Bounds and Inequalities for Single Server Loss Systems

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ABSTRACT

We consider a single server loss system in which arrivals occur according to a doubly stochastic Poisson process with a stationary ergodic intensity function λ_t. The service times are independent, exponentially distributed r.v.'s with mean μ^{-1}, and are independent of arrivals. We obtain monotonicity results for loss probabilities under time scaling as well as under amplitude scaling of λ_t. Moreover, using these results we obtain both lower and upper bounds for the loss probability.

Keywords: Ross's conjecture, loss systems, variability ordering, doubly stochastic Poisson processes

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1. Introduction

Consider a single-server loss system at which the customers arrive according to a doubly stochastic Poisson process with an intensity function $\lambda_t, (t \geq 0)$, where $\{\lambda_t, t \geq 0\}$ is a stochastic process with $\lim_{t\to\infty} \frac{1}{t}\int_0^t \lambda_s ds = \lambda$. Ross[14] conjectured that the loss probability of the system above is larger than the loss probability in a single server loss system in which the arrivals are Poisson with fixed rate λ. He assumed that the successive service times are independently drawn from a distribution G with mean μ^{-1}. In this paper, we consider a special case of the model just described in which the system has zero buffer capacity; the intensity function λ_t is positive, right continuous, and stationary ergodic; and the successive service times are independently drawn from an exponential distribution with mean μ^{-1}. By comparing two DSP/M/1 loss systems with intensities λ_t and $\tilde{\lambda}_t$, where $\tilde{\lambda}_t \stackrel{\text{def}}{=} \lambda_{\frac{t}{c}}$, and with loss probabilities P_l and $\tilde{P}_l$, we prove that $P_l \leq \tilde{P}_l$ for any positive integer c. By varying c, we obtain both a lower bound and an upper bound for the loss probability, i.e. we show that

$$\frac{\lambda}{\lambda+\mu} \leq P_l \leq E\Big(\frac{\lambda_0^2}{\lambda(\lambda_0+\mu)}\Big).$$

The upper bound obtained here is similar to the upper bound of the delay in DSP/GI/1 queueing systems (Rolski [13]). By comparing DSP/M/1 loss systems with intensities $\lambda_t^{(d)} = \lambda + d \cdot Z_t$, where Z_t is a bounded ergodically stationary process, we also prove P_l is a nondecreasing function of $|d|$ as long as $\lambda_t^{(d)} \geq 0$.

This problem has been treated in the literature before. Fond and Ross [5] considered a single server exponential loss system in which the arrival and service rates alternate between the pairs (λ_1,μ_1) and (λ_2,μ_2) spending an exponential amount of time with rate α_i/c in state (λ_i,μ_i), i=1 and 2, and showed that the loss probability is a nondecreasing function of c. This c corresponds to our c with the exception that Fond and Ross did not assume integrality. For $c \to 0$, their system behaves as an M/M/1 loss system and it serves as a lower bound for the loss probability of their model. Niu [10] considered a similar loss system that is in one sense more general: instead of assuming an exponential service distribution, he considered arbitrary service time distributions G_1 and G_2. In another sense it is somewhat more restricted as he assumes $\lambda_2 = 0$. Niu [10] showed that the loss probability of this system is also a nondecreasing function of c , but he gave counterexamples for Ross's conjecture when arrivals and services are arbitrary and nonstationary. Heyman [8] also found a counter-example showing that the assumption of exponential service times cannot be relaxed by considering a class of DSP/D/1 loss systems. Rolski [12] obtained the lower bound for our model as well as for DSP/M/2 loss systems. Svoronos and Green [18] considered DSP/M/1 loss systems with intensities $\lambda_t = \lambda + d \cdot f(t)$, where d is a constant and $f(t)$ is a bounded periodic function of which the integral over the period equals 0. They showed P_l is increasing convex in d. In this paper, we provide a simple method based on stochastic comparison techniques in order to deal with the loss probabilities of DSP/M/1 loss systems and show how some of their results can be obtained and generalized.

