{H}{\sigma^2}\gamma^2 &= -\inf_{\alpha>\gamma} \Lambda_Y^*(\alpha) \\
&\leq \liminf_{n\to\infty} \frac{1}{n^{2(1-H)}} \log \mathsf{P}[Y(n)/n > \gamma] \\
&\leq \limsup_{n\to\infty} \frac{1}{n^{2(1-H)}} \log \mathsf{P}[Y(n)/n \geq \gamma] \\
&= -\inf_{\alpha\geq\gamma} \Lambda_Y^*(\alpha) = -\frac{H}{\sigma^2}\gamma^2.
\end{aligned}$$

Thus,

$$\lim_{n\to\infty} \frac{1}{n^{2(1-H)}} \log \mu_X^{(n)}(\Gamma_D) = -\Lambda_Y^*(\gamma). \tag{20}$$

On the other hand, from the LDP(sp) upper bound for $\mu_X^{(n)}$, we have

$$\lim_{n\to\infty} \frac{1}{n^{2(1-H)}} \log \mu_X^{(n)}(B_\epsilon \cap \Gamma_D) \leq -\inf_{\phi\in B_\epsilon\cap\Gamma_D} I_X(\phi) \ . \tag{21}$$

Combining (20) and (21), and making use of Lemma 2.8 of Shwartz and Weiss [31], we have, for some $\delta > 0$,

$$\limsup_{n\to\infty} \frac{1}{n^{2(1-H)}} \log \Big(\frac{\mu_X^{(n)}(B_\epsilon \cap \Gamma_D)}{\mu_X^{(n)}(\Gamma_D)} \Big) \leq -\frac{\delta^2}{2\sigma^2},$$

which, in turn, leads to the desired limit in (19).

(ii) We have shown that the mapping from ϕ to ψ is continuous in the proof of Theorem 3.1. The desired weak convergence then follows from (i). ∎

4 Queues with Long-Range Dependent Input

Here we focus on queues with an input process $\{Y(t)\}$, which, in turn, is the output of a linear filter subject to the input process $\{X(t)\}$, following (16). Hence in general, the input to the queues, the Y process, has long-range dependence.

Throughout this section, we assume that the conditions in Theorem 3.1 are in force. Consider $V(t) = \sup_{0\leq s\leq t}(Y(t) - Y(s) - c(t-s))$. Note that $V(t)$ is the *state* process of the queue with input Y and a deterministic service mechanism that depletes work at rate c. In queueing literature, $V(t)$ is often referred to as the virtual waiting time process, or the workload/inventory process. See, for instance, Chang [5], Duffield and O'Connell [12], Harrison [18], and Sigman and Yao [32], among others.

Let $V^{(n)}(t) = V(nt)/n$; in particular, $V^{(n)}(1) = V(n)/n$. Below are the main theorems of this section.

(Rewrite Theorem 4.1.)

Theorem 4.1 Suppose the conditions in Theorem 3.1 hold. For $\gamma \geq 0$,

$$\lim_{n\to\infty} \frac{1}{n^{2(1-H)}} \log \mathsf{P}(V(n)/n \geq \gamma) = -\Lambda_V^*(\gamma),$$

where

$$\Lambda_V^*(\gamma) = \frac{H}{\sigma^2} \inf_{0\leq s\leq 1} \frac{(c(1-s)+\gamma)^2}{v(s)+(1-s)^{2H}}, \tag{22}$$

and

$$v(s) = 2H \int_0^s [(1-\tau)^{H-\frac{1}{2}} - (s-\tau)^{H-\frac{1}{2}}]^2 d\tau.$$

