, , 1–29 ()

Large Deviations for Large Capacity Loss Networks with Fixed Routing and Polyhedral Admission Sets

CHENG-SHANG CHANG AND HUNG-JEN WANG cschang@ee.nthu.edu.tw

Department of Electrical Engineering, National Tsing Hua University, Hsinchu, 30043, Taiwan, R.O.C.

Abstract. In this paper, we study large deviations of large capacity loss networks with fixed routing. We use two-level modelling for the loss networks: the call level and the cell level. At the call level, a call request is accepted if it succeeds an admission test. The test is based on a polyhedral set of the number of calls in progress when a new call arrives. After being accepted, a call then transmits a sequence of cells (random variables) during its holding period. We show that the fluid limits and the conditional central limit theorems in Kelly [1991] can be extended to the large deviation regime. Moreover, there are corresponding fluid flow explanations for our large deviation results. In particular, we derive the exponential decay rates of the call blocking probability and the cell loss probability. These decay rates are obtained by solving primal and dual convex programming problems.

Keywords: Large deviations, loss networks, call blocking probability, cell loss probability

1. Introduction

As technology advances, integrated telecommunication networks have moved into the high speed regime, where the theory of large deviation principle (LDP) is applicable. To serve a large number of different sources in such high speed networks, information from different sources is divided into short and fixed length cells, e.g., 53 bytes in a cell of Asynchronous Transfer Mode (ATM) networks. One common design requirement is to have an extremely small cell loss probability in such a network, e.g., 10^{-9}.

There are two common approaches to achieve a small cell loss probability: large buffer and large capacity (or the combination of both). These two approaches also correspond to two different approaches to obtain large deviations. For the large buffer case, many large deviation results were obtained for a single node and intree networks (see e.g., Chang [1995] and Whitt [1993] and references therein). However, there are few large deviation results for general networks with feedback. The essential difficulty is the interaction among nodes due to buffering. We will not pursue this issue in the paper.

In this paper, we concentrate on the large capacity approach. To limit the interaction among queues, we only consider unbuffered networks and call these *loss networks* (see e.g., Kelly [1991] and Ross [1995]). As the multilayer concept outlined in Hui [1988], we use two-level modelling for the loss networks: the call level and the cell level.

1.1. Call level modelling and call blocking probability

Consider a loss network with R fixed routes, $r = 1, \ldots, R$. Calls requesting route r form a Poisson process with rate ν_r. Let $n_r(t)$ be the number of calls in progress at time t on route r. Define the R-vectors $\nu = (\nu_1, \ldots, \nu_R)$ and $n(t) = (n_1(t), \ldots, n_R(t))$, and the M-vector $b = (b_1, \ldots, b_M)$. The acceptance of a call requesting route r at time t is determined by an $M \times R$ matrix A with $A_{jr} \in I\!R_+$ and the following admission test

$$A(n(t) + e_r)^T \leq b^T, \tag{1}$$

where the superscript T denotes the transpose of a vector (matrix) and e_r is a R-vector with its r^{th} element being 1 and all other elements being 0. If the test is successful, the call request is accepted for the holding period of the call. Otherwise, it is blocked. Assume the holding periods of the route r calls are independent (of all else) and identically distributed with mean $1/\mu_r$. As in Kelly [1991], the stochastic process $(n(t), t \geq 0)$ has a unique stationary distribution

$$\pi(n) = \mathrm{P}(n(t) = n) = G(b^T)^{-1} \prod_{r=1}^{R} \frac{\rho_r^{n_r}}{n_r!}, \tag{2}$$

where $\rho_r = \nu_r / \mu_r$ and

$$G(b^T) = \sum_{An^T \leq b^T} \prod_{r=1}^{R} \frac{\rho_r^{n_r}}{n_r!} \tag{3}$$

is the normalization constant. In particular, the call block probability of route r is given by

$$\mathrm{P}_{CBP,r} = 1 - G(b^T)^{-1} G(b^T - Ae_r^T). \tag{4}$$

To avoid triviality, throughout the paper we assume $b > 0$. If $b_j = 0$ for some j, then all the routes with $A_{jr} > 0$ will be rejected all the time. One may then discard all these routes and reduce the problem to another one with $b > 0$.

To interpret our results in the call level, sometimes we find it convenient to map our model to a multirate loss network with M nodes (see e.g. Kelly [1991]). View b_j as the number of links of the j^{th} node, and A_{jr} as the number of links used by a route r call at the j^{th} node. An arrival of a route r call grabs A_{jr} links simultaneously along the nodes in its route (if they are available) and releases them simultaneously after the holding period.

There are many approximation results for large capacity loss networks based on the Erlang loss formula (see e.g., Kelly [1991], Labourdette and Hart [1992], Gazdzicki et al. [1993], and Mitra and Morrison [1994]). Among them, the approach by Kelly [1991] is of particular interest to us.

Kelly considered a sequence of loss networks, indexed by $m = 1, 2, \ldots$. In the m^{th} network, replace ν by $\nu_m = m\nu$, and b by $b_m = mb$. Denote by $\mathrm{P}_{CBP,r,m}$ the call blocking probability of the r^{th} route in the m^{th} network. He showed that

$$\lim_{m\to\infty} \mathrm{P}_{CBP,r,m} = \prod_{i=1}^{M} (1 - B_i)^{A_{ir}},$$

where the vector $(B_1, \ldots, B_M)$ with $B_j, j = 1, \ldots, M \in [0, 1)$, is a feasible solution of the following equations:

$$\sum_{r=1}^{R} A_{jr}\rho_r \prod_{i=1}^{M} (1 - B_i)^{A_{ir}} \begin{cases} = b_j & \text{if } B_j > 0, \\ \le b_j & \text{if } B_j = 0, \end{cases} \quad j = 1, \ldots, M. \tag{5}$$

Kelly [1991] had a very nice interpretation of these equations in terms of fluid flow for the multirate loss network with M nodes. It is as if nodes block independently, node j with probability B_j. An external flow of ρ_r on route r is thinned by a factor $(1 - B_j)^{A_{jr}}$ at each node, $j = 1, \ldots, M$, and thus a flow of $\rho_r \prod_{i=1}^{M} (1 - B_i)^{A_{ir}}$ remains. Let $\alpha_r^0 = \rho_r \prod_{i=1}^{M} (1 - B_i)^{A_{ir}}$ be the remaining flow on route r. Also, let $\alpha^0 = (\alpha_1^0, \ldots, \alpha_M^0)$. The vector α^0 is called the most "probable" state in Kelly [1991]. In the case that $A\rho^T \le b^T$ (the light load case), then an obvious solution for (5) is $B_j = 0, j = 1, \ldots, M$. Thus, we have $\alpha_r^0 = \rho_r$, $r = 1, \ldots, M$, and $\mathrm{P}_{CBP,r,m} \to 0$ as $m \to \infty$.

In this paper, we extend Kelly's results to the large deviation regime. Let $\alpha = (\alpha_1, \ldots, \alpha_M)$ and

$$\Lambda_N^*(\alpha) = \sum_{r=1}^{R} \alpha_r \log \alpha_r - \alpha_r \log \rho_r - \alpha_r + \rho_r, \quad \alpha \ge 0. \tag{6}$$

Let $n_m(t) = (n_{m,1}(t), \ldots n_{m,R}(t))$ be the number of calls in progress at time t in the m^{th} network. We first establish in Theorem 3 the LDP for $n_m(t)/t$. Roughly speaking, the probability for $n_m(t)/t$ to behave as if it is a vector fluid flow with the rate vector α is as small as $\exp(-m(\Lambda_N^*(\alpha) - \Lambda_N^*(\alpha^0)))$.

The LDP result is then used in this paper to obtain the exponential decay rate of the call blocking probability as follows:

$$\lim_{m\to\infty} \frac{1}{m} \log \mathrm{P}_{CBP,r,m} = -\min_{j: A_{jr} > 0} [\Lambda_N^*(\alpha^j)] - \Lambda_N^*(\alpha^0), \tag{7}$$

where $\alpha^j = (\alpha_1^j, \ldots, \alpha_M^j)$, $j = 1, \ldots, R$, with

$$\alpha_r^j = \rho_r e^{\theta_j A_{jr}} \prod_{i=1}^{M} (1 - B_i)^{A_{ir}}, \tag{8}$$

and the vector $(B_1, \ldots, B_M, \theta_j)$ with $B_\ell \in [0, 1), \ell = 1, \ldots, M$, is a feasible solution of the following equations:

$$\sum_{r=1}^{R} A_{\ell r}\rho_r e^{\theta_j A_{jr}} \prod_{i=1}^{M}(1-B_i)^{A_{ir}} \begin{cases} = b_\ell & \text{if } B_\ell > 0, \\ \le b_\ell & \text{if } B_\ell = 0, \end{cases} \quad \ell = 1,\ldots,M, \tag{9}$$

and

$$\sum_{r=1}^{R} A_{jr}\rho_r e^{\theta_j A_{jr}} \prod_{i=1}^{M}(1-B_i)^{A_{ir}} = b_j. \tag{10}$$

Extending Kelly's fluid flow interpretation, call blocking of route r is the most likely to occur at node j when the arrival process of route r calls, $r = 1,\ldots,R$, behaves as if a fluid flow with rate $\rho_r e^{\theta_j A_{jr}}$. The route r flow is then thinned by a factor $(1-B_i)^{A_{ir}}$ for all $i = 1,\ldots,M$ and a flow of $\alpha_r^j = \rho_r e^{\theta_j A_{jr}} \prod_{i=1}^{M}(1-B_i)^{A_{ir}}$ remains. Thus, the total flow into the j^{th} node is

$$\sum_{r=1}^{R} \alpha_r^j A_{jr} = \sum_{r=1}^{R} \rho_r e^{\theta_j A_{jr}} \prod_{i=1}^{M}(1-B_i)^{A_{ir}} A_{jr}$$

which is equal to the (normalized) number of links b_j. According the LDP result, the probability for this to occur is approximately $\exp(-m(\Lambda_N^*(\alpha^j) - \Lambda_N^*(\alpha^0)))$. This provides the intuition for the result in (7). A formal argument for such a result will be given in Section 3.1.

We note that the large deviation upper bound (Chernoff's bound) for the complete sharing policy (the matrix A is a row vector) at the call level was previously derived by Hui [1988], and Labourdette and Hart [1992]. In Gazdzicki et al. [1993], local limit theorems were used to refine the upper bound.

Hunt and Kelly [1989] also considered expansions from the most "probable" state α^0 and derived conditional central limit theorems (see also Kelly [1991] and Reiman [1991]). This corresponds to the critically loaded case in the loss network. Analogously, we consider sublinear expansions slightly faster than the expansion in Kelly [1991], and derive various LDP results with sublinear speed in Section 3.2. Our LDP results in Section 3.2 are parallel to those conditional central limit theorems in Kelly [1991] and Hunt and Kelly [1989].

1.2. Cell level modelling and cell loss probability

At the cell level, we model the network as a discrete-time network with J nodes (J and M may not be identical). The capacity of the j^{th} node is c_j, $j = 1,\ldots,J$, i.e., at most c_j cells can be transmitted per unit of time. Define the J-vector $c = (c_1,\ldots,c_J)$. Let D be the routing matrix with $D_{jr} = 1$, $j = 1,\ldots,J$, $r = 1,\ldots,R$, if the j^{th} node is visited by route r, and $D_{jr} = 0$ otherwise. When a call of route r is accepted, it transmits a sequence of cells, independent of all else, through its route during its holding period. All the sequences of cells of route r calls are assumed to have the same joint distribution as a *stationary* sequence of random variables $\{a_r(t), t = 1,2,\ldots\}$. Also, let $\bar{a}_r = \mathsf{E}a_r(1)$ and $\Lambda_{a,r}(\theta_r) = \log \mathsf{E}e^{\theta_r a_r(1)}$, $r = 1,\ldots,R$. Assume that $\Lambda_{a,r}(\theta_r) < \infty$ for all θ_r, $r = 1,\ldots,R$. As in the call level, we assume there is *no buffer* for the cells in the nodes. Thus, a cell loss occurs immediately when the number of arriving cells at a node exceeds its capacity. To simplify our model, we also assume zero propagation delay between any two nodes in the network.