2. Preliminary Results

In this section, we state some definitions and propositions which we need in what follows.

Definition 2.1. A random variable (r.v.) X_1 with distribution function $G_1(x)$ is less variable ($\leq_v$) than a r.v. X_2 with distribution function $G_2(x)$ if

$$\int_{-\infty}^{\infty} h(x) dG_1(x) \leq \int_{-\infty}^{\infty} h(x) dG_2(x),$$

for all h increasing convex.

This ordering is referred to as variability ordering (see Stoyan [16] or Ross [15]). In what follows, we use both notations $X_1 \leq_v X_2$ and $G_1(x) \leq_v G_2(x)$.

For notions of stability, we basically follow Rolski's [11] definitions.

Definition 2.2. A sequence of not necessarily independent r.v's T=$\{T_1, T_2, \ldots\}$ is called stable if for any set E $\in$ $Borel(R^\infty)$,

$$\lim_{n\to\infty} \frac{1}{n} \Sigma_{i=1}^{n} P(S^i T \in E) = P_{T^0}(E)$$

exists, where P_{T^0} is a probability measure on R^∞ and $S^iT = S^{i-1}(ST)$ with ST=$\{T_2,T_3,...\}$.■

We call P_{T^0} the stationary distribution of T. Moreover, if S is ergodic on (R^∞,$Borel(R^\infty)$,P_{T^0}), then T is said to be ergodically stable.

Definition 2.3. Two stable sequences of r.v.'s T and $\tilde{T}$ are called stably equivalent if they have a common stationary distribution.

Now we consider a G/M/1 loss system with arbitrary arrivals and independent exponential service times with mean μ^{-1}. We assume the service times to be independent of the arrivals but successive interarrival times not necessarily to be independent. Since there is no buffer for customers, a customer when he arrives is either accepted and served immediately if he finds the system empty or lost if he finds the system busy.

Let T=$\{T_1,T_2,...\}$ be the sequence of interarrival times. Among all the sequences stably equivalent to T, the stationary sequence is of importance; T^0=$\{T_1^0,T_2^0,...\}$ is a stationary sequence of T if T^0 has the probability distribution P_{T^0}. Clearly, ergodic stability does not imply stationarity. However, an ergodically stable sequence T has associated with it a stationary sequence T^0. Let $I_i, i \geq 1$ be an indicator function defined as follows:

$$I_i = \begin{cases} 1, & \text{if the i-th customer is lost;} \\ 0, & \text{otherwise.} \end{cases}$$

Finally, $P^{(i)}$=EI_i denote the probability of the i-th customer being lost. Rolski ([12], Remark 2.14) states the following stability results.

Proposition 2.4. If an arrival process for a G/M/1 loss system is ergodically stable, then the loss probability

$$P_l = \lim_{n\to\infty} \frac{1}{n}\Sigma_{i=1}^n I_i \quad a.s.$$

By the bounded convergence theorem, $P_l = \lim_{n\to\infty} \frac{1}{n}\Sigma_{i=1}^n P^{(i)}$. ■

Lemma 2.5. $P^{(i)} = \int_0^\infty P(T_i < x)\mu e^{-\mu x}dx$

Proof: This follows from the fact that if the (i-1)th customer is accepted, then the system starts an exponential busy period at the instant when the (i-1)th customer arrives and if the (i-1)th customer is lost, the system must be in a busy period. By the memoryless property, the excess duration of this busy period is also exponential with the same mean. Hence, the probability that the i-th customer is lost is just the probability that the interarrival time is shorter than the exponential busy period started at the arrival of the (i-1)th customer which is the probability stated in the lemma. ∎

Lemma 2.6. If an arrival process for a G/M/1 loss system is ergodically stable, then

$$P_l = \int_0^\infty G(x)\mu e^{-\mu x} dx \ ,$$

where G(x)$\stackrel{\text{def}}{=}$ $P_{T^0}([-\infty, x) \times [-\infty, \infty] \times [-\infty, \infty] \ldots)$=P$(T_1^0 < x)$ is the distribution of the first interarrival time in the stationary sequence of interarrival times T^0.