Theorem 4.2 (i) Let $\hat{\mu}_X^{(n)}$ denote the law of $\{n^{H-\frac{1}{2}} X^{(n)}(\cdot) | V^{(n)}(1) \geq \gamma\}$ for some $\gamma \geq 0$. Suppose s^* achieves the minimum in (22). Then, $\hat{\mu}_X^{(n)}$ converges in distribution on $\mathcal{D}$ to $\delta_{\hat{\phi}_\gamma(\cdot)}$, where

$$\hat{\phi}_\gamma(t) = \begin{cases} 2H \frac{c(1-s^*)+\gamma}{v(s^*)+(1-s^*)^{2H}} \int_0^t [(1-\tau)^{H-\frac{1}{2}} - (s^*-\tau)^{H-\frac{1}{2}}] d\tau & \text{if } 0 \leq t \leq s^* \\ \hat{\phi}_\gamma(s^*) + 2H \frac{c(1-s^*)+\gamma}{v(s^*)+(1-s^*)^{2H}} \int_{s^*}^t (1-\tau)^{H-\frac{1}{2}} d\tau & \text{if } s^* < t \leq 1 \end{cases}.$$

(ii) Let $\hat{\mu}_V^{(n)}$ denote the law of $\{V^{(n)}(\cdot) | V^{(n)}(1) \geq \gamma\}$. Then $\hat{\mu}_V^{(n)}$ converges in distribution on $\mathcal{D}$ to $\delta_{\hat{\psi}_\gamma(\cdot)}$, where

$$\hat{\psi}_\gamma(t) = \sup_{0\leq s\leq t} [\int_0^t (t-\tau)^{H-1/2} d\hat{\phi}_\gamma(\tau) - \int_0^s (s-\tau)^{H-1/2} d\hat{\phi}_\gamma(\tau) - c(t-s)] \quad t \in [0,1].$$

Note that both theorems above are *transient* results. Also note that the conditional limits in Theorem 4.2, which illustrate how the queue builds up when the input has long-range dependence, are qualitatively different from the more standard case of input with short-range dependence (e.g., [1, 6, 10]), where the most likely path is linear. Here, the most likely path is *nonlinear*. In Figures 1 and 2 we have plotted the functions $\hat{\phi}_\gamma$ and $\hat{\psi}_\gamma$ for differing values of H.

In the rest of this section we first prove Theorem 4.1, and then prove a lemma, before proving Theorem 4.2.

(Rewrite the proof for Theorem 4.1.)

Proof. [Theorem 4.1]

Here we only prove the upper bound. The proof of the lower bound is similar. Observe from the reflection mapping that

$$V^{(n)}(t) = \sup_{0\le s\le t}\{Y^{(n)}(t) - Y^{(n)}(s) - c(t-s)\}, \quad t\in[0,1].$$

From the large deviation upper bound in Theorem 3.1, it suffices to show that

$$\Lambda_V^*(\gamma) = \inf\{I_X(\phi) : \phi\in\hat{F}\},$$

where

$$\hat{F} = \{\phi\in\mathcal{AC} : \sup_{0\le s\le 1}[\int_0^1 (1-\tau)^{H-\frac{1}{2}}\phi'(\tau)d\tau - \int_0^s (s-\tau)^{H-\frac{1}{2}}\phi'(\tau)d\tau - c(1-s)] \ge \gamma\}.$$

(Replace it by an inequality.) Now denote

$$\begin{aligned}\hat{F}_s &= \{\phi\in\mathcal{AC} : \phi(0)=0, \quad \int_0^s [(1-\tau)^{H-\frac{1}{2}} - (s-\tau)^{H-\frac{1}{2}}]\phi'(\tau)d\tau \\ &\quad + \int_s^1 (1-\tau)^{H-\frac{1}{2}}\phi'(\tau)d\tau \ge c(1-s)+\gamma\};\end{aligned}$$

(Replace it by an inequality.)