As before, we consider a sequence of networks described above, indexed by $m = 1, 2, \ldots$. In the m^{th} network, we replace c by $c_m = mc$, ν by $\nu_m = m\nu$, and b by $b_m = mb$. To derive the large deviation approximation of the cell loss probability, we first find a feasible solution for the vector $(B_1, \ldots, B_M, \eta_j)$ with $B_\ell \in [0, 1), \ell = 1, \ldots, M$, that satisfies the following conditions

$$\sum_{r=1}^{R} A_{\ell r} \rho_r e^{D_{jr}\Lambda_{a,r}(\eta_j)} \prod_{i=1}^{M} (1 - B_i)^{A_{ir}} \begin{cases} = b_\ell & \text{if } B_\ell > 0, \\ \leq b_\ell & \text{if } B_\ell = 0, \end{cases} , \ell = 1, \ldots, M, \tag{11}$$

and

$$\sum_{r=1}^{R} \rho_r e^{D_{jr}\Lambda_{a,r}(\eta_j)} \prod_{i=1}^{M} (1 - B_i)^{A_{ir}} D_{jr} \Lambda'_{a,r}(\eta_j) = c_j. \tag{12}$$

One may explain these equations by Kelly's fluid flow interpretation. Congestion is the most likely to occur at node j when the arrival process of route r calls, $r = 1, \ldots, R$, behaves as if a fluid flow with rate $\rho_r e^{D_{jr}\Lambda_{a,r}(\eta_j)}$, and the sequences of cells of route r calls behave as if fluid flows with rate $\Lambda'_{a,r}(\eta_j)$. The route r flow is then thinned by a factor $(1 - B_i)^{A_{ir}}$ for all $i = 1, \ldots, M$ and a flow of $\rho_r e^{D_{jr}\Lambda_{a,r}(\eta_j)} \prod_{i=1}^{M} (1 - B_i)^{A_{ir}}$ remains. Thus, the total flow into the j^{th} node is

$$\sum_{r=1}^{R} \rho_r e^{D_{jr}\Lambda_{a,r}(\eta_j)} \prod_{i=1}^{M} (1 - B_i)^{A_{ir}} D_{jr} \Lambda'_{a,r}(\eta_j)$$

which is equal to the capacity c_j.

From the vector $(B_1, \ldots, B_M, \eta_j)$ in (11) and (12), we define the vector $\bar{\alpha}^j = (\bar{\alpha}^j_1, \ldots, \bar{\alpha}^j_M)$, $j = 1, \ldots, R$, with

$$\bar{\alpha}^j_r = \rho_r e^{D_{jr}\Lambda_{a,r}(\eta_j)} \prod_{i=1}^{M} (1 - B_i)^{A_{ir}}. \tag{13}$$

As explained earlier, $\bar{\alpha}^j_r$ is the remaining route r flow when congestion occurs at node j. According to the LDP result for $n_m(t)/m$, the probability for $n_m(t)/m$ to behave as if it is a vector fluid flow with the rate vector $\bar{\alpha}^j$ is approximately $\exp(-m(\Lambda^*_N(\bar{\alpha}^j) - \Lambda^*_N(\alpha^0)))$. On the other hand, conditioning on that $n_m(t)/m$ behaves as if it is a vector fluid flow with the rate vector $\bar{\alpha}^j$ and that route r visits node j ($D_{jr} = 1$), the probability for the $n_{m,r}(t)$ independent sequences of cells of route r calls to behave as if fluid flow with rate $\Lambda'_{a,r}(\eta_j)$ is approximately $\exp(-m\bar{\alpha}^j_r D_{jr}(\eta_j \Lambda'_{a,r}(\eta_j) - \Lambda_{a,r}(\eta_j)))$, according to the Cramér's theorem (see e.g.

Dembo and Zeitouni [1992]). Thus, the probability for congestion to occur at node j is approximately

$$e^{-m(\Lambda_N^*(\bar{\alpha}^j)-\Lambda_N^*(\alpha^0))}\prod_{r=1}^{R} e^{-m\bar{\alpha}_r^j D_{jr}(\eta_j\Lambda'_{a,r}(\eta_j)-\Lambda_{a,r}(\eta_j))}.$$

This leads to the following approximation for the cell loss probability of the m^{th} network

$$e^{-m\min_{j=1,\ldots,J}[(\Lambda_N^*(\bar{\alpha}^j)-\Lambda_N^*(\alpha^0))+\sum_{r=1}^{R}\bar{\alpha}_r^j D_{jr}(\eta_j\Lambda'_{a,r}(\eta_j)-\Lambda_{a,r}(\eta_j))]}. \tag{14}$$

A formal argument for such a result will be given in Section 4.

We note that the solution of (11) and (12) can be found by the following "water-filling" algorithm under the light load conditions $\sum_{r=1}^{R} D_{jr}\alpha_r^0\bar{a}_r < c_j$, $j=1,\ldots,J$. One may start from $\eta_j = 0$. Find the Kelly's fluid limit $(B_1,\ldots,B_M)$ for that η_j in (11) (as noted in Kelly [1991], the solution of (11) may not be unique). Then use the left hand side of (12) to compute the total flow into the j^{th} node. Since $\sum_{r=1}^{R} D_{jr}\alpha_r^0\bar{a}_r < c_j$, the total flow into the j^{th} node is less than the capacity c_j when $\eta_j = 0$. Clearly, increasing η_j increases the external flows into the network, which in turn increases the total flow into the j^{th} node. The solution of (12) can then be found by keeping on pouring water into the network until the total flow into the j^{th} node reaches its capacity c_j.

Observe that the set of conditions in (9) and (10) is a special case of the set of conditions in (11) and (12) by taking $A_{jr} = D_{jr}\bar{a}_r$ and $\Lambda_{a,r}(\eta_j) = \bar{a}_r\eta_j$ ($a_r(1)$ is a constant). Thus, the exponential decay rate of the call blocking probabilities can also be obtained by the same water-filling algorithm. In particular, when considering the complete sharing policy as a special case, we have $B_j = 0$ for all j and $D_{jr} = 1$ for all j and r in (12) (under the assumption that $\sum_{r=1}^{R} D_{jr}\alpha_r^0\bar{a}_r < c_j$, $j=1,\ldots,J$). This recovers the key equations in Hui [1988], Labourdette and Hart [1992], Gazdzicki et al. [1993], and Mitra and Morrison [1994].

2. Review of large deviations

In this section, we briefly review the definition of the large deviation principle. See e.g., Dembo and Zeitouni [1992] for more details.

Definition 1. A sequence of probability measures $\mu^{(n)}$ on a metric space $\mathcal{S}$ satisfies the LDP with speed $\delta_n \uparrow \infty$ and good rate function $I(\cdot) : \mathcal{S} \to [0,\infty]$ if

(i) the level sets $\{x \in \mathcal{S} : I(x) \le \alpha\}$, $\alpha \in \mathbb{R}$, are compact,

(ii) for any closed set F,

$$\limsup_{n\to\infty}\frac{1}{\delta_n}\log\mu^{(n)}(F) \le -\inf_{x\in F} I(x), \tag{15}$$

(iii) for any open set G,

$$\liminf_{n\to\infty} \frac{1}{\delta_n} \log \mu^{(n)}(G) \geq - \inf_{x\in G} I(x). \tag{16}$$

A sequence of random variables is said to satisfy the LDP if the corresponding sequence of distributions does.

A set F is called I-continuous if

$$\inf_{x\in F^o} I(x) = \inf_{x\in \bar{F}} I(x), \tag{17}$$

where F^0 is its interior and $\bar{F}$ is its closure. In this case, one has

$$\lim_{n\to\infty} \frac{1}{\delta_n} \log \mu^{(n)}(F) = - \inf_{x\in F} I(x). \tag{18}$$

We will use the following theorem to establish the LDP in the paper. For more details of the theorem, see e.g. Dembo and Zeitouni [1992], Theorem 2.3.6 and Exercise 2.3.20. Let

$$\{Y_n = (Y_{n,1}, \ldots, Y_{n,d}), n \geq 1\}$$

be a sequence of (not necessarily independent) random vectors in R^d and $\phi_n(\theta) = \frac{1}{\delta_n} \log Ee^{\delta_n \theta Y_n^T}$, where $\theta = (\theta_1, \ldots, \theta_d)$ is a vector in R^d and $\theta Y_n^T = \sum_{i=1}^d \theta_i Y_{n,i}$ is the inner product.

THEOREM 1 *(Gärtner [1977] and Ellis [1984]) Assume*

(A1) $\lim_{n\to\infty} \phi_n(\theta) = \phi(\theta) < \infty$ *for all* $\theta \in I\!\!R^d$,

(A2) $\phi(\theta)$ *is differentiable for all* $\theta \in I\!\!R^d$.

Let $I(\alpha) = \sup_\theta [\theta\alpha^T - \phi(\theta)]$ *be the convex transform (or Legendre transform) of* $\phi(\theta)$. *Then* $\{Y_n, n \geq 1\}$ *satisfies the LDP with the good rate function* $I(\alpha)$ *and the speed* δ_n.

Another important theorem for the LDP is the contraction principle (continuous mapping theorem) [see e.g. Dembo and Zeitouni [1992], Theorem 4.2.1].

THEOREM 2 **(Contraction Principle)** *Let* f *be a continuous function from a metric space* $\mathcal{S}_1$ *to another metric space* $\mathcal{S}_2$, *i.e.,* $f : \mathcal{S}_1 \mapsto \mathcal{S}_2$. *If* $\mu_1^{(n)}$ *satisfies the LDP with the good rate function* $I_1(\phi)$ *in* $\mathcal{S}_1$, *then the induced sequence of measures* $\mu_2^{(n)} = \mu_1^{(n)} \circ f^{-1}$ *in* $\mathcal{S}_2$ *satisfies the LDP with the good rate function* $I_2(\psi)$, *where*

$$I_2(y) = \inf\{I_1(x) : y = f(x), x \in \mathcal{S}_1\}, \tag{19}$$

with the convention that the infimum over an empty set is ∞.

3. Large deviations at the call level

3.1. Large deviations with linear speed

Consider a sequence of networks in Section 1.1, indexed by $m = 1, 2, \ldots$. Our objective in this section is to establish the LDP results for such networks as $m \to \infty$. Define the row vectors $\theta = (\theta_1, \ldots, \theta_R)$, $\rho = (\rho_1, \ldots, \rho_R)$ and $\alpha = (\alpha_1, \ldots, \alpha_R)$. Let

$$\Lambda_N(\theta) = \sup_\alpha [\theta\alpha^T - \Lambda_N^*(\alpha)] = \sum_{r=1}^R \rho_r (e^{\theta_r} - 1), \tag{20}$$

where $\Lambda_N^*(\alpha)$ is defined in (6). Note that $\Lambda_N(\theta)$ and $\Lambda_N^*(\alpha)$ are convex conjugates and they are related by the convex transform i.e., $\Lambda_N^*(\alpha) = \sup_\theta[\theta\alpha^T - \Lambda_N(\theta)]$. Moreover, $\Lambda_N^*(\alpha) = \theta\alpha^T - \Lambda_N(\theta)$ if and only if α is the gradient of $\Lambda_N(\theta)$, i.e., $\alpha = \nabla\Lambda_N(\theta)$ (see e.g., Rockafellar [1970], Theorem 26.1, or Hiriart-Urruty and Lemaaréchal [1991], Corollary X.1.4.4). Also, let

$$\Lambda_1(\theta) = \sup_{\alpha \geq 0, A\alpha^T \leq b^T} [\theta\alpha^T - \Lambda_N^*(\alpha)]. \tag{21}$$

The following theorem establishes the LDP for the number of calls in progress at time t. The proof of the theorem will be given at the end of this section.

THEOREM 3 *Let $n_m(t) = (n_{m,1}(t), \ldots n_{m,R}(t))$ be the number of calls in progress at time t in the m^{th} network. Then*

$$\Lambda_n(\theta) \equiv \lim_{m\to\infty} \frac{1}{m} \log \mathsf{E} e^{\theta n_m(t)^T} = \Lambda_1(\theta) - \Lambda_1(\mathbf{0}), \tag{22}$$

and $\{n_m(t)/m, m = 1, 2, \ldots\}$ satisfies the LDP with the speed m and the good rate function $\Lambda_n^*(\cdot) : I\!R^R \mapsto I\!R$

$$\Lambda_n^*(\alpha) = \sup_\theta [\theta\alpha^T - \Lambda_n(\theta)] = \begin{cases} \Lambda_N^*(\alpha) + \Lambda_1(\mathbf{0}) & \textit{if } \alpha \geq 0, A\alpha^T \leq b^T \\ \infty & \textit{otherwise} \end{cases}, \tag{23}$$

where $\mathbf{0}$ is the vector with all its elements being 0.

