Functional characterizations of some positively dependent bivariate random vectors

Cheng-Shang Chang
IBM Research Division

POSITIVELY DEPENDENT RANDOM VECTORS

Cheng-Shang Chang
IBM Research Division
T.J. Watson Research Center
H2-K06, P.O. Box 704
Yorktown Heights, NY 10598

Abstract

In this paper, we consider three different notions of positive dependence for a bivariate random vector: (i) total positivity of order 2, (ii) stochastic increasingness and (iii) positive quadrant dependence. By defining three classes of arrangement-increasing functions, we show that these three different notions can be unified by functional inequalities. Using these functional inequalities, we derive the relations between the positively dependent notions (i) and (iii) and their corresponding counterparts in stochastic majorization orderings. Moreover, these classes of functions also lead to equivalent characterizations of two random vectors with independent components ordered in the sense of likelihood ratio ordering and stochastic ordering.

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1 Introduction

Various interpretations of different notions of positive dependence of bivariate random vectors have been reported in the literature ([1, 3, 4, 9, 10, 13, 16, 17, 20, 21, 24, 25, 26, 29, 31, 33, 34] among many others). Using inequalities on distribution functions, Yanagimoto [34] and Metry and Sampson [20] were able to construct systematically various notions of positive dependence as well as positive dependent orderings. Block, Chhetry, Fang and Sampson [4] showed equivalent characterizations for various notions of positive dependence via permutation orderings. Nelsen [21] provided geometric interpretations via copula. Whitt [33] showed connections between positive dependence and conditional stochastic orderings. By comparing covariances, Alzaid [1], Bergmann [3] and Shaked [25] provided general theories for notions of positive dependence weaker than positive quadrant dependence and association. Tchen [29] characterized the notion of positive quadrant dependence by supermodular functions. However, as pointed out in Metry and Sampson [20], it is not clear if there exist functional characterizations for different notions of positive dependence and positive orderings besides the one considered by Tchen. One of our objectives in this paper is to provide a unified characterization by functional inequalities for three different notions of positive dependence: (i) total positivity of order 2, (ii) stochastic increasingness and (iii) positive quadrant dependence.

Kendall [15] introduced a measure of correlation for a bivariate random vector $\boldsymbol{X} = (X_1, X_2)$ by comparing two pairs of independent random vectors $\boldsymbol{X}'$ and $\boldsymbol{X}''$ with the same joint distribution as $\boldsymbol{X}$. The measure, known as Kendall's τ, is defined as follows:

$$\tau \stackrel{\text{def}}{=} P((X_1' - X_1'')(X_2' - X_2'') > 0) - P((X_1' - X_1'')(X_2' - X_2'') < 0). \tag{1}$$

The intuition behind this is that the value of τ is 1 if (X_1', X_2') and (X_1'', X_2'') are concordant, i.e., either $X_1' \le X_1''$ and $X_2' \le X_2''$ or $X_1' \ge X_1''$ and $X_2' \ge X_2''$, and -1 if (X_1', X_2') and (X_1'', X_2'') are discordant, i.e., either $X_1' \le X_1''$ and $X_2' \ge X_2''$ or $X_1' \ge X_1''$ and $X_2' \le X_2''$. The value of τ is 0 if X_1 and X_2 are independent. In view of the concordance in (1), one notices that the Hardy-Littlewood-Polya (HLP) inequality [11] plays a key role in Kendall's τ, i.e.,

$$X_1'X_2' + X_1''X_2'' \ge X_1'X_2'' + X_1''X_2'$$

if $\boldsymbol{X}'$ and $\boldsymbol{X}''$ are concordant. This suggests that one might be able to use the HLP inequality to extend the notion of Kendall's τ from a measure of positive dependence to characterizations of positive dependence (see [32, 29] for other interesting connections between the HLP inequality and positive dependence).