and write

$$\hat{F} = \cup_{\{0\le s\le 1\}}\hat{F}_s.$$

Applying Lemma 3.3(ii) yields

$$\inf_{\phi\in\hat{F}_s}\int_0^1 \frac{1}{2\sigma^2}(\phi'(\tau))^2 d\tau = \frac{H}{\sigma^2}\frac{(c(1-s)+\gamma)^2}{v(s)+(1-s)^{2H}},$$

where

$$v(s) = 2H\int_0^s [(1-\tau)^{H-\frac{1}{2}} - (s-\tau)^{H-\frac{1}{2}}]^2 d\tau.$$

Also, the minimum is uniquely achieved by

$$\phi'(\tau) = \begin{cases} 2H\frac{c(1-s)+\gamma}{v(s)+(1-s)^{2H}}[(1-\tau)^{H-\frac{1}{2}} - (s-\tau)^{H-\frac{1}{2}}] & \text{if } 0\le\tau<s \\ 2H\frac{c(1-s)+\gamma}{v(s)+(1-s)^{2H}}(1-\tau)^{H-\frac{1}{2}} & \text{if } s\le\tau\le 1 \end{cases}.$$

Finally, the minimization over $\hat{F}$ is as follows:

$$\Lambda_V^*(\gamma) = \frac{H}{\sigma^2} \inf_{0 \le s \le 1} \frac{(c(1-s)+\gamma)^2}{v(s)+(1-s)^{2H}}.$$

■

Lemma 4.3 The solution to the minimization problem in (22), s^*, is unique.

Proof. Write the objective function in (22) as

$$R(s) := [c(1-s)+\gamma]^2/f(s),$$

where

$$\begin{aligned} f(s) &= v(s)+(1-s)^{2H} \\ &= 2H\int_0^s [(1-\tau)^{2H-1}+(s-\tau)^{2H-1}]d\tau - 4H\int_0^s [(1-\tau)(s-\tau)]^{H-\frac{1}{2}}d\tau + (1-s)^{2H} \\ &= 1+s^{2H} - 4H\int_0^s [(1-\tau)(s-\tau)]^{H-\frac{1}{2}}d\tau \\ &= 1+s^{2H} - 4H\int_0^s [u(1-s+u)]^{H-\frac{1}{2}}du. \end{aligned}$$

The derivative (w.r.t. s) of $R(s)$ is:

$$R'(s) = h(s)[c(1-s)+\gamma]/f^2(s),$$

where

$$h(s) := -2cf(s) - [c(1-s)+\gamma]f'(s).$$

It is easy to verify that $f(0)=1$ and $f(1)=0$, and hence

$$R(0) = (c+\gamma)^2, \quad \text{and} \quad R(1) = +\infty.$$

In addition, $h(0) = -2c$, and hence $R'(0) < 0$. In other words, the minimum of $R(s)$ cannot be reached at either 0 or 1.

Therefore, it suffices to argue that $h(s)$, and hence $R'(s)$, has a unique zero over the interval $(0,1)$. Taking derivatives, we have

$$h'(s) = -2cf'(s) + cf'(s) - [c(1-s)+\gamma]f''(s) = -cf'(s) - [c(1-s)+\gamma]f''(s),$$

and

$$h''(s) = -cf''(s) + cf''(s) - [c(1-s)+\gamma]f^{(3)}(s) = -[c(1-s)+\gamma]f^{(3)}(s),$$

where

$$f'(s) = 2Hs^{2H-1} - 4H[s^{H-\frac{1}{2}} - (H-\frac{1}{2})\int_0^s u^{H-\frac{1}{2}}(1-s+u)^{H-\frac{3}{2}}du],$$

$$\begin{aligned} f''(s) &= 2H(2H-1)s^{2H-2} - 4H(H-\frac{1}{2})[s^{H-\frac{3}{2}} - s^{H-\frac{1}{2}} - (\frac{3}{2}-H)\int_0^s u^{H-\frac{1}{2}}(1-s+u)^{H-\frac{5}{2}}du] \\ &= 2H(2H-1)[s^{2H-2} - s^{H-\frac{3}{2}} + s^{H-\frac{1}{2}} + (\frac{3}{2}-H)\int_0^s u^{H-\frac{1}{2}}(1-s+u)^{H-\frac{5}{2}}du], \end{aligned}$$