We note that (i) $\Lambda_n(\theta) < \infty$ for all θ, (ii) $\Lambda_n(\theta)$ and $\Lambda_n^*(\alpha)$ are convex conjugates, and (iii) $\Lambda_n(\theta)$ is differentiable for all θ. To see (i), observe $\Lambda_1(\theta) \leq \sup_\alpha[\theta\alpha^T - \Lambda_N^*(\alpha)] = \Lambda_N(\theta) < \infty$ for all θ. For (ii), observe from (21) that $\Lambda_1(\theta)$ can be written as a convex transform of the function

$$\Lambda_1^*(\alpha) = \begin{cases} \Lambda_N^*(\alpha) & \text{if } \alpha \geq 0, A\alpha^T \leq b^T \\ \infty & \text{otherwise} \end{cases}. \tag{24}$$

Thus, $\Lambda_1(\theta)$ is a closed convex function. So is $\Lambda_n(\theta)$. This implies the convex transform of the convex transform of $\Lambda_n(\theta)$ is itself (see e.g. Rockafellar [1970], Theorem 23.5). For (iii), observe that $\Lambda_n^*(\alpha)$ is strictly convex in its effective domain $\alpha \geq 0, A\alpha^T \leq b^T$ since $\Lambda_N^*(\alpha)$ is strictly convex. The strict convexity of $\Lambda_n^*(\alpha)$ implies that $\Lambda_n(\theta)$ is differentiable for all θ (see e.g. Rockafellar [1970], Theorem 26.3 or Hiriart-Urruty and Lemaaréchal [1991], Theorem X.4.1.1).

Since $\Lambda_n(\theta)$ is differentiable, let $\alpha^*(\theta)$ be the gradient of $\Lambda_n(\theta)$, i.e.,

$$\alpha^*(\theta) = \bigtriangledown\Lambda_n(\theta) = \bigtriangledown\Lambda_1(\theta). \tag{25}$$

The quantity $\alpha^*(\theta)$ is the key element in this paper. From the convex transform in (23), one has that

$$\Lambda_n^*(\alpha^*(\theta)) = \theta\alpha^*(\theta)^T - \Lambda_n(\theta) \tag{26}$$

and that $\alpha^*(\theta)$ is in the effective domain of $\Lambda_n^*(\alpha)$, i.e., $\alpha^*(\theta) \geq 0$ and $A\alpha^*(\theta)^T \leq b^T$ for all θ. The identity in (26) is then equivalent to

$$\Lambda_1(\theta) = \theta\alpha^*(\theta)^T - \Lambda_N^*(\alpha^*(\theta)). \tag{27}$$

Thus, $\alpha^*(\theta)$ uniquely achieves the maximum in the following optimization problem:

$$\begin{aligned} &\text{Maximize} \quad \theta\alpha^T - \Lambda_N^*(\alpha) \\ &\text{subject to} \quad \alpha \geq 0, A\alpha^T \leq b^T. \end{aligned} \tag{28}$$

When taking $\theta = \mathbf{0}$ in (27), one has from (23) that

$$\Lambda_n^*(\alpha^*(\mathbf{0})) = \Lambda_N^*(\alpha^*(\mathbf{0})) + \Lambda_1(\mathbf{0}) = 0. \tag{29}$$

As a direct consequence of the LDP in Theorem 3 and the strict convexity of $\Lambda_n^*(\alpha)$,

$$\mathrm{P}(\max_{r=1,\dots,R} |n_{m,r}(t)/m - \alpha_r^*(\mathbf{0})| \geq \epsilon) \tag{30}$$

goes to zero exponentially fast. Thus, one has (from the Borel-Cantelli lemma) the following strong law for $\{n_m(t)/m, m \geq 1\}$.

Corollary 1 $\lim_{n\to\infty} \frac{n_m(t)}{m} = \alpha^*(\mathbf{0}) \quad a.s.$

To find $\alpha^*(\theta)$ from the optimization problem in (28), we follow the argument in Kelly [1991]. Consider the Lagrangian form

$$L(\alpha, z; y) = \theta\alpha^T - \Lambda_N^*(\alpha) + y(b^T - A\alpha^T - z^T), \tag{31}$$

where $y = (y_1, \dots, y_M)$ is a vector of the Lagrange multipliers and $z = (z_1, \dots, z_M)$ is the vector of slack variables. To maximize $L(\alpha, z; y)$ over the cone $\alpha, z \geq 0$, one requires $y \geq 0$, $yz^T = 0$, and differentiating with respect to α_r,

$$\theta_r + \log\rho_r - \log\alpha_r - \sum_{j=1}^{M} y_j A_{jr} = 0. \tag{32}$$

Thus, the maximizing α_r is

$$\alpha_r = \rho_r e^{\theta_r - \sum_{j=1}^{M} y_j A_{jr}}, \tag{33}$$

and we have the dual problem

$$\begin{aligned} &\text{Minimize} \quad \Lambda_N(\theta - yA) + yb^T \\ &\text{subject to} \quad y \geq 0. \end{aligned} \tag{34}$$

To solve the optimization problem, one may choose values for the Lagrange multipliers $y = \bar{y}$ so that $\alpha = \bar{\alpha}$, $\bar{y}$ are primal and dual feasible, i.e.,

$$\bar{\alpha} \geq 0, \quad \bar{z}^T = b^T - A\bar{\alpha}^T \geq 0, \quad \bar{y} \geq 0, \tag{35}$$

and the complementary slackness constraint

$$\bar{y}\bar{z}^T = 0. \tag{36}$$

As in Kelly [1991], we use the transformation $B_j = 1 - exp(-y_j)$, $j = 1, \ldots, M$. This yields the following set of conditions:

$$\sum_{r=1}^{R} A_{jr}\rho_r e^{\theta_r} \prod_{i=1}^{M} (1 - B_i)^{A_{ir}} \begin{cases} = b_j & \text{if } B_j > 0, \\ \leq b_j & \text{if } B_j = 0, \end{cases} \tag{37}$$

and $B_1, \ldots, B_M \in [0, 1)$. In conjunction with (33),

$$\alpha_r^*(\theta) = \rho_r e^{\theta_r} \prod_{j=1}^{M} (1 - B_j)^{A_{jr}}, \quad r = 1, \ldots, R. \tag{38}$$

In particular, when $\theta = \mathbf{0}$, one has $\alpha^*(\mathbf{0}) = \alpha^0$ as the most "probable" state for the original loss network. Moreover, $\Lambda_1(\mathbf{0}) = -\Lambda_N^*(\alpha^0)$ from (26). If, furthermore, $A\rho^T \leq b^T$, then an obvious solution for (37) is $B_j = 0, j = 1, \ldots, M$, and we have $\alpha^*(\mathbf{0}) = \rho$. From (29), it then follows that $\Lambda_1(\mathbf{0}) = -\Lambda_N^*(\rho) = 0$.

To explore further connections between Kelly's fluid results and our LDP results, we consider the following objective function for the m^{th} network

$$W^{(m)}(\rho, A, b) = \mathsf{E} \exp(\sum_{r=1}^{R} \theta_r n_{m,r}(t)). \tag{39}$$

To interpret the objective function, suppose that a route r call generates a revenue at the rate $\theta_r \geq 0$, $r = 1, \ldots, R$. Then the rate of the return from the m^{th} loss network at time t is $\sum_{r=1}^{R} \theta_r n_r(t)$. The objective function is then the expected exponential rate of return. This objective function, though different from the expected rate of return in Kelly [1991], Hunt [1989] ,and Reiman [1991], was previously used in Whittle [1990,1991a,1991b].

From Theorem 3 and (27), it follows that

$$\begin{aligned} &\lim_{m \to \infty} \frac{1}{m} \log W^{(m)}(\rho, A, b) = \Lambda_n(\theta) \\ &= \sup_{\alpha \geq 0, A\alpha^T \leq b^T} [\theta\alpha^T - \Lambda_N^*(\alpha)] - \Lambda_1(\mathbf{0}) && (40) \\ &= \theta\alpha^*(\theta)^T - \Lambda_N^*(\alpha^*(\theta)) - \Lambda_1(\mathbf{0}). && (41) \end{aligned}$$

Thus, for large m, one has (up to log equivalence)

$$W^{(m)}(\rho, A, b) \approx e^{m(\theta\alpha^*(\theta)^T - (\Lambda_N^*(\alpha^*(\theta)) + \Lambda_1(\mathbf{0})))}$$
$$= e^{m\theta\alpha^*(\theta)^T} e^{-m(\Lambda_N^*(\alpha^*(\theta)) + \Lambda_1(\mathbf{0}))}. \tag{42}$$

This shows that $W^{(m)}(\rho, A, b)$ is earned mostly by the most "profitable" state $\alpha^*(\theta)$. The first term $e^{m\theta\alpha^*(\theta)^T}$ is the gain from the state $\alpha^*(\theta)$ and the second term $e^{-m(\Lambda_N^*(\alpha^*(\theta)) + \Lambda_1(\mathbf{0}))}$ is the probability of being in that state. See also the maximum principle in Whittle [1990]. For a multirate loss network in Kelly [1991], one may explain intuitively that $W^{(m)}(\rho, A, b)$ is earned mostly when the external arrival process of the route r behaves as if a flow with rate $\rho_r e^{\theta_r}$, $r = 1, \ldots, R$. The probability for this to occur is approximately $e^{-m(\Lambda_N^*(\alpha^*(\theta)) + \Lambda_1(\mathbf{0}))}$.

The condition $An(t)^T \leq b^T$ corresponds to the complete sharing policy in a multirate loss network. One common question is whether one should impose further conditions to improve the gain. When $A\rho^T \leq b^T$, $\Lambda_1(\mathbf{0}) = 0$. For our objective function, one can see from (40) that further linear conditions only impair the gain of the network for large m. This implies complete sharing is asymptotically optimal for our objective function in light load when the admission test is based on a polyhedral admission set. This is intuitive, since in light load, capacity is ample and there is no need to limit the access of any call.

In Theorems 4 and 5, we show the LDP for the number of free links in the M nodes of the multirate loss network, and the LDP for the call blocking probabilities.

THEOREM 4 *Let $f_m(t)^T = mb^T - An_m(t)^T$, $\tilde{\alpha} = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_M)$, and $\tilde{\theta} = (\tilde{\theta}_1, \ldots, \tilde{\theta}_M)$. Then $\{f_m(t)/m, m = 1, 2, \ldots\}$ satisfies the LDP with the speed m and the good rate function $\Lambda_f^*(\cdot) : I\!R^M \mapsto I\!R$*

$$\Lambda_f^*(\tilde{\alpha}) = \begin{cases} \inf_{\{\alpha \geq 0, A\alpha^T = b^T - \tilde{\alpha}^T\}} [\Lambda_N^*(\alpha) + \Lambda_1(\mathbf{0})] & \text{if } \tilde{\alpha} \geq 0 \\ \infty & \text{otherwise.} \end{cases} \tag{43}$$

Moreover,

$$\lim_{m\to\infty} \frac{1}{m} \log \mathsf{E} e^{\tilde{\theta} f_m(t)^T} = \Lambda_1(-\tilde{\theta}A) - \Lambda_1(\mathbf{0}) + \tilde{\theta} b^T, \quad \text{for all } \tilde{\theta} \in I\!R^M. \tag{44}$$

THEOREM 5 *Let $P_{CBP,r,m}$ be the call blocking probability of the r^{th} route in the m^{th} network. Then*

$$\lim_{m\to\infty} \frac{1}{m} \log P_{CBP,r,m}$$
$$= -\min_{j: A_{jr} > 0} \Big[\inf_{\alpha \geq 0, A\alpha^T \leq b^T, \sum_{r=1}^R A_{jr}\alpha_r = b_j} (\Lambda_N^*(\alpha) + \Lambda_1(\mathbf{0})) \Big]. \tag{45}$$

Now we show the optimization problem in (45) can be solved as in (7). Note that

$$\inf_{\alpha \geq 0, A\alpha^T \leq b^T, \sum_{r=1}^R A_{jr}\alpha_r = b_j} (\Lambda_N^*(\alpha) + \Lambda_1(\mathbf{0}))$$
$$= \inf_{\sum_{r=1}^R A_{jr}\alpha_r = b_j} \Lambda_n^*(\alpha) = -\sup_{\sum_{r=1}^R A_{jr}\alpha_r = b_j} [-\Lambda_n^*(\alpha)]. \tag{46}$$

Thus, we have the optimization problem

$$\begin{aligned} &\text{Maximize} \quad && -\Lambda_n^*(\alpha) \\ &\text{subject to} \quad && \sum_{r=1}^{R} A_{jr}\alpha_r = b_j. \end{aligned}$$

Analogous to the argument in (34), the problem can be solved by maximizing the following Lagrangian form over all α and the Lagrange multiplier θ_j

$$\begin{aligned} L(\alpha, \theta_j) &= -\Lambda_n^*(\alpha) - \theta_j(b_j - \sum_{r=1}^{R} A_{jr}\alpha_r) \\ &= \sum_{r=1}^{R} \theta_j A_{jr}\alpha_r - \Lambda_n^*(\alpha) - \theta_j b_j. \end{aligned} \tag{47}$$

Since $\Lambda_n(\theta)$ and $\Lambda_n^*(\alpha)$ are convex conjugates,

$$\sup_{\alpha} L(\alpha, \theta_j) = \Lambda_n(\theta_j A_{j1}, \ldots, \theta_j A_{jR}) - \theta_j b_j. \tag{48}$$

As there is no restriction for the Lagrange multiplier θ_j, we differentiate (48) with respect to θ_j and set it to 0. In view of (25), this yields

$$\sum_{r=1}^{R} \alpha_r^*(\theta_j A_{j1}, \ldots, \theta_j A_{jR}) A_{jr} = b_j. \tag{49}$$

In conjunction with the conditions in (37) and (38), the optimization problem in (45) can be solved by finding a feasible solution for the vector $(B_1, \ldots, B_M, \theta_j)$ with $B_\ell \in [0, 1), \ell = 1, \ldots, M$, that satisfies the following conditions:

$$\sum_{r=1}^{R} A_{\ell r}\rho_r e^{\theta_j A_{jr}} \prod_{i=1}^{M} (1 - B_i)^{A_{ir}} \begin{cases} = b_\ell & \text{if } B_\ell > 0, \\ \le b_\ell & \text{if } B_\ell = 0, \end{cases} \quad \ell = 1, \ldots, M, \tag{50}$$

and

$$\sum_{r=1}^{R} A_{jr}\rho_r e^{\theta_j A_{jr}} \prod_{i=1}^{M} (1 - B_i)^{A_{ir}} = b_j. \tag{51}$$

The rest of section is devoted to proofs of Theorems 3, 4 and 5. We will need the following proposition for the proof of Theorem 3.