Proof: Since the arrival process is ergodically stable by assumption , we have from Definition 2.2 that

$$\lim_{n\to\infty} \frac{1}{n}\Sigma_{i=1}^n P(T_i < x) = P(T_1^0 < x) \tag{1}$$

By Lemma 2.5, Fubini's theorem and the bounded convergence theorem,

$$\lim_{n\to\infty} \frac{1}{n}\Sigma_{i=1}^n P^{(i)} = \lim_{n\to\infty} \int_0^\infty \{\frac{1}{n}\Sigma_{i=1}^n P(T_i < x)\}\mu e^{-\mu x} dx$$
$$= \int_0^\infty P(T_1^0 < x)\mu e^{-\mu x} dx = \int_0^\infty G(x)\mu e^{-\mu x} dx < \infty. \tag{2}$$

Hence, by (2) and Proposition 2.4,

$$P_l = \lim_{n\to\infty} \frac{1}{n}\Sigma_{i=1}^n P^{(i)} = \int_0^\infty G(x)\mu e^{-\mu x} dx \quad ∎$$

Theorem 2.7. If T and $\tilde{T}$ are two ergodically stable input sequences of G/M/1 loss systems with $G(x)\stackrel{\text{def}}{=}P(T_1^0 \le x)$, $\tilde{G}(x) \stackrel{\text{def}}{=}$ P$(\tilde{T}_1^0 \le x)$, and $\int xdG(x)$=$\int xd\tilde{G}(x)$, then $G(x) \le_v \tilde{G}(x)$ implies $P_l \le \tilde{P}_l$.

Proof: Note that $G(0^+) = 0$. Integration by parts yields

$$\int_0^\infty G(x)\mu e^{-\mu x} dx = \int_0^\infty e^{-\mu x} dG(x).$$

Since $e^{-\mu x}$ is a convex function, the theorem follows from Corollary 8.5.2 of Ross [15] and Lemma 2.6. ∎

Remark 2.8. Observe that the loss probability is the Laplace-Stieltjes transform of the one dimensional stationary distribution $G(x)$. The ordering by Laplace-Stieltjes transform (Stoyan [16], Section 1.8) is sufficient for the proof of Theorem 2.7.

3. Main Results

Let N(t) be a doubly stochastic Poisson process (DSP). We introduce the following assumptions for the intensity λ_t with respect to N(t).

(A.1) The intensity function $\lambda_t, t \geq 0$ is stationary ergodic.

(A.2) The sample path (realization) of λ_t is right continuous.

(A.3) The intensity function λ_t is positive, i.e., P($\lambda_t > 0$)=1, t≥0.

Since λ_t is stationary ergodic, N(t) is also stationary ergodic (Theorems 1.4 and 1.7 of Grandell [7]). Let T denote the sequence of the interarrival times of N(t). By the cross ergodic theorem((8.3.2) of Baccelli and Bremaud [1]), it can be shown that T is ergodically stable. Let T^0=[T_1^0 ,T_2^0 ,. . .] be a stationary sequence which is stably equivalent to T. Thus we can define G(x) as the stationary distribution of interarrival times:

$$G(x) \stackrel{\text{def}}{=} P(T_1^0 < x). \tag{3}$$

Notice that (A.1) implies that

$$\lambda = E\int_0^1 \lambda_s ds = E\lambda_t = E\lambda_0 = \lim_{t\to\infty}\frac{1}{t}\int_0^t \lambda_s ds \ \ a.s. \tag{4}$$

where λ_0 is λ_t at time 0.