and in particular,

$$\begin{aligned} f^{(3)}(s) &= 2H(2H-1)\{-(2-2H)s^{2H-3} + (\frac{3}{2}-H)s^{H-\frac{5}{2}} + (H-\frac{1}{2})s^{H-\frac{3}{2}} \\ &\quad +(\frac{3}{2}-H)[s^{H-\frac{1}{2}} + (\frac{5}{2}-H)\int_0^s u^{H-\frac{1}{2}}(1-s+u)^{H-\frac{7}{2}}du]\}. \end{aligned}$$

Note that since $s \in (0,1)$ and $H \in (\frac{1}{2}, 1)$, we have

$$\begin{aligned} (\frac{3}{2}-H)s^{H-\frac{5}{2}} - (2-2H)s^{2H-3} &\geq (\frac{3}{2}-H)(s^{H-\frac{5}{2}} - s^{2H-3}) \\ &= (\frac{3}{2}-H)s^{H-\frac{5}{2}}(1-s^{H-\frac{1}{2}}) \\ &> 0. \end{aligned}$$

Hence, $f^{(3)}(s) > 0$ for $s \in (0,1)$. This implies that $h(s)$ is strictly concave for $s \in (0,1)$. Since $h(0) = -2c < 0$ and

$$h(1) = -2cf(1) - \gamma f'(0) = 0,$$

we know that $h(s)$ has at most one zero over $(0,1)$. But it is impossible for $h(s)$ not to have any zero over $(0,1)$. For this would imply that $h(s)$ is strictly negative for all $s \in (0,1)$, and hence $R(s)$ would keep decreasing, contradicting the fact that $R(s)$ eventually increases to $+\infty$ as $s \to 1$. Therefore, $h(s)$ must have a unique zero over the interval $(0,1)$. ■

(From the above proof, it is clear that the optimal solution, s^*, can be easily identified, for instance, through a bisectional search for the (unique) zero of $h(s)$ over the interval (0,1).)

Proof. [Theorem 4.2]

As in the proof of Theorem 3.2, it suffices to show that for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$\inf_{\phi \in B_\epsilon \cap \hat{F}} I_X(\phi) \geq \Lambda_V^*(\gamma) + \delta \ , \tag{23}$$

where

$$B_\epsilon = \{\phi \in \mathcal{D} : ||\phi - \hat{\phi}_\gamma||_\infty \geq \epsilon\} \ ,$$

and $\hat{F}$ is defined as in the proof of Theorem 4.1. We note that, due to the definition of I_X, we may consider instead of $\hat{F}$ the set

$$\{\phi \in \mathcal{D} : \sup_{0 \leq s \leq 1} [\int_0^1 (H-1/2)(1-\tau)^{H-\frac{3}{2}}\phi(\tau)d\tau - \int_0^s (H-1/2)(s-\tau)^{H-\frac{3}{2}}\phi(\tau)d\tau - c(1-s)] \geq \gamma\} \ .$$

(Replace it by an inequality.) The desired (23) now follows from Lemma 2.8 of Shwartz and Weiss [31].

For part (ii), make use of the continuity of the mapping as in the proof of Theorem 3.2 (ii).

■

5 A Steady-State Result

In standard queueing theory, the steady-state limit, $V(\infty)$, is obtained through the limit of another process, $M(t) = \sup_{0 \leq s \leq t}(Y(s) - cs)$, the maximum of a random walk. The key linkage between the two is the stationary increment property of the process Y, which guarantees that $V(\infty)$ and $M(\infty)$ are equal in distribution. This fact is crucial in characterizing the steady-state limit, since whereas $M(t)$ is an increasing process (in t), $V(t)$ is not; and short of the linkage to $M(\infty)$, there is no easy way to even ensure the existence of $V(\infty)$.