PROPOSITION 1 *Let* $N(m\rho) = (N_1(m\rho_1), \ldots, N_R(m\rho_R))$, *where* $N_r(m\rho_r)$, $r = 1, \ldots, R$ *are* independent *Poisson random variables with parameters* $m\rho_r$, $r = 1, \ldots, R$. *Then* $\{N(m\rho)/m, m = 1, 2, \ldots\}$ *satisfies the LDP with the rate function* $\Lambda_N^*(\alpha)$ *defined in (6).*

Proof. This is a direct consequence of the Gärtner-Ellis theorem. One first computes the following limit

$$\lim_{m\to\infty} \frac{1}{m} \log \mathsf{E} e^{\theta N(m\rho)^T} = \sum_{r=1}^{R} \rho_r (e^{\theta_r} - 1), \tag{52}$$

and then derives its convex transform

$$\Lambda_N^*(\alpha) = \sup_{\theta} [\theta\alpha^T - \sum_{r=1}^{R} \rho_r (e^{\theta_r} - 1)]. \tag{53}$$

Q.E.D.

Let $S = \{\alpha \in I\!R^R : \alpha > 0, A\alpha^T < b^T\}$. Since we assume $A_{jr} \in I\!R_+$ and $b > 0$, S is a nonempty open set and the set $\bar{S} = \{\alpha \in I\!R^R : \alpha \geq 0, A\alpha^T \leq b^T\}$ is its closure. Note that $\Lambda_N^*(\alpha)$ is continuous for $\alpha > 0$ and thus continuous for $\alpha \in S$. This implies

$$\sup_{\alpha\in S} [\theta\alpha^T - \Lambda_N^*(\alpha)] = \sup_{\alpha\in \bar{S}} [\theta\alpha^T - \Lambda_N^*(\alpha)] = \Lambda_1(\theta) \tag{54}$$

for all θ. In particular, when taking $\theta = \mathbf{0}$,

$$\inf_{\alpha\in S} \Lambda_N^*(\alpha) = \inf_{\alpha\in \bar{S}} \Lambda_N^*(\alpha) = -\Lambda_1(\mathbf{0}), \tag{55}$$

and S is a $\Lambda_N^*(\alpha)$ continuous set.
[*Proof of Theorem 3*] Note from (2) that

$$\begin{aligned}
\mathrm{P}(n_m(t) = n) &= G(mb^T)^{-1} \prod_{r=1}^{R} \frac{(m\rho_r)^{n_r}}{n_r!} \\
&= (\exp(m\sum_{r=1}^{R}\rho_r) G(mb^T)^{-1}) \exp(-m\sum_{r=1}^{R}\rho_r) \prod_{r=1}^{R} \frac{(m\rho_r)^{n_r}}{n_r!} \\
&= \frac{\mathrm{P}(N(m\rho) = n)}{\mathrm{P}(AN(m\rho)^T \leq mb^T)},
\end{aligned} \tag{56}$$

where $N(m\rho) = (N_1(m\rho_1), \ldots, N_R(m\rho_R))$ are independent Poisson random variables with parameters $m\rho_r$, $r = 1, \ldots, R$. Thus,

$$\begin{aligned}
&\lim_{m\to\infty} \frac{1}{m} \log \mathsf{E} e^{\theta n_m(t)^T} \\
&= \lim_{m\to\infty} \frac{1}{m} \log \sum_{An^T \leq mb^T} e^{\theta n^T} \frac{\mathrm{P}(N(m\rho) = n)}{\mathrm{P}(AN(m\rho)^T \leq mb^T)} \\
&= \lim_{m\to\infty} \frac{1}{m} \log \sum_{An^T \leq mb^T} e^{\theta n^T} \mathrm{P}(N(m\rho) = n) \\
&\quad - \lim_{m\to\infty} \frac{1}{m} \log \mathrm{P}(AN(m\rho)^T \leq mb^T)
\end{aligned} \tag{57}$$

From the LDP for independent Poisson random variables in Proposition 1 and (55),

$$\lim_{m\to\infty}\frac{1}{m}\log \mathrm{P}(AN(m\rho)^T \le mb^T) = \Lambda_1(\mathbf{0}). \tag{58}$$

Note that

$$\begin{aligned}&\lim_{m\to\infty}\frac{1}{m}\log\sum_{An^T\le mb^T} e^{\theta n^T}\mathrm{P}(N(m\rho)=n)\\ &=\lim_{m\to\infty}\frac{1}{m}\log \mathsf{E}e^{\theta N(m\rho)^T}\mathbf{1}\{AN(m\rho)^T\le mb^T\},\end{aligned} \tag{59}$$

where $\mathbf{1}\{E\}$ is an indicator random variable that takes value 1 when the event E occurs and 0 otherwise. From the Varadhan integral lemma (see Dembo and Zeitouni [1992], Exercise 4.3.11, which was taken from Deuschel and Stroock [1989], Exercise 2.1.24), one has

$$\lim_{m\to\infty}\frac{1}{m}\log \mathsf{E}e^{\theta N(m\rho)^T}\mathbf{1}\{AN(m\rho)^T\le mb^T\} \le \sup_{\alpha\in\bar{S}}[\theta\alpha^T-\Lambda_N^*(\alpha)] = \Lambda_1(\theta). \tag{60}$$

Thus,

$$\lim_{m\to\infty}\frac{1}{m}\log \mathsf{E}e^{\theta n_m(t)^T} \le \Lambda_1(\theta)-\Lambda_1(\mathbf{0}). \tag{61}$$

Similarly, one can show the following lower bound

$$\lim_{m\to\infty}\frac{1}{m}\log \mathsf{E}e^{\theta n_m(t)^T} \ge \sup_{\alpha\in S}[\theta\alpha^T-\Lambda_N^*(\alpha)]-\Lambda_1(\mathbf{0}) = \Lambda_1(\theta)-\Lambda_1(\mathbf{0}), \tag{62}$$

where we use (54) for the equality. From both the upper and lower bounds in (61) and (62), the Gärtner-Ellis limit in (22) holds. Since $\Lambda_1(\theta)$ is differentiable for all θ ($\bigtriangledown\Lambda_1(\theta)=\alpha^*(\theta)$), it then follows from the Gärtner-Ellis theorem that $\{n_m(t)/m, m=1,2,\ldots\}$ satisfies the LDP with the speed m and the good rate function $\Lambda_n^*(\alpha)=\sup_\theta[\theta\alpha^T-\Lambda_n(\theta)]$. To see that $\Lambda_n^*(\alpha)$ indeed has the form in (23), note from the primal and dual convex programming in (28) and (34) that

$$\Lambda_1(\theta) = \sup_{\alpha\ge 0, A\alpha^T\le b^T}[\theta\alpha^T-\Lambda_N^*(\alpha)] = \inf_{y\ge 0}[\Lambda_N(\theta-yA)+yb^T]. \tag{63}$$

Thus,

$$\begin{aligned}&\sup_\theta[\theta\alpha^T-\Lambda_1(\theta)] = \sup_\theta\left[\theta\alpha^T-\inf_{y\ge 0}[\Lambda_N(\theta-yA)+yb^T]\right]\\ &=\sup_{y\ge 0}\left[\sup_\theta[(\theta-yA)\alpha^T-\Lambda_N(\theta-yA)]+y(A\alpha^T-b^T)\right]\\ &=\sup_\theta[\theta\alpha^T-\Lambda_N(\theta)]+\sup_{y\ge 0}[y(A\alpha^T-b^T)].\end{aligned} \tag{64}$$

Since $\Lambda_N(\theta)$ and $\Lambda_N^*(\alpha)$ are convex conjugates, $\sup_\theta[\theta\alpha^T - \Lambda_N(\theta)] = \Lambda_N^*(\alpha)$ if $\alpha \geq 0$, and ∞ otherwise. Also, $\sup_{y\geq 0}[y(A\alpha^T - b^T)] = 0$ if $A\alpha^T \leq b^T$, and ∞ otherwise. Thus,

$$\sup_\theta[\theta\alpha^T - \Lambda_1(\theta)] = \begin{cases} \Lambda_N^*(\alpha) & \text{if } \alpha \geq 0, A\alpha^T \leq b^T \\ \infty & \text{otherwise} \end{cases}. \tag{65}$$

Finally, note from (22) that

$$\begin{aligned} \Lambda_n^*(\alpha) &= \sup_\theta[\theta\alpha^T - \Lambda_n(\theta)] \\ &= \sup_\theta[\theta\alpha^T - \Lambda_1(\theta)] + \Lambda_1(\mathbf{0}). \end{aligned} \tag{66}$$

Q.E.D.

[*Proof of Theorem 4*] Note that $f_m(t)^T/m = b^T - An_m(t)^T/m$ is a continuous function from $n_m(t)/m$ to $f_m(t)/m$. It then follows from the contraction principle and Theorem 3 that

$$\begin{aligned} \Lambda_f^*(\tilde{\alpha}) &= \inf_{\{A\alpha^T = b^T - \tilde{\alpha}^T\}} \Lambda_n^*(\alpha) \\ &= \inf_{\{\alpha\geq 0, A\alpha^T \leq b^T, A\alpha^T = b^T - \tilde{\alpha}^T\}} [\Lambda_N^*(\alpha) + \Lambda_1(\mathbf{0})]. \end{aligned} \tag{67}$$

To see (43), note that the set in the infimum is empty when $\tilde{\alpha}_j < 0$ for some $j \in \{1, \ldots, M\}$. On the other hand, the set is the same as $\{\alpha \geq 0, A\alpha^T = b^T - \tilde{\alpha}^T\}$ when $\tilde{\alpha} \geq 0$.

That (44) holds follows immediately from (22). Q.E.D.

[*Proof of Theorem 5*] Let $S_{jr}^{(m)} = \{\alpha : b_j - A_{jr}/m < \sum_{r=1}^R A_{jr}\alpha_r \leq b_j\}$, $j = 1, \ldots, M$, $r = 1, \ldots, R$, and $S_r^{(m)} = \cup_{\{j:A_{jr}>0\}}[S_{jr}^{(m)} \cap \bar{S}]$, where the set $\bar{S} = \{\alpha : \alpha \geq 0, A\alpha^T \leq b^T\}$ as defined before. Then

$$\mathrm{P}_{CBP,r,m} = \mathrm{P}(n_m(t)/m \in S_r^{(m)}). \tag{68}$$

Let $S_{jr} = \{\alpha : \sum_{r=1}^R A_{jr}\alpha_r = b_j\}$ and $S_r = \cup_{\{j:A_{jr}>0\}}[S_{jr} \cap \bar{S}]$. We need to show

$$\begin{aligned} \lim_{m\to\infty} \frac{1}{m} \log \mathrm{P}_{CBP,r,m} &= -\min_{j:A_{jr}>0}[\inf_{S_{jr}\cap\bar{S}}(\Lambda_N^*(\alpha) + \Lambda_1(\mathbf{0}))] \\ &= -\inf_{S_r}(\Lambda_N^*(\alpha) + \Lambda_1(\mathbf{0})). \end{aligned} \tag{69}$$

To prove the upper bound, applying the union bound yields

$$\begin{aligned} \mathrm{P}_{CBP,r,m} &\leq \sum_{j:A_{jr}>0} \mathrm{P}(n_m(t)/m \in S_{jr}^{(m)} \cap \bar{S}) \\ &\leq M \max_{j:A_{jr}>0} \mathrm{P}(n_m(t)/m \in S_{jr}^{(m)} \cap \bar{S}). \end{aligned} \tag{70}$$

It remains to show that

$$\lim_{m\to\infty} \frac{1}{m} \log \mathrm{P}(n_m(t)/m \in S_{jr}^{(m)} \cap \bar{S}) \le - \inf_{S_{jr}\cap\bar{S}} [\Lambda_N^*(\alpha) + \Lambda_1(\mathbf{0})]. \tag{71}$$

Let $S_{jr,\delta} = \{\alpha : b_j - \delta \le \sum_{r=1}^R A_{jr}\alpha_r \le b_j\}$, $j = 1, \ldots, M$. Note that $S_{jr}^{(m)} \subset S_{jr,\delta}$ for all $\delta > 0$ and m sufficiently large. From the LDP upper bound in Theorem 3, one has for all $\delta > 0$

$$\lim_{m\to\infty} \frac{1}{m} \log \mathrm{P}(n_m(t)/m \in S_{jr}^{(m)} \cap \bar{S}) \le - \inf_{S_{jr,\delta}\cap\bar{S}} \Lambda_n^*(\alpha) = - \inf_{S_{jr,\delta}} \Lambda_n^*(\alpha), \tag{72}$$

as $\Lambda_n^*(\alpha) = \infty$ when $\alpha \notin \bar{S}$. Since $\Lambda_n^*(\alpha)$ in (23) is a good rate function and the sets $S_{jr,\delta}$ are (nested) closed for all $\delta > 0$, letting $\delta \to 0$ yields (see e.g., Dembo and Zeitouni [1992], Lemma 4.1.6)

$$\begin{aligned} \lim_{\delta\to 0} \inf_{S_{jr,\delta}} \Lambda_n^*(\alpha) &= \inf_{S_{jr}} \Lambda_n^*(\alpha) = \inf_{S_{jr}\cap\bar{S}} \Lambda_n^*(\alpha) \\ &= \inf_{S_{jr}\cap\bar{S}} [\Lambda_N^*(\alpha) + \Lambda_1(\mathbf{0})]. \end{aligned} \tag{73}$$

This proves the upper bound.