A generalization of the HLP inequality is known as Day's inequalities [8]. For two concordant vectors (x_1, x_2) and (y_1, y_2),

$$h\Big(\phi(x_1, x_2), \phi(y_1, y_2)\Big) \geq h\Big(\phi(x_1, y_2), \phi(y_1, x_2)\Big)$$

for all ϕ monotone and supermodular and h increasing and Schur convex. Based on recently developed rearrangement inequalities and stochastic majorization orderings (Shanthikumar and Yao [27], Chang [6]), Chang and Yao [7] derived stochastic versions of Day's inequalities [8]: for four independent random variables X_1, X_2, Y_1 and Y_2 and ϕ monotone and supermodular,

$$Eh\Big(\phi(X_1, X_2), \phi(Y_1, Y_2)\Big) \geq Eh\Big(\phi(X_1, Y_2), \phi(Y_1, X_2)\Big)$$

for (i) h increasing and Schur convex if X_1 and X_2 are less than Y_1 and Y_2 in the sense of *likelihood ratio ordering (lr)* respectively, and (ii) h increasing, symmetric, submodular and convex in each variables if X_1 and X_2 are less than Y_1 and Y_2 in the sense of *stochastic ordering (st)* respectively. These two inequalities are identified as two different stochastic majorization orderings $E_1^{\uparrow}$ and $E_5^{\uparrow}$ and these two classes of functions of h are called $\mathbf{C}_1^{\uparrow}$ in [19] and $\mathbf{C}_5^{\uparrow}$ in [6].

Using Kendall's idea of comparing two independent copies $\boldsymbol{X}'$ and $\boldsymbol{X}''$ of a bivariate random vectors $\boldsymbol{X}$, we show in this paper that for ϕ monotone and supermodular and h in $\mathbf{C}_i^{\uparrow}$, $i = 1$ and 5, respectively

$$Eh\Big(\phi(X_1', X_2'), \phi(X_1'', X_2'')\Big) \geq Eh\Big(\phi(X_1', X_2''), \phi(X_1'', X_2')\Big)$$

if the joint density of $\boldsymbol{X}$, is (i) totally positive of order 2, and (ii) positively quadrant dependent. To generalize these results, we define three new classes of arrangement-increasing functions AI_1, $\mathrm{AI}_{5,1}$ and AI_5 in this paper. We show that the three notions of positive dependence, (i) total positivity of order 2, (ii) stochastic increasingness and (iii) positive quadrant dependence, hold if and only if for any two independent copies $\boldsymbol{X}'$ and $\boldsymbol{X}''$ of $\boldsymbol{X}$,

$$E\psi\Big((X_1', X_2'), (X_1'', X_2'')\Big) \geq E\psi\Big((X_1', X_2''), (X_1'', X_2')\Big)$$

for all ψ in AI_1, $\mathrm{AI}_{5,1}$ and AI_5, respectively. By identifying the closure properties and examples of these three classes of functions, we also derive other equivalent characterizations for these three notions of positive dependence.

As a generalization of the notions of concordance and discordance, we consider two partial orderings, a and b, for random variables Two independent random vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ with independent components are said to be concordant with respect to (a, b) if either $X_1 \leq_a Y_1$ and $X_2 \leq_b Y_2$ or $X_1 \geq_a Y_1$ and $X_2 \geq_b Y_2$. For these three classes of functions, (i) AI_1, (ii) $\mathrm{AI}_{5,1}$ and (iii) AI_5, we show that the *expected* function value for two concordant vectors with respect to (a, b) is not less than that of two corresponding discordant vectors when (i) $(a, b) = (lr, lr)$, (ii) $(a, b) = (lr, st)$ and (iii) $(a, b) = (st, st)$. These functional characterizations of two partially ordered random vectors with independent components not only lead to extensions of the rearrangement inequalities in [7] but also provide converse statements for these inequalities.