Define $G_l(x)$ and $G_u(x)$ as follows:

$$\begin{aligned} G_l(x) &\stackrel{\text{def}}{=} 1 - e^{-\lambda x}, \\ G_u(x) &\stackrel{\text{def}}{=} E(1 - \frac{\lambda_0}{\lambda}e^{-\lambda_0 x}). \end{aligned} \tag{5}$$

In order to obtain more insight into the distribution $G_u(x)$, consider the case that λ_0 takes values on a countable space ($\lambda_i,\ i = 1, 2 \ldots$), thus $G_u(x)$ then can be viewed as a mixture of exponential distributions with means λ_i^{-1} and mixing probabilities $\frac{\lambda_i P(\lambda_0=\lambda_i)}{\lambda}$, where $\lambda = E\lambda_0 = \sum_i \lambda_i P(\lambda_0 = \lambda_i)$. The reason for assuming λ_t strictly positive is to make $G_u(x)$ a distribution function that is a mixture of exponential distributions.

Define the integral of intensity $\Lambda[t_1, t_2]$ as follows:

$$\Lambda[t_1, t_2] \stackrel{\text{def}}{=} \int_{t_1}^{t_2} \lambda_s ds. \tag{6}$$

We now show that the r.v. $\Lambda[0, t]$ plays a key role in the variability ordering of interarrival time distributions.

Lemma 3.1.

$$\int_a^\infty (1 - G(x))dx = \frac{1}{\lambda} E e^{-\Lambda[0,a]}.$$

Proof: Again, it follows from the cross ergodic theorem ((8.3.2) of Baccelli and Bremaud [1]) that $P_{T^0}(E) = P_N^0(T \in E)$ for all $E \in Borel(R^\infty)$, where P_N^0 is the Palm probability (defined in (3.1.2) of Baccelli and Bremaud [1]). From the inversion formula ((4.2.4b) of Baccelli and Bremaud [1], Corollary 1.2.10 of Franken et al. [6]) or the Palm-Khintchine equations of Cox & Isham ([4], pp. 29-30), it follows that

$$\begin{aligned}\int_a^\infty (1 - G(x))dx &= \frac{1}{\lambda} P(T_1 > a) \\ &= \frac{1}{\lambda} P(\textit{there is no arrival in } (0, a)).\end{aligned}$$

From (1.9) of Bremaud [2], it follows that

$$P(\textit{there is no arrival in } (0, a)) = E e^{-\Lambda[0,a]}. \quad \blacksquare$$

From Lemma 3.1, it follows that $G_l(x)$, $G_u(x)$ and $G(x)$ have the same mean, i.e.,

$$\frac{1}{\lambda} = \int_0^\infty x dG(x) = \int_0^\infty x dG_l(x) = \int_0^\infty x dG_u(x)$$

Consider a DSP/M/1 loss system with input N(t) and independent exponential service times with mean μ^{-1}. By Lemma 2.6, the loss probability P_l is well defined and has the form of Lemma 2.6. Let $\tilde{N}(t)$ be a second DSP process with intensity $\tilde{\lambda}_t$, $\tilde{\Lambda}[t_1,t_2]\stackrel{\text{def}}{=}\int_{t_1}^{t_2}\tilde{\lambda}_s ds$, $\tilde{G}(x)$ being the stationary distribution of its interarrival times and $\tilde{P}_l$ being the loss probability for the system with input $\tilde{N}(t)$.

Theorem 3.2. If $E\Lambda[0,t]{=}E\tilde{\Lambda}[0,t]$ and $\Lambda[0,t]\leq_v\tilde{\Lambda}[0,t]$ for all t, then under (A.1) $G(x)\leq_v\tilde{G}(x)$,and $P_l\leq\tilde{P}_l$.