For the lower bound, we note that there may not exist a nonempty open set in $S_r^{(m)}$ for all m. Thus, the LDP lower bound in Theorem 3 may not be applicable. Instead, we use a direct approach. If S_r is empty, then the infimum in (69) is infinity and the lower bound in (69) becomes trivial. Thus, we assume S_r is nonempty. Since S_r (as a union of a finite number of closed sets) is a closed set and $\Lambda_N^*(\alpha)$ is a closed convex function, there exists a vector $\alpha^o = (\alpha_1^o, \ldots, \alpha_R^o) \in S_r$ that achieves the infimum in (69). Suppose there exists a sequence of vectors $n_m^o = (n_{m,1}^o, \ldots, n_{m,R}^o) \in \mathcal{Z}_+^R$ such that $n_m^o/m \in S_r^{(m)}$ and

$$\max_{r=1,\ldots R} |n_{m,r}^o - m\alpha_r^o| \le K \tag{74}$$

for some finite constant K. Then it follows from (56) that

$$\begin{aligned} \mathrm{P}_{CBP,r,m} &= \mathrm{P}(n_m(t)/m \in S_r^{(m)}) \ge \mathrm{P}(n_m(t) = n_m^o) \\ &= \frac{\mathrm{P}(N(m\rho) = n_m^o)}{\mathrm{P}(AN(m\rho)^T \le mb^T)}, \end{aligned} \tag{75}$$

where $N(m\rho)$ is a vector of independent Poisson random variables with parameters $m\rho_r$, $r = 1, \ldots, R$. Using Stirling's formula $\log m! = m \log m - m + O(\log m)$, one can easily see from the joint probability mass function of independent Poisson random variables that

$$\lim_{m\to\infty} \frac{1}{m} \log \mathrm{P}(N(m\rho) = n_m^o) = -\Lambda_N^*(\alpha^o), \tag{76}$$

as $n_m^o = m\alpha^o + O(1)$ from (74). In conjunction with (58) and (75), one has

$$\begin{aligned} \lim_{m\to\infty} \frac{1}{m} \log \mathrm{P}(n_m(t)/m \in S_r^{(m)}) &\ge -\Lambda_N^*(\alpha^o) - \Lambda_1(\mathbf{0}) \\ &= - \inf_{S_r} [\Lambda_N^*(\alpha) + \Lambda_1(\mathbf{0})]. \end{aligned} \tag{77}$$

It remains to find such a sequence of vectors n_m^o's. Let $n_{m,r}^* = \lfloor m\alpha_r^o \rfloor$, $r = 1, \ldots, R$, and $n_m^* = (n_{m,1}^*, \ldots, n_{m,R}^*)$. Using the fact that $x - 1 < \lfloor x \rfloor \le x$ and $\alpha^o \in S_r$, one can verify the following

$$\max_{r=1,\ldots,R} |n_{m,r}^* - m\alpha_r^o| \le 1, \tag{78}$$

$$A{n_m^*}^T \le mb^T, \tag{79}$$

$$mb_j - \sum_{r=1}^R A_{jr} < \sum_{r=1}^R A_{jr} n_{m,r}^*, \quad \text{for some } j \in \{j : A_{jr} > 0\}. \tag{80}$$

Now we use the following algorithm to find n_m^o.

1. Note that n_m^*/m is already in $\bar{S}$. If n_m^*/m is also in $S_r^{(m)}$, i.e., $mb_j - A_{jr} < \sum_{r=1}^R A_{jr} n_{m,r}^*$ for some $j \in \{j : A_{jr} > 0\}$, we can simply let $n_m^o = n_m^*$ and we are done.

2. Otherwise, we have $mb_j - A_{jr} \ge \sum_{r=1}^R A_{jr} n_{m,r}^*$ for all $j \in \{j : A_{jr} > 0\}$. Thus, we may increase $n_{m,r}^*$ by 1 and the new n_m^* still satisfies (79). Repeat Step 1 until n_m^* is in $S_r^{(m)}$.

The lower bound in (80) ensures the number of loops in the algorithm is bounded above by the finite constant $K = \max_{j:A_{jr}>0} \lceil \frac{\sum_{r=1}^R A_{jr}}{A_{jr}} \rceil$. In conjunction with (78), one has (74).

Q.E.D.

3.2. Large deviations with sublinear speed : the case at critical load

Consider a sequence of networks as described in Section 3, indexed by $m = 1, 2, \ldots$. In the m^{th} network, replace ν by $\nu_m = m\nu$, and b^T by $b_m^T = mA\rho^T + m^{1-\beta}\tilde{b}^T$ for some $\beta \in (0, 1/2)$ and some $\tilde{b} = (\tilde{b}_1, \ldots, \tilde{b}_M)$. Note that we do not assume $\tilde{b} > 0$ here and $\tilde{b}_j$ can be negative for some $j = 1, \ldots, M$. For the model of a multirate loss network, this expansion at $A\rho^T$ corresponds to the case at critical load. However, in contrast to the usual assumption $\beta = 1/2$ in Kelly [1991], Hunt and Kelly [1989], and Reiman [1991] where central limit theorems were established, we assume $0 < \beta < 1/2$ and derive the corresponding LDP for the above expansion. Our LDP results are parallel to the conditional normal distributions in Kelly [1991] and Hunt and Kelly [1989].

Define the row vectors $\theta = (\theta_1, \ldots, \theta_R)$, $\rho = (\rho_1, \ldots, \rho_R)$ and $\alpha = (\alpha_1, \ldots, \alpha_R)$. Let

$$\tilde{\Lambda}_N^*(\alpha) = \sum_{r=1}^R \frac{\alpha_r^2}{2\rho_r}, \tag{81}$$

$$\tilde{\Lambda}_N(\theta) = \sum_{r=1}^{R} \frac{\rho_r \theta_r^2}{2}, \tag{82}$$

and

$$\tilde{\Lambda}_1(\theta) = \sup_{A\alpha^T \le \tilde{b}^T} [\theta\alpha^T - \tilde{\Lambda}_N^*(\alpha)]. \tag{83}$$

Note that $\tilde{\Lambda}_N^*(\alpha)$ and $\tilde{\Lambda}_N(\theta)$ are convex conjugates, and that $\tilde{\Lambda}_N^*(\alpha)$ is the exponent of the joint density of independent normal random variables $N(0, \rho)$.

THEOREM 6 *Let $n_m(t) = (n_{m,1}(t), \ldots n_{m,R}(t))$ be the number of calls in progress at time t in the m^{th} network and $\tilde{n}_m(t) = m^{\beta-1}(n_m(t) - m\rho)$ for some $\beta \in (0, 1/2)$. Then*

$$\tilde{\Lambda}_n(\theta) \equiv \lim_{m\to\infty} \frac{1}{m^{1-2\beta}} \log \mathsf{E} e^{m^{1-2\beta}\theta\tilde{n}_m(t)^T} = \tilde{\Lambda}_1(\theta) - \tilde{\Lambda}_1(\mathbf{0}), \tag{84}$$

and $\{\tilde{n}_m(t), m = 1, 2, \ldots\}$ satisfies the LDP with the speed $m^{1-2\beta}$ and the good rate function

$$\tilde{\Lambda}_n^*(\alpha) = \sup_{\theta}[\theta\alpha^T - \tilde{\Lambda}_n(\theta)] = \begin{cases} \tilde{\Lambda}_N^*(\alpha) + \tilde{\Lambda}_1(\mathbf{0}) & \text{if } A\alpha^T \le \tilde{b}^T \\ \infty & \text{otherwise} \end{cases} . \tag{85}$$

The proof of this theorem will be given at the end of this section. As for Theorem 3, we note that (i) $\tilde{\Lambda}_n(\theta) < \infty$ for all θ, (ii) $\tilde{\Lambda}_n(\theta)$ and $\tilde{\Lambda}_n^*(\alpha)$ are convex conjugates, and (iii) $\tilde{\Lambda}_n(\theta)$ is differentiable for all θ.

To see the connection between Theorem 6 and the conditional normal distribution in Hunt and Kelly [1989], Theorem 2.1, note that Theorem 6 implies

$$\mathrm{P}(\tilde{n}_m(t) \in (\alpha, \alpha + d\alpha)) \approx \exp\Big(- m^{1-2\beta}(\tilde{\Lambda}_N^*(\alpha) + \tilde{\Lambda}_1(\mathbf{0}))\Big). \tag{86}$$

Write

$$b_m^T = mA\rho^T + \sqrt{m}(m^{1/2-\beta}\tilde{b}^T), \tag{87}$$

and

$$\mathrm{P}(\tilde{n}_m(t) \in (\alpha, \alpha + d\alpha)) = \mathrm{P}(\frac{n_m(t) - m\rho}{\sqrt{m}} \in m^{1/2-\beta}(\alpha, \alpha + d\alpha)). \tag{88}$$

Note that this expansion in (87) does not satisfy the condition in Hunt and Kelly [1989]. However, if the conditional normal distribution in Hunt and Kelly [1989], Theorem 2.1, is still applicable, then $\mathrm{P}(\tilde{n}_m(t) \in (\alpha, \alpha+d\alpha))$ can be approximated by the joint density $f(m^{1/2-\beta}\alpha)$ of the multivariate normal random variables $N(\mathbf{0}, \rho)$ conditioning on $Am^{1/2-\beta}\alpha^T \le m^{1/2-\beta}\tilde{b}^T$. This yields a log equivalent approximation to that in (86), taking into account $\tilde{\Lambda}_1(\mathbf{0}) = \sup_{A\alpha^T \le \tilde{b}}[-\tilde{\Lambda}_N^*(\alpha)]$.

Analogous to the proof of Theorem 4, one has the following LDP result for the normalized number of free links.