As remarked in Section 3, the filter that relates X to Y unfortunately does not yield a process with stationary increments even when X has stationary increments. Hence, in this context, we cannot claim that $M(\infty)$ gives a characterization of the steady-state limit of the queueing process. The interest of the theorem below, which concerns $M(\infty)$, however, lies in at least two regards: (a) insofar as it gives a MDP for the maximum of a random walk, it is analogous to the study of Duffield and O'Connell [12], and it does more (in our context): the starting point being the MDP of the X process, instead of the Y process (as in the case of [12]); (b) when X is the Brownian motion, our result illustrates the difference in the tail behaviour of $M(\infty)$ when considering Lévy's FBM rather than the model considered in [12].

Theorem 5.1 Let $M(t) = \sup_{0\le s\le t}(Y(s) - cs)$ for some $c > 0$. Then, $M(t)$ converges almost surely to a finite random variable $M(\infty)$, which has a subexponential tail distribution. Specifically,

$$\lim_{x\to\infty} \frac{1}{x^{2(1-H)}} \log \mathsf{P}[M(\infty) > x] = -\theta^*,$$

where

$$\theta^* = \frac{H}{\sigma^2} \frac{c^{2H}}{H^{2H}(1-H)^{2(1-H)}}.$$

To prove Theorem 5.1, we need two lemmas.

Lemma 5.2 For all $\gamma \ge 0$

$$\lim_{n\to\infty} \frac{1}{n^{2(1-H)}} \log \mathsf{P}[M(n)/n > \gamma] = -g(\gamma),$$

where

$$g(\gamma) = \begin{cases} \frac{H}{\sigma^2}(c+\gamma)^2 & \text{if } \gamma \ge \frac{c(1-H)}{H} \\ \frac{H}{\sigma^2}\frac{c^{2H}}{H^{2H}(1-H)^{2(1-H)}}\gamma^{2(1-H)} & \text{if } \gamma < \frac{c(1-H)}{H} \end{cases}.$$

Proof. Let $M^{(n)}(t) = M(nt)/n$. Clearly,

$$M^{(n)}(t) = \sup_{0\le s\le t}(Y^{(n)}(s) - cs).$$

Also, the mapping

$$\{Y^{(n)}(t), t\in[0,1]\} \in \mathcal{D} \;\mapsto\; \{M^{(n)}(t), t\in[0,1]\} \in \mathcal{D}$$

is continuous. Thus, it follows from the contraction principle that the probability law of $\{M^{(n)}(t), t\in[0,1]\}$ satisfies the LDP(sp) with speed $n^{2(1-H)}$ and a good rate function $I_M(\psi)$: $\mathcal{D} \mapsto I\!R$,

$$I_M(\psi) = \inf\{I_X(\phi) : \psi(t) = \sup_{0\le s\le t}[\int_0^s (s-\tau)^{H-\frac{1}{2}}\phi'(\tau)d\tau - cs],\ t\in[0,1]\}.$$

In addition, from the contraction principle and $M^{(n)}(1) = M(n)/n$, we know that $\{M(n)/n, n \ge 0\}$ satisfies the LDP with speed $n^{2(1-H)}$ and a good rate function

$$\Lambda_M^*(\gamma) = \inf_{\psi(1)=\gamma} I_M(\psi).$$

In particular, we have

$$\limsup_{n\to\infty} \frac{1}{n^{2(1-H)}} \mathsf{P}[M(n)/n \geq \gamma] \leq -\inf_{\alpha\geq\gamma} \Lambda_M^*(\alpha)$$
$$= -\inf_F \int_0^1 \frac{1}{2\sigma^2}(\phi'(\tau))^2)d\tau,$$

where

$$F = \{\phi \in \mathcal{AC} : \sup_{0\leq s\leq 1} [\int_0^s (s-\tau)^{H-\frac{1}{2}}\phi'(\tau)d\tau - cs] \geq \gamma\}.$$