Proof: Since $E\Lambda[0,t]{=}E\tilde{\Lambda}[0,t]$, by the well known properties of variability ordering (Proposition 8.5.1 and Corollary 8.5.2 of Ross [15]) and Lemma 3.1, we conclude $G(x)\leq_v\tilde{G}(x)$. The last part of the statement follows from Theorem 2.7. ∎

In what follows, we show variability ordering of $\Lambda[0,t]$ under both time scaling and amplitude scaling. Then we use Theorem 3.2 to find comparison results for loss probabilities.

Theorem 3.3. If $\tilde{\lambda}_t\stackrel{\text{def}}{=}\lambda_{\frac{t}{c}}$ $(\tilde{\lambda}_t\stackrel{\text{def}}{=}\lambda_{ct})$, where c is a positive integer, then under (A.1), $\Lambda[0,t]$ $\leq_v(\geq_v)$ $\tilde{\Lambda}[0,t]$ for all $t\geq0$, and henceforth $G(x)\leq_v(\geq_v)$ $\tilde{G}(x)$, $P_l\leq(\geq)$ $\tilde{P}_l$.

Proof: It suffices to show the theorem for $\tilde{\lambda}_t{=}\lambda_{\frac{t}{c}}$. We have $E\Lambda[0,t]{=}E\tilde{\Lambda}[0,t]{=}\lambda t$ by (A.1) and Fubini's Theorem. By Corollary 8.5.2 of Ross[15], we need to show $Eh(\Lambda[0,t])\leq Eh(\tilde{\Lambda}[0,t])$ for all convex functions h. Observe that

$$\Lambda[0,t]=\int_0^t\lambda_s ds=\frac{1}{c}\int_0^{ct}\lambda_{\frac{s}{c}}ds=\frac{1}{c}\sum_{i=0}^{c-1}\tilde{\Lambda}[it,(i+1)t].$$

By the convexity of h,

$$h(\Lambda[0,t])\leq\frac{1}{c}\sum_{i=0}^{c-1}h(\tilde{\Lambda}[it,(i+1)t]),\tag{7}$$

and by stationarity (A.1),

$$Eh(\tilde{\Lambda}[it,(i+1)t])=Eh(\tilde{\Lambda}[0,t]).$$

By taking expectation on (7), we arrive at the desired result. ∎

Theorem 3.3 yields stochastic comparison results for time scaled DSP/M/1 loss systems. The larger (smaller) c, where $\tilde{\lambda}_t=\lambda_{\frac{t}{c}}$ $(\tilde{\lambda}_t=\lambda_{ct})$, the larger the loss probability is. As Ross [14] explained, c regulates how fast the arrival rate changes and intuitively the larger c ($\tilde{\lambda}_t = \lambda_{\frac{t}{c}}$) is the less "stationary" λ_t is. However, we cannot show this to be true for arbitrary noninteger $c \geq 0$. Consider the following example which violates the variability ordering for interarrival time distributions.

Example 3.4. (Svoronos and Green [17])) Let

$$f(t) = \begin{cases} 1, & \text{if } 0 \leq t < 1; \\ 2, & \text{if } 1 \leq t < 2. \end{cases},$$

and $f(t) = f(t+2)$,

$$\lambda_t = \begin{cases} \lambda - a, & \text{if } f(t+\theta) = 1; \\ \lambda + a, & \text{if } f(t+\theta) = 2. \end{cases},$$

where θ is a r.v. with uniform distribution over [0,2] and λ, a ($\lambda \geq a$) are positive constants. For each draw of the r.v. θ, we generate a sample path λ_t. Then λ_t is ergodically stationary and periodic for each sample path. Now consider the time scaled process $\tilde{\lambda}_t = \lambda_{\frac{2}{3}t}$ ($c = \frac{3}{2}$). Observe that $\tilde{\Lambda}[0,3] = \int_0^3 \lambda_{\frac{2}{3}s} ds = \frac{3}{2}\Lambda[0,2] = 3\lambda$ which is a constant and $\Lambda[0,3]$ is not.