THEOREM 7 *Let $\tilde{f}_m(t)^T = \tilde{b}^T - A\tilde{n}_m(t)^T$, $\tilde{\alpha} = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_M)$, and $\tilde{\theta} = (\tilde{\theta}_1, \ldots, \tilde{\theta}_M)$. Then $\{\tilde{f}_m(t), m = 1, 2, \ldots\}$ satisfies the LDP with the speed $m^{1-2\beta}$ and the good rate function*

$$\tilde{\Lambda}_f^*(\tilde{\alpha}) = \begin{cases} \inf_{\{A\alpha^T = \tilde{b} - \tilde{\alpha}\}}[\tilde{\Lambda}_N^*(\alpha) + \tilde{\Lambda}_1(\mathbf{0})] & \text{if } \tilde{\alpha} \geq 0 \\ \infty & \text{otherwise} \end{cases} \tag{89}$$

Moreover, for all $\tilde{\theta} \in I\!R^M$,

$$\lim_{m\to\infty} \frac{1}{m^{1-2\beta}} \log \mathsf{E} e^{m^{1-2\beta}\tilde{\theta}\tilde{f}_m(t)^T} = \tilde{\Lambda}_1(-\tilde{\theta}A) - \tilde{\Lambda}_1(\mathbf{0}) + \tilde{\theta}\tilde{b}^T. \tag{90}$$

Theorem 7 is parallel to that in Hunt and Kelly [1989], Theorem 2.2. To see this, note that $\inf_{\{A\alpha^T = \tilde{b} - \tilde{\alpha}\}} \tilde{\Lambda}_N^*(\alpha)$ can be solved explicitly by considering the following maximization problem.

$$\begin{array}{ll} \text{Maximize} & -\tilde{\Lambda}_N^*(\alpha) \\ \text{subject to} & A\alpha^T \leq \tilde{b}^T - \tilde{\alpha}^T. \end{array}$$

Analogous to the argument for (34), its dual problem (taking $\theta = \mathbf{0}$) is as follows:

$$\begin{array}{ll} \text{Minimize} & \tilde{\Lambda}_N(-yA) + y(\tilde{b}^T - \tilde{\alpha}^T) \\ \text{subject to} & y \quad \text{unrestricted.} \end{array} \tag{91}$$

Let $\Delta(\rho)$ be the $R \times R$ diagonal matrix with its r^{th} diagonal element being ρ_r, $r = 1, \ldots, R$. In view of (82), $\tilde{\Lambda}_N(-yA) = \frac{1}{2} yA\Delta(\rho)A^T y^T$. Thus, the minimum in (91) occurs when

$$y = (\tilde{\alpha} - \tilde{b})(A\Delta(\rho)A^T)^{-1}. \tag{92}$$

Replacing (92) in (91) yields

$$\inf_{\{A\alpha^T = \tilde{b}^T - \tilde{\alpha}^T\}} \tilde{\Lambda}_N^*(\alpha) = \frac{1}{2}(\tilde{\alpha} - \tilde{b})(A\Delta(\rho)A^T)^{-1}(\tilde{\alpha} - \tilde{b})^T. \tag{93}$$

This is the exponent of the joint density of the multivariate normal random variables with the mean vector $\tilde{b}$ and the covariance matrix $A\Delta(\rho)A^T$. We may also write

$$\begin{aligned} -\tilde{\Lambda}_1(0) &= \inf_{A\alpha^T \leq \tilde{b}^T} \tilde{\Lambda}_N^*(\alpha) = \inf_{\tilde{\alpha} \geq 0} \inf_{A\alpha^T = \tilde{b}^T - \tilde{\alpha}} \tilde{\Lambda}_N^*(\alpha) \\ &= \inf_{\tilde{\alpha} \geq 0} \frac{1}{2}(\tilde{\alpha} - \tilde{b})(A\Delta(\rho)A^T)^{-1}(\tilde{\alpha} - \tilde{b})^T. \end{aligned} \tag{94}$$

Now the connection to the conditional normal distribution for the normalized number of free links in Hunt and Kelly [1989], Theorem 2.2 is apparent.

In the following theorem, we derive the LDP for the call blocking probability.

THEOREM 8 *Let $P_{CBP,r,m}$ be the call blocking probability of the r^{th} route in the m^{th} network. Then*

$$\begin{aligned} &\lim_{m\to\infty} \frac{1}{m^{1-2\beta}} \log P_{CBP,r,m} \\ &= -\min_{j:A_{jr}>0} [\inf_{A\alpha^T \le \tilde{b}^T, \sum_{r=1}^R A_{jr}\alpha_r = \tilde{b}_j} (\tilde{\Lambda}_N^*(\alpha) + \tilde{\Lambda}_1(\mathbf{0}))]. \end{aligned} \tag{95}$$

The rest of this section is devoted to the proofs of Theorems 6 and 8. We will need the following proposition for the proof of Theorem 6.

PROPOSITION 2 *Let $N_r(m\rho_r)$, $r = 1, \ldots, R$, be* independent *Poisson random variables with parameters $m\rho_r$, $r = 1, \ldots, R$, and $N(m\rho) = (N_1(m\rho_1), \ldots, N_R(m\rho_R))$. Define $\tilde{N}(m\rho) = m^{\beta-1}(N(m\rho) - m\rho)$ for some $\beta \in (0, 1/2)$. Then $\{\tilde{N}(m\rho), m = 1, 2, \ldots\}$ satisfies the LDP with the rate function $\tilde{\Lambda}_N^*(\alpha) = \sum_{r=1}^R \frac{\alpha_r^2}{2\rho_r}$ and the speed $m^{1-2\beta}$.*

Proof. This is also a direct consequence of the Gärtner-Ellis theorem. One first computes the following limit

$$\lim_{m\to\infty} \frac{1}{m^{1-2\beta}} \log \mathsf{E} e^{m^{1-2\beta}\theta \tilde{N}(m\rho)^T} = \sum_{r=1}^R \frac{\rho_r \theta_r^2}{2} = \tilde{\Lambda}_N(\theta), \tag{96}$$

and then derives its convex transform

$$\tilde{\Lambda}_N^*(\alpha) = \sup_\theta [\theta\alpha^T - \tilde{\Lambda}_N(\theta)]. \tag{97}$$

Q.E.D.

Note that $\tilde{\Lambda}_N^*(\alpha)$ is continuous for all α. Thus,

$$\sup_{A\alpha^T < \tilde{b}^T} [\theta\alpha^T - \tilde{\Lambda}_N^*(\alpha)] = \sup_{A\alpha^T \le \tilde{b}^T} [\theta\alpha^T - \tilde{\Lambda}_N^*(\alpha)] = \tilde{\Lambda}_1(\theta) \tag{98}$$

for all θ. In particular, when taking $\theta = \mathbf{0}$,

$$\inf_{A\alpha^T < \tilde{b}^T} \tilde{\Lambda}_N^*(\alpha) = \inf_{A\alpha^T \le \tilde{b}^T} \tilde{\Lambda}_N^*(\alpha) = -\tilde{\Lambda}_1(\mathbf{0}). \tag{99}$$

[*Proof of Theorem 6*] As in the proof of Theorem 3, note that

$$\mathrm{P}(n_m(t) = n) = \frac{\mathrm{P}(N(m\rho) = n)}{\mathrm{P}(AN(m\rho)^T \le mA\rho^T + m^{1-\beta}\tilde{b}^T)}, \tag{100}$$

where $N(m\rho) = (N_1(m\rho_1), \ldots, N_R(m\rho_R))$ are independent Poisson random variables with parameters $m\rho_r$, $r = 1, \ldots, R$.

Thus,

$$
\begin{aligned}
&\lim_{m\to\infty}\frac{1}{m^{1-2\beta}}\log \mathsf{E}e^{m^{1-2\beta}\theta\tilde{n}_m(t)^T}\\
&=\lim_{m\to\infty}\frac{1}{m^{1-2\beta}}\log\sum_{An^T\le mA\rho^T+m^{1-\beta}\tilde{b}^T}\exp(\frac{\theta n^T-m\theta\rho^T}{m^\beta})\\
&\quad\frac{\mathrm{P}(N(m\rho)=n)}{\mathrm{P}(AN(m\rho)^T\le mA\rho^T+m^{1-\beta}\tilde{b}^T)}\\
&=\lim_{m\to\infty}\frac{1}{m^{1-2\beta}}\log\sum_{An^T\le mA\rho^T+m^{1-\beta}\tilde{b}^T}\exp(\frac{\theta n^T-m\theta\rho^T}{m^\beta})\mathrm{P}(N(m\rho)=n)\\
&\quad-\lim_{m\to\infty}\frac{1}{m^{1-2\beta}}\log \mathrm{P}(AN(m\rho)^T\le mA\rho^T+m^{1-\beta}\tilde{b}^T) \qquad (101)
\end{aligned}
$$

From the LDP for independent Poisson random variables in Proposition 2 and (99)

$$
\begin{aligned}
&\lim_{m\to\infty}\frac{1}{m^{1-2\beta}}\log \mathrm{P}(AN(m\rho)^T\le mA\rho^T+m^{1-\beta}\tilde{b}^T)\\
&=\lim_{m\to\infty}\frac{1}{m^{1-2\beta}}\log \mathrm{P}(A\tilde{N}(m\rho)^T\le\tilde{b}^T)=\tilde{\Lambda}_1(\mathbf{0}). \qquad (102)
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\lim_{m\to\infty}\frac{1}{m^{1-2\beta}}\log\sum_{An^T\le mA\rho^T+m^{1-\beta}\tilde{b}^T}\exp(\frac{\theta n^T-m\theta\rho^T}{m^\beta})\mathrm{P}(N(m\rho)=n)\\
&=\lim_{m\to\infty}\frac{1}{m^{1-2\beta}}\log \mathsf{E}e^{m^{1-2\beta}\theta\tilde{N}(m\rho)^T}\mathbf{1}\{A\tilde{N}(m\rho)^T\le\tilde{b}^T\}. \qquad (103)
\end{aligned}
$$

Once again, it follows from the Varadhan integral lemma (Dembo and Zeitouni [1992], Exercise 4.3.11) that

$$
\begin{aligned}
&\lim_{m\to\infty}\frac{1}{m^{1-2\beta}}\log \mathsf{E}e^{m^{1-2\beta}\theta\tilde{N}(m\rho)^T}\mathbf{1}\{A\tilde{N}(m\rho)^T\le\tilde{b}^T\}\\
&\le\sup_{A\alpha^T\le\tilde{b}^T}[\theta\alpha^T-\tilde{\Lambda}^*_N(\alpha)]=\tilde{\Lambda}_1(\theta). \qquad (104)
\end{aligned}
$$

Thus,

$$
\lim_{m\to\infty}\frac{1}{m^{1-2\beta}}\log \mathsf{E}e^{m^{1-2\beta}\theta\tilde{n}_m(t)^T}\le\tilde{\Lambda}_1(\theta)-\tilde{\Lambda}_1(\mathbf{0}). \qquad (105)
$$

The lower bound can be proved by the same argument and (98). Q.E.D.

[*Proof of Theorem 8*] The proof follows the same argument as that in Theorem 5. Let $\tilde{S}=\{\alpha : A\alpha^T\le\tilde{b}^T\}$, $\tilde{S}^{(m)}_{jr}=\{\alpha : \tilde{b}_j-A_{jr}/m^{1-\beta}<\sum_{r=1}^R A_{jr}\alpha\le\tilde{b}_j\}$, $j=1,\ldots,M$, $r=1,\ldots,R$, and $\tilde{S}^{(m)}_r=\cup_{\{j:A_{jr}>0\}}[\tilde{S}^{(m)}_{jr}\cap\tilde{S}]$. Then

$$
\mathrm{P}_{CBP,r,m}=\mathrm{P}(\tilde{n}_m(t)\in\tilde{S}^{(m)}_r). \qquad (106)
$$

Also, let $\tilde{S}_{jr} = \{\alpha : \sum_{r=1}^{R} A_{jr}\alpha = \tilde{b}_j\}$, $j = 1, \dots, M$, $r = 1, \dots, R$, and $\tilde{S}_r = \cup_{\{j:A_{jr}>0\}}[\tilde{S}_{jr} \cap \tilde{S}]$. One needs to show that

$$\lim_{m\to\infty} \frac{1}{m^{1-2\beta}} \log \mathrm{P}_{CBP,r,m} = -\min_{j:A_{jr}>0}[\inf_{\tilde{S}_{jr}\cap\tilde{S}}(\tilde{\Lambda}_N^*(\alpha) + \tilde{\Lambda}_1(\mathbf{0}))]$$
$$= -\inf_{\tilde{S}_r}(\tilde{\Lambda}_N^*(\alpha) + \tilde{\Lambda}_1(\mathbf{0})). \tag{107}$$

As in the proof of Theorem 5, one has from the union bound that

$$\mathrm{P}_{CBP,r,m} \le M \max_{j:A_{jr}>0} \mathrm{P}(\tilde{n}_m(t) \in \tilde{S}_{jr}^{(m)} \cap \tilde{S}). \tag{108}$$

Applying the LDP upper bound in Theorem 6 and using the δ argument in Theorem 5 yield

$$\lim_{m\to\infty} \frac{1}{m^{1-2\beta}} \log \mathrm{P}(\tilde{n}_m(t) \in \tilde{S}_{jr}^{(m)} \cap \tilde{S}) \le -\inf_{\tilde{S}_{jr}\cap\tilde{S}}[\tilde{\Lambda}_N^*(\alpha) + \tilde{\Lambda}_1(\mathbf{0})]. \tag{109}$$

In view of (108), one proves the upper bound in (107).

For the lower bound, we follow the direct approach in Theorem 5. Suppose that $\alpha^o = (\alpha_1^o, \dots, \alpha_R^o) \in \tilde{S}_r$ achieves the infimum in (107). It suffices to show there exists a sequence of vectors n_m^o's, $n_m^o = (n_{m,1}^o, \dots, n_{m,R}^o) \in \mathcal{Z}_+^R$ such that $\frac{n_m^o - m\rho}{m^{1-\beta}} \in \tilde{S}_r^{(m)}$ and $\max_{r=1,\dots R} |n_{m,r}^o - m\rho_r - m^{1-\beta}\alpha_r^o| \le K$ for some finite constant K.