Now, denote

$$F_s = \{\phi \in \mathcal{AC} : \int_0^s (s-\tau)^{H-\frac{1}{2}}\phi'(\tau)d\tau - cs \geq \gamma\};$$

and write $F = \cup_{0\leq s\leq 1} F_s$. Similar to the proof of Corollary 3.4, we have from Lemma 3.3(i) that for $\gamma \geq 0$

$$\inf_{F_s} \int_0^s \frac{1}{2\sigma^2}(\phi'(\tau))^2 d\tau = \frac{H}{\sigma^2}\frac{(cs+\gamma)^2}{s^{2H}}.$$

Note that

$$\inf_{F_s} \int_0^1 \frac{1}{2\sigma^2}(\phi'(\tau))^2 d\tau = \inf_{F_s} \int_0^s \frac{1}{2\sigma^2}(\phi'(\tau))^2 d\tau$$

since we can simply set $\phi'(\tau) = 0$ for $s < \tau \leq 1$. Hence,

$$\inf_F \int_0^1 \frac{1}{2\sigma^2}(\phi'(\tau))^2 d\tau = \inf_{0\leq s\leq 1} \inf_{F_s} \int_0^1 \frac{1}{2\sigma^2}(\phi'(\tau))^2 d\tau$$
$$= \inf_{0\leq s\leq 1} \frac{H}{\sigma^2}\frac{(cs+\gamma)^2}{s^{2H}}.$$

To solve the above minimization problem, it suffices to minimize the function $f(s) := (cs+\gamma)/s^H$. It is easy to verify that $f'(s)$ has a single zero at $s^* = H\gamma/[c(1-H)]$, and that $f(s)$ is decreasing when $s < s^*$ and increasing when $s > s^*$. Hence, when $\gamma < c(1-H)/H$, the optimal solution is $s = s^* < 1$; and when $\gamma \geq c(1-H)/H$, the optimal solution is $s = 1$. The desired optimal value $g(\gamma)$ then follows.

To summarize, we have proved

$$\limsup_{n\to\infty} \frac{1}{n^{2(1-H)}} \mathsf{P}[M(n)/n \geq \gamma] \leq -g(\gamma).$$

Completely analogously, we can prove

$$\liminf_{n\to\infty} \frac{1}{n^{2(1-H)}} \mathsf{P}[M(n)/n \geq \gamma] \geq -g(\gamma),$$

and hence the desired conclusion. ■

Lemma 5.3 Let $Z(n) = \sup_{n-1<s\leq n}(Y(s) - cs)$, $n = 1, 2, \ldots$. For all $\gamma \geq 0$,

$$\lim_{n\to\infty} \frac{1}{n^{2(1-H)}} \log \mathsf{P}[Z(n)/n > \gamma] = -\frac{H}{\sigma^2}(c+\gamma)^2.$$

Proof. Note that $Z(n) \geq Y(n) - cn$. Hence, the lower bound follows from the liminf part of the LDP for $Y(n)$ in Corollary 3.4.

To establish the upper bound, we have

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{n^{2(1-H)}} \log \mathsf{P}[Z(n)/n \geq \gamma] &= \limsup_{n\to\infty} \frac{1}{n^{2(1-H)}} \log \mathsf{P}[\sup_{1-1/n<s\leq 1} (Y^{(n)}(s) - cs) \geq \gamma] \\
&\leq \limsup_{n\to\infty} \frac{1}{n^{2(1-H)}} \log \mathsf{P}[\sup_{1-\delta\leq s\leq 1} (Y^{(n)}(s) - cs) \geq \gamma],
\end{aligned}$$

for any $0 < \delta < 1$. Following the same argument as in the proof of Lemma 5.2, we have