This illustrates the fact that there is not necessarily a variability ordering between $G(x)$ and $\tilde{G}(x)$ for arbitrary c. But Lemma 3.1 provides a sufficient condition and what is needed for the loss probability is the Laplace transform of the interarrival time distribution. Although variability ordering does not hold, the inequality of loss probabilities still holds for arbitrary nonnegative constants c. Svoronos and Green [17] showed

$$\begin{aligned} 1-P_l \stackrel{\text{def}}{=} h(\mu,\lambda,a) &= \frac{\mu}{2\lambda}\left(\frac{\lambda - a}{\mu+\lambda-a} + \frac{\lambda+a}{\mu+\lambda+a}\right) \\ &+ \frac{2a^2\mu^2}{\lambda} \frac{(1-exp(-\mu-\lambda-a))(1-exp(-\mu-\lambda+a))}{(\mu+\lambda+a)^2(\mu+\lambda-a)^2(1-exp(-2\mu-2\lambda))}. \end{aligned} \quad (8)$$

Consider the case $\tilde{\lambda}_s \stackrel{\text{def}}{=} \lambda_{\frac{s}{c}}$ and run another clock $t' = \frac{t}{c}$ which is c times slower than the original clock. Hence the service rates and arrival rates are speeded up by a factor c under t'. Let $\tilde{\lambda}'_s$, μ', λ', a' be $\tilde{\lambda}_s$, μ, λ, a measured under t'. Then $\mu' = c\mu$, $\lambda' = c\lambda$, $a' = ca$, and

$$\lambda'_s = \begin{cases} \lambda' - a', & \text{if } f(s+\theta) = 1; \\ \lambda' + a', & \text{if } f(s+\theta) = 2. \end{cases}.$$

But the loss probabilities are not changed under t' because they only count percentages of lost customers. By (8), $1 - \tilde{P}_l$=h(μ',λ',a')=h ($c\mu$,$c\lambda$,ca)$\stackrel{\text{def}}{=} H(c)$. By differentiating H(c), it can be shown that H(c) is a decreasing function of $c \geq 0$ (Chang [3]), and henceforth $\tilde{P}_l$ is an increasing function of $c \geq 0$.

Now we can use the results of time scaling to obtain bounds. We do this by considering the limiting cases.

Lemma 3.5. Under (A.1), $G_l(x) \leq_v G(x)$.

Proof: From Lemma 3.1 and Jensen's inequality, it follows that

$$\int_a^\infty (1-G(x))dx = \frac{1}{\lambda} E e^{-\int_0^a \lambda_s ds} \geq \frac{1}{\lambda} e^{-\lambda a} = \int_a^\infty (1-G_l(x))dx.$$

We can also prove it using Theorem 3.3. Let $c \to \infty$ in the case $\tilde{\lambda}_t = \lambda_{ct}$ and then use the ergodicity of λ_t to show $\tilde{\Lambda}[0,t] \to \lambda t$. ∎

Lemma 3.6. Under (A.1), (A.2), and (A.3), $G(x) \leq_v G_u(x)$.

Proof: Let $\tilde{\lambda}_t \stackrel{\text{def}}{=} \lambda_{\frac{t}{c}}$. By Theorem 3.3 and Lemma 3.1,

$$\int_a^\infty (1-G(x))dx \leq \int_a^\infty (1-\tilde{G}(x))dx = \frac{1}{\lambda} E e^{-\int_0^a \tilde{\lambda}_s ds} = \frac{1}{\lambda} E e^{-c\int_0^{\frac{a}{c}} \lambda_s ds}. \tag{9}$$

Note that λ_t is right continuous by (A.2) and then by the mean value theorem and continuity of e^{-x},

$$\lim_{c\to\infty} e^{-c\int_0^{\frac{a}{c}} \lambda_s ds} = e^{-\lambda_0 a} \quad a.s. \,, \tag{10}$$

where λ_0 is λ_t at time 0.