Then it follows from (100) that

$$\mathrm{P}_{CBP,r,m} = \mathrm{P}(\frac{n_m(t) - m\rho}{m^{1-\beta}} \in \tilde{S}_r^{(m)}) \ge \mathrm{P}(n_m(t) = n_m^o)$$
$$= \frac{\mathrm{P}(N(m\rho) = n_m^o)}{\mathrm{P}(AN(m\rho)^T \le mA\rho^T + m^{1-\beta}\tilde{b}^T)}, \tag{110}$$

where $N(m\rho)$ is a vector of independent Poisson random variables with parameters $m\rho_r$, $r = 1, \dots, R$. Using Stirling's formula $\log m! = m\log m - m + O(\log m)$ and $\log(1+x) = x - x^2/2 + o(x^2)$, one can easily see from the joint probability mass function of independent Poisson random variables that

$$\lim_{m\to\infty} \frac{1}{m^{1-2\beta}} \log \mathrm{P}(N(m\rho) = n_m^o) = -\tilde{\Lambda}_N^*(\alpha^o), \tag{111}$$

as $n_m^o = m\rho + m^{1-\beta}\alpha^o + O(1)$. In conjunction with (102) and (110), one has

$$\lim_{m\to\infty} \frac{1}{m^{1-2\beta}} \log \mathrm{P}(\tilde{n}_m(t) \in \tilde{S}_r^{(m)}) \ge -\tilde{\Lambda}_N^*(\alpha^o) - \tilde{\Lambda}_1(\mathbf{0})$$
$$= -\inf_{\tilde{S}_r}[\tilde{\Lambda}_N^*(\alpha) + \tilde{\Lambda}_1(\mathbf{0})]. \tag{112}$$

To find such a sequence of vectors n_m^o's, we may use the algorithm in the proof of Theorem 5 with $n_{m,r}^* = \lfloor m\rho_r + m^{1-\beta}\alpha_r^o \rfloor$, $r = 1, \dots, R$, and $n_m^* = (n_{m,1}^*, \dots, n_{m,R}^*)$. Q.E.D.

4. Large deviations at the cell level

As in Section 1.2, we consider a sequence of networks, indexed by $m = 1, 2, \ldots$, in the cell level. Let $a_{i,r}(t)$, $i = 1, \ldots, n_{m,r}(t)$, be the number of cells that arrive at time t from the i^{th} route r call in progress (note that $a_{i,r}(t)$, $i = 1, \ldots, n_{m,r}(t)$, are i.i.d. random variables from our assumptions). Then the total number of the cells that arrive at the j^{th} node in the m^{th} network at time t is $x_{m,j}(t) = \sum_{r=1}^{R} D_{jr} \sum_{i=1}^{n_{m,r}(t)} a_{i,r}(t)$. Let $x_m(t) = (x_{m,1}(t), \ldots, x_{m,J}(t))$. Recall that $\bar{a}_r = \mathsf{E}a_r(1)$ and $\Lambda_{a,r}(\theta_r) = \log \mathsf{E}e^{\theta_r a_r(1)} < \infty$, $r = 1, \ldots, R$ for all θ_r, $r = 1, \ldots, R$. Define the row vectors $\bar{a} = (\bar{a}_1, \ldots, \bar{a}_R)$, $\gamma = (\gamma_1, \ldots, \gamma_J)$ and $\eta = (\eta_1, \ldots, \eta_J)$.

THEOREM 9 $\lim_{m\to\infty} \frac{1}{m} \log \mathsf{E} \exp(\eta x_m(t)^T) = \Lambda_x(\eta)$, *where*

$$\Lambda_x(\eta) = \Lambda_n(\Lambda_{a,1}(\sum_{j=1}^{J} D_{j1}\eta_j), \ldots, \Lambda_{a,R}(\sum_{j=1}^{J} D_{jR}\eta_j)) \tag{113}$$

and $\Lambda_n(\theta)$ is defined in Theorem 3. As a direct consequence of the Gärtner-Ellis theorem, $\{x_m(t)/m, m = 1, 2, \ldots\}$ satisfies the LDP with the speed m and the good rate function $\Lambda_x^*(\gamma) = \sup_\eta[\gamma\eta^T - \Lambda_x(\eta)]$.

As a corollary of Theorem 9, one has for $j = 1, \ldots, J$,

$$\lim_{m\to\infty} \frac{1}{m} \log \mathsf{E} \exp(\eta_j x_{m,j}(t)) = \Lambda_{x_j}(\eta_j), \tag{114}$$

where

$$\Lambda_{x_j}(\eta_j) = \Lambda_x(\eta_j e_j) = \Lambda_x(0, \ldots, 0, \eta_j, 0, \ldots, 0). \tag{115}$$

Moreover, $\{x_{m,j}(t)/m, m = 1, 2, \ldots\}$ satisfies the LDP with the speed m and the good rate function $\Lambda_{x_j}^*(\gamma_j) = \sup_{\eta_j}[\gamma_j\eta_j - \Lambda_{x_j}(\eta_j)]$.

In view of the strong law for $n_m(t)/m$ in Corollary 1, one also has the following strong law for $x_m(t)/m$.

COROLLARY 2 *Let $\bar{\rho}_r = \alpha_r^*(\mathbf{0})\bar{a}_r$, $r = 1, \ldots, R$ and $\bar{\rho} = (\bar{\rho}_1, \ldots, \bar{\rho}_R)$. Then*

$$\lim_{m\to\infty} \frac{x_m(t)}{m} = D\bar{\rho}^T \quad a.s. \tag{116}$$

Let $L_m(t)$ be the number of cells lost at time t in the m^{th} network. Define the cell loss probability in the m^{th} network at time t as follows:

$$\mathrm{P}_{CLP,m}(t) \equiv \frac{\mathsf{E}L_m(t)}{\mathsf{E}\sum_{r=1}^{R}\sum_{i=1}^{n_{m,r}(t)} a_{i,r}(t)}, \tag{117}$$

where $\sum_{r=1}^{R}\sum_{i=1}^{n_{m,r}(t)} a_{i,r}(t)$ is the total number of cells that arrive at the m^{th} network at time t. To avoid triviality, we assume $\mathsf{E}\sum_{r=1}^{R}\sum_{i=1}^{n_{m,r}(t)} a_{i,r}(t) > 0$. Note that the loss probability defined in (117) is identical to the long run loss ratio if the network is stationary and ergodic. Whether the network is stationary and ergodic depends on how the network drops the cells from these R routes when the number of cells arriving at a node exceeds its capacity. We will not pursue this issue in this paper.

THEOREM 10 $\liminf_{m\to\infty} \frac{1}{m} \log P_{CLP,m}(t) \geq -\min_{j=1,\dots,J} \inf_{\gamma_j > c_j} \Lambda^*_{x_j}(\gamma_j)$.
If $D\bar{\rho}^T < c$, then

$$\limsup_{m\to\infty} \frac{1}{m} \log P_{CLP,m}(t) \leq -\min_{j=1,\dots,J} \Lambda^*_{x_j}(c_j). \tag{118}$$

Note that $\Lambda^*_{x_j}(c_j)$ can be found from the convex transform

$$\Lambda^*_{x_j}(c_j) = \eta^*_j c_j - \Lambda_{x_j}(\eta^*_j), \tag{119}$$

if η^*_j is a solution of the equation

$$c_j = \Lambda'_{x_j}(\eta_j). \tag{120}$$

To find $\Lambda'_{x_j}(\eta_j)$, we note from (115) that

$$\begin{aligned}\Lambda_{x_j}(\eta_j) &= \Lambda_n(\Lambda_{a,1}(D_{j1}\eta_j), \dots, \Lambda_{a,R}(D_{jR}\eta_j)) \\ &= \Lambda_n(D_{j1}\Lambda_{a,1}(\eta_j), \dots, D_{jR}\Lambda_{a,R}(\eta_j))\end{aligned} \tag{121}$$

since D_{jr}'s are indicator variables and $\Lambda_{a,r}(0) = 0$ for all r. Let

$$\theta(\eta_j) = (D_{j1}\Lambda_{a,1}(\eta_j), \dots, D_{jR}\Lambda_{a,R}(\eta_j)). \tag{122}$$

It then follows from (25) that

$$\Lambda'_{x_j}(\eta_j) = \sum_{r=1}^{R} \alpha^*_r(\theta(\eta_j)) D_{jr} \Lambda'_{a,r}(\eta_j). \tag{123}$$

In particular,

$$\Lambda'_{x_j}(0) = \sum_{r=1}^{R} \alpha^*_r(\mathbf{0}) D_{jr} \bar{a}_r = D\bar{\rho}^T. \tag{124}$$

Moreover, we have from (22) and (27) that

$$\Lambda_{x_j}(\eta_j) = \Lambda_n(\theta(\eta_j)) = \theta(\eta_j)\alpha^*(\theta(\eta_j))^T - \Lambda^*_N(\alpha^*(\theta(\eta_j))) - \Lambda_1(\mathbf{0}). \tag{125}$$

In conjunction with (37) and (38), $\Lambda^*_{x_j}(c_j)$ can be obtained from (119) and (125) by finding a feasible solution for the vector $(B_1, \dots, B_M, \eta_j)$ with $B_\ell \in [0,1), \ell = 1, \dots, M$, that satisfies the following conditions

$$\sum_{r=1}^{R} A_{\ell r}\rho_r e^{D_{jr}\Lambda_{a,r}(\eta_j)} \prod_{i=1}^{M}(1-B_i)^{A_{ir}} \begin{cases} = b_\ell & \text{if } B_\ell > 0, \\ \le b_\ell & \text{if } B_\ell = 0, \end{cases} , \ell = 1, \ldots, M, \tag{126}$$

and

$$\sum_{r=1}^{R} \rho_r e^{D_{jr}\Lambda_{a,r}(\eta_j)} \prod_{i=1}^{M}(1-B_i)^{A_{ir}} D_{jr}\Lambda'_{a,r}(\eta_j) = c_j. \tag{127}$$

To derive the exponent in the large deviation approximation of the cell loss probability in (14) from Theorem 10, we note from (119), (125) and (127) that

$$\begin{aligned} \Lambda^*_{x_j}(c_j) &= \eta^*_j c_j - \Lambda_{x_j}(\eta^*_j) \\ &= \eta^*_j \sum_{r=1}^{R} \alpha^*_r(\theta(\eta^*_j)) D_{jr}\Lambda'_{a,r}(\eta^*_j) - \theta(\eta^*_j)\alpha^*(\theta(\eta^*_j))^T \\ &\quad + \Lambda^*_N(\alpha^*(\theta(\eta^*_j))) + \Lambda_1(\mathbf{0}) \\ &= \sum_{r=1}^{R} \alpha^*_r(\theta(\eta^*_j)) D_{jr}\eta^*_j \Lambda'_{a,r}(\eta^*_j) - \sum_{r=1}^{R} \alpha^*_r(\theta(\eta^*_j)) D_{jr}\Lambda_{a,r}(\eta^*_j) \\ &\quad + \Lambda^*_N(\alpha^*(\theta(\eta^*_j))) - \Lambda^*_N(\alpha^*(\mathbf{0})) \\ &= \sum_{r=1}^{R} \alpha^*_r(\theta(\eta^*_j)) D_{jr}(\eta^*_j \Lambda'_{a,r}(\eta^*_j) - \Lambda_{a,r}(\eta^*_j)) \\ &\quad + \Lambda^*_N(\alpha^*(\theta(\eta^*_j))) - \Lambda^*_N(\alpha^*(\mathbf{0})), \end{aligned}$$

with $\alpha^*_r(\theta(\eta^*_j)) = \bar{\alpha}^j_r$ and $\alpha^*_r(\mathbf{0}) = \alpha^0_r$, $r = 1, \ldots, R$.

Now we consider the cell loss probability for a particular route. Let $L_{m,r}(t)$ be the number of route r cells lost at time t in the m^{th} network. Define the cell loss probability of the r^{th} route in the m^{th} network at time t as follows:

$$\mathrm{P}_{CLP,m,r}(t) \equiv \frac{\mathsf{E}L_{m,r}(t)}{\mathsf{E}\sum_{i=1}^{n_{m,r}(t)} a_{i,r}(t)}, \tag{128}$$

where $\sum_{i=1}^{n_{m,r}(t)} a_{i,r}(t)$ is the total number of route r cells that arrive at the m^{th} network at time t.