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{n^{2(1-H)}} \log \mathsf{P}[\sup_{1-\delta\leq s\leq 1} (Y^{(n)}(s) - cs) \geq \gamma] &\leq -\inf_{\delta\leq s\leq 1} \frac{H}{\sigma^2} \frac{(cs+\gamma)^2}{s^{2H}} \\
&= -g_\delta(\gamma),
\end{aligned}$$

where

$$g_\delta(\gamma) = \begin{cases} \frac{H}{\sigma^2}(c+\gamma)^2 & \text{if } \gamma \geq \frac{c(1-H)}{H} \\ \frac{H}{\sigma^2} \frac{c^{2H}}{H^{2H}(1-H)^{2(1-H)}} \gamma^{2(1-H)} & \text{if } \frac{c(1-H)}{H}\delta < \gamma < \frac{c(1-H)}{H} \\ \frac{H}{\sigma^2} \frac{(c\delta+\gamma)^2}{\delta^{2H}} & \text{if } \frac{c(1-H)}{H}\delta \geq \gamma \end{cases}.$$

Since δ is arbitrary, letting $\delta \to 1$ yields the upper bound, taking into account

$$\lim_{\delta\to 1} g_\delta(\gamma) = \frac{H}{\sigma^2}(c+\gamma)^2.$$

■

Proof. [Theorem 5.1]

Since $M(t)$ is an increasing sequence and since $Y(n)/n \to 0$ almost surely (following Corollary 3.4) whereas $c > 0$, we know $M(t)$ converges almost surely to a finite random variable $M(\infty)$.

Below we show the MD limit by arguing that the liminf and the limsup are dominated from below and from above, respectively, by the same bound.

To establish the lower bound, we follow the approach in Chang [5]. Note that for all $\gamma > 0$

$$\begin{aligned}
\liminf_{x\to\infty} \frac{1}{x^{2(1-H)}} \log \mathsf{P}[M(\infty) > x] &= \liminf_{n\to\infty} \frac{1}{(\gamma n)^{2(1-H)}} \log \mathsf{P}[M(\infty) > \gamma n] \\
&\geq \liminf_{n\to\infty} \frac{1}{(\gamma n)^{2(1-H)}} \log \mathsf{P}[Y(n) - cn > \gamma n] \\
&\geq -\frac{H}{\sigma^2} \frac{(\gamma + c)^2}{\gamma^{2(1-H)}},
\end{aligned}$$

where we have used the LDP lower bound for $\{Y(n)/n, n \geq 0\}$ in Corollary 3.4. Thus,

$$\begin{aligned}
\liminf_{x\to\infty} \frac{1}{x^{2(1-H)}} \log \mathsf{P}[M(\infty) > x] &\geq -\inf_{\gamma>0} \frac{H}{\sigma^2} \frac{(\gamma + c)^2}{\gamma^{2(1-H)}} \\
&= -\theta^*.
\end{aligned}$$

To establish the upper bound, also note that for all $\gamma > 0$

$$\limsup_{x\to\infty} \frac{1}{x^{2(1-H)}} \log \mathsf{P}[M(\infty) \geq x] = \limsup_{n\to\infty} \frac{1}{(\gamma n)^{2(1-H)}} \log \mathsf{P}[M(\infty) \geq \gamma n].$$

Write

$$\begin{aligned}
\mathsf{P}[M(\infty) \geq \gamma n] &= \mathsf{P}[M(n) \geq \gamma n] + \mathsf{P}[M(n) < \gamma n, \sup_{s>n}(Y(s) - cs) \geq \gamma n] \\
&\leq \mathsf{P}[M(n) \geq \gamma n] + \mathsf{P}[\sup_{m\geq n+1} Z(m) \geq 0], \qquad (24)
\end{aligned}$$

where $Z(n)$ follows the definition in Lemma 5.3.