By (9),(10), the bounded convergence theorem, (A.3) and Fubini's theorem,

$$\int_a^\infty (1-G(x))dx \leq \frac{1}{\lambda} E e^{-\lambda_0 a} = \frac{1}{\lambda} \int_a^\infty E\lambda_0 e^{-\lambda_0 x} dx \tag{11}$$

Thus, by (11) and (5),

$$\int_a^\infty (1-G(x))dx \leq \int_a^\infty (1-G_u(x))dx \quad \blacksquare \tag{12}$$

In the following theorem we present bounds on the loss probabilities.

Theorem 3.7. (i) Under (A.1), $P_l \geq \frac{\lambda}{\lambda+\mu}$.

(ii) Under (A.1),(A.2) and (A.3), $P_l \leq E\big(\frac{\lambda_0^2}{\lambda(\lambda_0+\mu)}\big)$.

Proof: Observe that

$$\int_0^\infty \mu e^{-\mu x} G_l(x)dx = \frac{\lambda}{\lambda+\mu}$$

and

$$\int_0^\infty \mu e^{-\mu x} G_u(x)dx = \int_0^\infty e^{-\mu x} dG_u(x)$$

$$= \int_0^\infty e^{-\mu x}\big(E(\frac{\lambda_0^2}{\lambda}e^{-\lambda_0 x})\big)dx = E\big(\frac{\lambda_0^2}{\lambda(\lambda_0+\mu)}\big).$$

The theorem follows from Lemma 3.5 and 3.6 and Theorem 2.7. ∎

Consider a family of DSP/M/1 loss systems with intensities scaled in amplitude , i.e. $\lambda_t^{(d)}$ $=\lambda+d\cdot Z_t$, where Z_t is a bounded ($\sup_t |Z_t| \leq M$) ergodically stationary process with mean 0 ($EZ_t = 0$). Obviously, $\lambda_t^{(d)}$ satisfies (A.1). Let $\Lambda^{(d)}[0,t] \stackrel{\text{def}}{=} \int_0^t \lambda_t^{(d)}ds$ and let $P_l^{(d)}$ be the corresponding loss probability.

Theorem 3.8. If $0 \leq d_1 \leq d_2 \leq \frac{\lambda}{M}$ then $\Lambda^{(d_1)}[0,t] \leq_v \Lambda^{(d_2)}[0,t]$ and henceforth $P_l^{(d_1)} \leq P_l^{(d_2)}$.

Proof: Define $Y_t \stackrel{\text{def}}{=} d_1 \cdot \int_0^t Z_s ds$. Then EY_t=0 by Fubini's Theorem. Rewrite $\Lambda^{(d_1)}[0,t] = \lambda t + Y_t$ and $\Lambda^{(d_2)}[0,t] = \lambda t + d_3 Y_t$, where $d_3 = \frac{d_2}{d_1} \geq 1$. By Problem 8.29 of Ross [15], Y_t $\leq_v$ $d_3 Y_t$. Note λt is a constant and hence independent of Y_t. The result follows from the fact that the variability ordering is preserved under summation of independent r.v.'s. ∎

Corollary 3.9. Let Z_t $=f(t+\theta)$, and $\sup_t |f(t)| \leq \lambda$, $f(t+1) = f(t)$, $\int_0^1 f(t)dt = 0$, θ a r.v. with uniform distribution on $[0,1]$. Then $EZ_t = 0$ and Theorem 3.8 holds. This model was considered by Svoronos and Green [18].