Theorem 11 *If $D\bar{\rho}^T < c$, then*

$$\limsup_{m\to\infty} \frac{1}{m} \log P_{CLP,m,r}(t) \le -\min_{j:D_{jr}>0} \Lambda^*_{x_j}(c_j). \tag{129}$$

Unlike Theorem 10, the upper bound for the cell loss probability of the r^{th} route may not be tight. This is due to the fact that there might exist a particular route with no cell loss while there are cell losses in the whole network. To see this, consider a two-node tandem network. There are two routes: the first route visits both nodes, and the second route only visits the second node. Let $c_1 = 1$, $c_2 = 2$,

$A_{11} = 1$, $A_{12} = 0$, $A_{21} = 1$, $A_{22} = 1$, $b_1 = 1$ and $b_2 = 2$. Thus, at most one route 1 call can be accepted, and at most two calls from both routes can be accepted. An accepted route 1 call transmits 2 cells per unit of time with probability 1/2 and 0 cell with probability 1/2. An accepted route 2 call transmits exactly 1 cell per unit of time. Due to the limitation of the capacity at the first node, there is at most 1 cell departing from the first node. Thus, there is no cell loss for the second route in the second node at any time, but there are cell losses for the first route cells in the first node if a route 1 call is accepted.

The rest of this section is devoted to the proofs of Theorems 9, 10 and 11.

[*Proof of Theorem 9*] Since we assume $\{a_{i,r}(t), t = 1, 2, \ldots\}, i = 1, \ldots, n_{m,r}(t)$, are *independent* (of all else) sequences of random variables that have the same joint distribution as $\{a_r(t), t = 1, 2, \ldots\}$,

$$\begin{aligned}
&\lim_{m\to\infty} \frac{1}{m} \log \mathsf{E} e^{\eta x_m(t)^T} \\
&= \lim_{m\to\infty} \frac{1}{m} \log \mathsf{E} \exp(\sum_{j=1}^{J} \eta_j \sum_{r=1}^{R} D_{jr} \sum_{i=1}^{n_{m,r}(t)} a_{i,r}(t)) \\
&= \lim_{m\to\infty} \frac{1}{m} \log \mathsf{E} \exp(\sum_{r=1}^{R} \sum_{j=1}^{J} D_{jr}\eta_j \sum_{i=1}^{n_{m,r}(t)} a_{i,r}(t)) \\
&= \lim_{m\to\infty} \frac{1}{m} \log \mathsf{E}[\mathsf{E} \exp(\sum_{r=1}^{R} \sum_{j=1}^{J} D_{jr}\eta_j \sum_{i=1}^{n_{m,r}(t)} a_{i,r}(t))|n_{m,r}(t), r = 1, \ldots, R] \\
&= \lim_{m\to\infty} \frac{1}{m} \log \mathsf{E} \prod_{r=1}^{R} [\mathsf{E} \exp(\sum_{j=1}^{J} D_{jr}\eta_j a_r(1))]^{n_{m,r}(t)} \\
&= \Lambda_n(\Lambda_{a,1}(\sum_{j=1}^{J} D_{j1}\eta_j), \ldots, \Lambda_{a,R}(\sum_{j=1}^{J} D_{jR}\eta_j)),
\end{aligned} \tag{130}$$

where we apply Theorem 3 in the last equality. Q.E.D.

[*Proof of Theorem 10*] We first note that

$$\lim_{m\to\infty} \frac{1}{m} \log \mathsf{E} \sum_{r=1}^{R} \sum_{i=1}^{n_{m,r}(t)} a_{i,r}(t) = 0 \tag{131}$$

since $\mathsf{E} \sum_{r=1}^{R} \sum_{i=1}^{n_{m,r}(t)} a_{i,r}(t)$ is increasing in m and

$$\mathsf{E} \sum_{r=1}^{R} \sum_{i=1}^{n_{m,r}(t)} a_{i,r}(t) = \sum_{r=1}^{R} \mathsf{E} n_{m,r}(t) \mathsf{E} a_r(1) \le \sum_{r=1}^{R} m\rho_r \bar{a}_r. \tag{132}$$

As we assume zero propagation delay between any two nodes in the network, there will be no cell loss at time t if $x_{m,j}(t) \le mc_j$, $j = 1, \ldots, J$. Though we do not specify how we drop cells from these R routes when the total number of cells in a node exceeds its capacity, we can still find bounds for $L_m(t)$. Note that the number of cell losses at time t at the j^{th} node cannot be larger than $(x_{m,j}(t) - mc_j)^+$. Thus,

$$L_m(t) \leq \sum_{j=1}^{J} [x_{m,j}(t) - mc_j]^+. \tag{133}$$

To find a lower bound for $L_m(t)$, consider a sequence of auxiliary networks with $c_\ell = \infty, \ell \neq j$. In this sequence of auxiliary networks, cell losses occur only at the j^{th} node and the number of cell losses at time t in the m^{th} auxiliary network is $(x_{m,j}(t) - mc_j)^+$, which cannot be larger than $L_m(t)$ in the original network. Thus, for all j

$$[x_{m,j}(t) - mc_j]^+ \leq L_m(t). \tag{134}$$

In view of (133) and (134), we have

$$\max_{j=1,\ldots,J} \mathsf{E}[x_{m,j}(t) - mc_j]^+ \leq \mathsf{E}L_m(t) \leq J \max_{j=1,\ldots,J} \mathsf{E}[x_{m,j}(t) - mc_j]^+. \tag{135}$$

From (131), we have

$$\lim_{m\to\infty} \frac{1}{m} \log \mathrm{P}_{CLP,m}(t) = \max_{j=1,\ldots,J} \lim_{m\to\infty} \frac{1}{m} \log \mathsf{E}[x_{m,j}(t) - mc_j]^+. \tag{136}$$

Now we prove the lower bound. In view of (136), it suffices to show that

$$\lim_{m\to\infty} \frac{1}{m} \log \mathsf{E}(x_{m,j}(t) - mc_j)^+ \geq -\inf_{\gamma_j > c_j} \Lambda^*_{x_j}(\gamma_j). \tag{137}$$

Since

$$\mathsf{E}(x_{m,j}(t) - mc_j)^+ \geq 1 \cdot \mathrm{P}(x_{m,j}(t) > mc_j), \tag{138}$$

applying the LDP lower bound for $\{x_{m,j}(t)/m, m = 1, 2, \ldots\}$ yields

$$\liminf_{m\to\infty} \frac{1}{m} \log \mathsf{E}(x_{m,j}(t) - mc_j)^+ \geq -\inf_{\gamma_j > c_j} \Lambda^*_{x_j}(\gamma_j). \tag{139}$$

To prove the upper bound, we have from Chernoff's bound that for all $\eta_j > 0$

$$\begin{aligned} \mathsf{E}(x_{m,j}(t) - mc_j)^+ &= \sum_{\ell=1}^{\infty} \mathrm{P}((x_{m,j}(t) - mc_j)^+ \geq \ell) \\ &= \sum_{\ell=1}^{\infty} \mathrm{P}(x_{m,j}(t) - mc_j \geq \ell) \\ &\leq \sum_{\ell=1}^{\infty} e^{-\eta_j \ell} \mathsf{E} \exp(\eta_j x_{m,j}(t) - m\eta_j c_j). \end{aligned} \tag{140}$$

Note that $\sum_{\ell=1}^{\infty} e^{-\eta_j \ell} = e^{-\eta_j}(1-e^{-\eta_j})^{-1} < \infty$ for all $\eta_j > 0$. Thus, in view of (114)

$$\lim_{m\to\infty} \frac{1}{m} \log \mathsf{E}(x_{m,j}(t) - mc_j)^+ \leq \Lambda_{x_j}(\eta_j) - \eta_j c_j \tag{141}$$

for all $\eta_j > 0$. Taking the best bound yields

$$\lim_{m\to\infty} \frac{1}{m} \log \mathsf{E}(x_{m,j}(t) - mc_j)^+ \leq - \sup_{\eta_j>0} [\eta_j c_j - \Lambda_{x_j}(\eta_j)]. \tag{142}$$

From (124) and the assumption $D\bar{\rho}^T < c$, one has $\Lambda'_{x_j}(0) < c_j$ for all j. Thus, the supremum of $\sup_{\eta_j>0}[\eta_j c_j - \Lambda_{x_j}(\eta_j)]$ cannot be attained for $\eta_j \leq 0$. This implies

$$\sup_{\eta_j>0} [\eta_j c_j - \Lambda_{x_j}(\eta_j)] = \sup_{\eta_j} [\eta_j c_j - \Lambda_{x_j}(\eta_j)] = \Lambda^*_{x_j}(c_j). \tag{143}$$

Q.E.D.

[*Proof of Theorem 11*] Since we do not specify how we drop cells when the total number of cells in a node exceeds its capacity, we only have the following upper bound for $L_{m,r}(t)$ (cf. the derivation for the upper bound in (133))

$$L_{m,r}(t) \leq \sum_{j=1}^{J} D_{jr}[x_{m,j}(t) - mc_j]^+ \leq J \max_{j:D_{jr}>0} [x_{m,j}(t) - mc_j]^+. \tag{144}$$

The rest of the proof is completely analogous to that for Theorem 10. Q.E.D.

Acknowledgements
The author would like to thank the referees for their careful reading and detailed comments that have greatly improved the presentation of this paper. The research is supported in part by the National Science Council, Taiwan, R.O.C., under Contract NSC85-2121-M007-035.

References

Chang, C.S. 1995. Sample path large deviations and intree networks. *Queueing Systems*, 20, pp. 7–36.

Dembo. A., and Zeitouni, O. 1992. *Large Deviations Techniques and Applications*. Boston: Jones and Barlett Publishers.

Deuschel, J.D., and Stroock, D.W. 1989. *Large Deviations*. Boston: Academic Press.

Ellis, R. 1984. Large deviations for a general class of random vectors. *Ann. Probab.*, 12, pp. 1–12.

Gärtner, J. 1977. On large deviations from invariant measure. *Theory Probab. Appl.*, 22, pp. 24–39.

Gazdzicki, P., Lambadaris, I., and Mazumdar, R.R. 1993. Blocking probabilities for large multirate Erlang loss systems. *Adv. Appl. Probab.*, 25, pp. 997–1009.

Hiriart-Urruty, J.-B., and Lemaaréchal, C. 1991. *Convex Analysis and Minimization Algorithms II.* Springer-Verlag, New York.

Hui, J.Y. 1988. Resource allocation for broadband networks. *IEEE Select. Areas Commun.*, 6, pp. 1598–1608.

Hunt, P.J. 1989. Implied cost in loss networks. *Adv. Appl. Probab.*, 21, pp. 661–680.

Hunt, P.J., and Kelly, F.P. 1989. On critically loaded loss networks. *Adv. Appl. Probab.*, 21, pp. 831–841.

Kelly, F.P. 1991. Loss networks. *Ann. Appl. Probab.*, 1, pp. 319–378.

Labourdette, J.-F.P., and Hart, G.W. 1992. 'Blocking probabilities in multitraffic loss systems: insenstivity, asymptotic behavior and approximations. *IEEE Trans. on Commum.*, 40, pp. 1355–1366.

Mitra, D., and Morrison, J.A. 1994. Erlang capacity and uniform approximations for shared unbuffered resources. *IEEE/ACM Trans. Networking*, 2, pp. 558–570.

Reiman, M.I. 1991. A critically loaded multiclass Erlang loss system. *Queueing Systems*, 9, pp. 65–82.

Rockafeller, R.T. 1970. *Convex Analysis.* Princeton: Princeton University Press.

Ross, K.W. 1995. *Multiservice Loss Models for Broadband Telecommunication Networks.* Springer-Verlag: New York.

Whitt, W. 1993. Tail probability with statistical multiplexing and effective bandwidths in multi-class queues. *Telecommunication Systems*, 2, pp. 71–107.

Whittle, P. 1990. A risk-sensitive maximum principle. *Systems & Control Letters*, 15, pp. 183–192.

Whittle, P. 1991a. A risk-sensitive maximum principle: the case of imperfect state observation. *IEEE Trans. Automat. Contr.*, 36, pp. 793–801.

Whittle, P. 1991b. Likelihood and cost as path integrals. *J. R. Statist. Soc, B*, 53, pp. 505–538.

Contributing Authors

Cheng-Shang Chang received the B.S. degree from the National Taiwan University, Taipei, Taiwan, in 1983, and the M.S. and Ph.D. degrees from Columbia University, New York, NY, in 1986 and 1989, respectively, all in Electrical Engineering. From 1989 to 1993 he was employed as a Research Staff Member at the IBM Thomas J. Watson Research Center, Yorktown Heights, N.Y. In 1993, he joined the Department of Electrical Engineering at National Tsing Hua University, Taiwan, R.O.C. as an associate professor. His current research interests are concerned with queueing theory, stochastic scheduling and performance evaluation of telecommunication networks and parallel processing systems. Dr. Chang received the IBM Outstanding Innovation Award in 1992, and he is currently Associate Editor for *Operations Research.*

Hung-Jen Wang received the B.S. degree in power mechnical engineering from the National Tsing Hua University, Hsinchu, Taiwan, in 1994, and the M.S. degree in electrical engineering from the National Tsing Hua University, Hsinchu, Taiwan, in 1996.