The key now is to argue (cf. [12]) that γ can be made small enough so that "large excursions" are at least as likely to occur in the first term on the right side of (24) as in the second term. Specifically, if $\gamma < \frac{c(1-H)}{H}$, then from Lemma 5.2, we have

$$\limsup_{n\to\infty} \frac{1}{(\gamma n)^{2(1-H)}} \log \mathsf{P}[M(n) \geq \gamma n] = -\theta^*. \qquad (25)$$

It remains to show that γ can also be chosen small enough such that the second term is no larger than the first term. In view of the upper bound for $\{Z(n)/n, n \geq 0\}$ in Lemma 5.3, for every $\epsilon > 0$ there exists an n_0 such that for all $n \geq n_0$ we have

$$\mathsf{P}(Z(n)/n \geq 0) \leq \exp\Big(-n^{2(1-H)}\big(\frac{H}{\sigma^2}c^2 - \epsilon\big)\Big).$$

Now choose ϵ such that $0 < \epsilon < \frac{H}{2\sigma^2}c^2$, and then choose γ small enough such that

$$\gamma^{2(1-H)}\theta^* + \epsilon < \frac{H}{\sigma^2}c^2 - \epsilon \quad \text{and} \quad \gamma < \frac{c(1-H)}{H}.$$

This way, we have, for $n \geq n_0$,

$$\begin{aligned}
\mathsf{P}[\sup_{m \geq n+1} Z(m) \geq 0] &= \mathsf{P}[\sup_{m \geq n+1} Z(m)/m \geq 0] \\
&\leq \sum_{m=n+1}^{\infty} \mathsf{P}[Z(m)/m \geq 0] \\
&\leq \sum_{m=n+1}^{\infty} \exp\Big(- n^{2(1-H)}(\gamma^{2(1-H)}\theta^* + \epsilon)\Big). \qquad (26)
\end{aligned}$$

Simple algebra yields

$$\limsup_{n\to\infty} \frac{1}{(\gamma n)^{2(1-H)}} \log \sum_{m=n+1}^{\infty} \exp\Big(- n^{2(1-H)}(\gamma^{2(1-H)}\theta^* + \epsilon)\Big) \leq -\theta^*. \qquad (27)$$

(Note that for $\alpha > 0, z > 0$, $n > (\frac{2}{\alpha z})^{1/\alpha}$

$$ne^{-n^\alpha z} = \int_n^\infty (\alpha z s^\alpha - 1)e^{-s^\alpha z} ds \geq \int_n^\infty e^{-s^\alpha z} ds.$$

Choose $\alpha = 2(1-H)$, $z = \gamma^{2(1-H)}\theta^* + \epsilon$.) Therefore, combining (24), (25), (26) and (27), we have established the desired upper bound,

$$\limsup_{x\to\infty} \frac{1}{x^{2(1-H)}} \log \mathsf{P}(M(\infty) \geq x) \leq -\theta^*.$$

■

In closing, we remark that when Y follows Lévy's model of FBM, with $X(t) \equiv \sqrt{2H}B(t)$, and $B(t)$, the standard Brownian motion, we have $\mathsf{E}[Y^2(t)] = t^{2H}$. This is consistent with the relation in (57) of [12], where a different model of FBM is studied, with integration taken over $(-\infty, t)$ (hence producing a process with stationary increments). In addition, following Dembo [9], we know that X satisfies the conditions in Theorem 3.1 with $\sigma^2 = 2H$. Comparing the minimization problem in the proof of Lemma 5.2 with that of (58) of [12], we observe that in the analogue of Theorem 5.1 in [12], the constant θ^* is replaced by $\frac{\tilde{H}}{H}\theta^*$, where

$$\tilde{H} = \frac{1}{2}[\int_0^1 (1-s)^{2(H-1/2)} ds + \int_{-\infty}^0 [(1-s)^{H-1/2} - (-s)^{H-1/2}]^2 ds]^{-1}.$$

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