Remark 3.10. Rolski [11] gave two examples to show that the stationarity of (A.1) can be relaxed. One is a Markovian modulated Poisson process with countable state space, the other is a DSP with periodic deterministic intensity function. Here we argue that the λ_t being strictly positive assumption (A.3) can also be relaxed if λ_t is bounded, i.e. P($\lambda_t <$ M, t$\geq$0)=1, for some constant M$< \infty$. In this case, we can choose a sequence $\lambda_t^{(n)}$ positive and bounded which converges to λ_t pointwise, and (by Bremaud [2] Chapter VI(2.25)) the loss probability $P_l^{(n)}$ converges to P_l. Hence the inequality still holds for λ_t nonnegative if

λ_t is bounded. This result allows us to extend the result for interrupted Poisson processes which are also renewal processes. The property of stochastic convexity of the distribution of interarrival times can be used for proving both the lower bound and upper bound for the delays of infinite capacity queueing models and tandem queueing models(Niu [10]) if the arrival processes are interrupted Poisson processes. Similarly, it can also be applied to GI/M/S loss systems. Kokotushkin [9] showed the lower bounds of the loss probabilities for interrupted Poisson arrivals. The upper bounds obtained by Theorem 3.7 are the loss probabilities of the systems without arrival interruptions.

Acknowledgment

We would like to thank Prof. N.U. Prabhu , Prof. T. Rolski, and the referees for their useful suggestions and comments.

References

[1] F. Baccelli and P. Bremaud, *Palm Probabilities and Stationary Queues* (Springer-Verlag, New York, 1987).

[2] P. Bremaud, *Point Processes and Queues, Martingale Dynamics* (Springer-Verlag, New York, 1981).

[3] C.S. Chang, Comparison Theorems for Queueing Systems and Their Applications to ISDN. PhD dissertation, Department of Electrical Engineering, Columbia University, New York, 1989.

[4] D.R. Cox and V. Isham, *Point Processes* (Chapman and Hall, New York, 1980).

[5] S. Fond and S.M. Ross, A Heterogeneous Arrival and Service Queueing Loss Model, Naval Research Logistics Quarterly 25 (1978) 483-488.

[6] P. Franken, D. Konig, U. Arndt, and V. Schmidt, *Queues and Point Processes* (Academie-Verlag, J. Wiley & Sons, Berlin, 1982).

[7] J. Grandell, *Doubly Stochastic Poisson Processes* (Springer-Verlag, New York, 1976).

[8] D.P. Heyman, On Ross's Conjecture about Queues with Nonstationary Poisson Arrivals, J. Appl. Prob. 19 (1982) 245-249.

[9] V.A. Kokotushkin, A Generalization of the Palm-Khinchin Theorem, Teor. Verojat Drimen 19 (1974) 622-625 (=Theory Prob. Appl. 19 (1974) 594-597).

[10] S.-C Niu, A Single Server Queueing Loss Model with Heterogeneous arrival and Service, Operat. Res. 28 (1980) 584-593.

[11] T. Rolski, Queues with Non-stationary Input Stream: Ross's Conjecture, Adv. Appl. Prob 13 (1981) 603-618.

[12] T. Rolski, Comparison Theorems for Queues with Dependent Inter-Arrival Times, Modelling and Performance Evaluation Proceeding of the International Seminar, Paris, France (Jan 24-26, 1982) 42-67.

[13] T. Rolski, Upper Bounds for Single Server Queues with Doubly Stochastic Poisson Arrivals, Math. of Operat. Res. 11 (1986) 442-450.

[14] S.M. Ross, Average Delay in Queues with Non-stationary Poisson Arrivals, J. Appl. Prob. 15 (1978) 602-609.

[15] S.M. Ross, *Stochastic Processes* (J. Wiley & Sons, New York, 1983).

[16] D. Stoyan, *Comparison Methods for Queues and Other Stochastic Models* (J. Wiley & Sons, New York, 1983).

[17] A. Svoronos and L. Green, The N-Seasons S-Servers Loss System, Naval Research Logistics 34 (1987) 579-591.

[18] A. Svoronos and L. Green, A Convexity Result for Single Server Exponential Loss Systems with Nonstationary Arrivals, J. Appl. Prob. 25 (1988) 224-227.