Briefly, the rest of the paper is organized as follows: In §2 we introduce various notions of positive dependence in [2, 30]. In §3 we review definitions of some classes of functions that are used in this paper and define new classes of functions for characterizing the notions of positive dependence introduced in §1. In §4 we prove the main theorems for characterizations of positive dependence. In §5, we discuss the relations between positive dependence and stochastic majorization. Then we show in §6 that these three classes of arrangement-increasing functions lead to functional characterizations for partially ordered bivariate random vectors with independent components. Finally, in §7 we conclude the paper by discussing possible extensions to negatively dependent bivariate random vectors and positively dependent multivariate random vectors with dimension larger than 2.

A few words on the notation. Unless otherwise specified, throughout the paper we will use lower case letters such as x and y to denote deterministic vectors (in bold face) or scalars. In the vector case, a subscript will index the component of a vector, e.g. x_i; on the other hand, a superscript will index an entire vector, e.g. $\boldsymbol{x}'$, $\boldsymbol{y}^1$, etc. Upper case letters will denote random vectors (in bold face) and random variables, with subscripts and superscripts following the same conventions as in the deterministic case.

2 Positive dependence

In this section, we introduce some common notions of positive dependence of bivariate random vectors that are used in this paper. For other notions of positive dependence, we refer to Barlow and Proschan [2], Chapter 5 and Tong [30], Chapter 5.

Definition 2.1 *Consider a bivariate random vector $\boldsymbol{X} = (X_1, X_2)$ with joint density $f(x_1, x_2)$.*

(i) *The joint density is* totally positive of order 2 *(TP_2) if*

$$det \begin{vmatrix} f(x_1, x_2) & f(x_1, y_2) \\ f(y_1, x_2) & f(y_1, y_2) \end{vmatrix} \geq 0, \tag{2}$$

for all $(x_1, x_2) \leq (y_1, y_2)$.

(ii) *X_2 is said to be* stochastically increasing *in X_1, denoted by $SI(X_2|X_1)$, if*

$$P(X_2 > x_2 | X_1 = x_1) \leq P(X_2 > x_2 | X_1 = y_1) \tag{3}$$

for all x_2 and $x_1 \le y_1$.

(iii) *X_1 and X_2 are* positively quadrant dependent *(PQD) if*

$$P(X_1 \le x_1, X_2 \le x_2) \ge P(X_1 \le x_1)P(X_2 \le x_2) \tag{4}$$

for all x_1 and x_2.

For examples with TP_2 densities, we refer to Lehmann [17] and Karlin and Rinott [13]. Examples with stochastic monotonicity can be found in Stoyan [28] and Ross [23]. For examples with PQD densities, we refer to Lehmann [17].

As usual, for two random variables X_1 and X_2, $X_1 \le_{st} X_2$ if $P(X_1 \ge x) \le P(X_2 \ge x)$ for all x and $X_1 \le_{lr} X_2$ if $P(X_1 = x)P(X_2 = y) \ge P(X_1 = y)P(X_2 = x)$ for all $x \le y$. For the properties of these two orderings, we refer to [28, 23]. Note that (2) can be rewritten as

$$\frac{P(X_2 = y_2|X_1 = y_1)}{P(X_2 = x_2|X_1 = y_1)} \ge \frac{P(X_2 = y_2|X_1 = x_1)}{P(X_2 = x_2|X_1 = x_1)}.$$

This shows that X_2 is increasing in X_1 in the sense of likelihood ratio ordering. Thus, the TP_2 property is also known as the positively likelihood ratio dependent (PLRD) property. We note that the $SI(X_2|X_1)$ property is also known as the *positively regression dependent* (PRD) property in [17].

Another commonly used notion of positive dependence is *association* by Esary, Proshan and Walkup [9].

Definition 2.2 *X_1 and X_2 are said to be* associated *if for all h_1 and h_2 increasing,*

$$Eh_1(X_1, X_2)h_2(X_1, X_2) \ge Eh_1(X_1, X_2)Eh_2(X_1, X_2).$$

Unlike the other notions of positive dependence, association seems to be the only one that is defined by functional inequalities instead of inequalities of distributions (though an equivalent definition by inequalities of distributions is provided in [9], Theorem 3.1). Using the Lorentz inequality [18], Tchen [29] showed that $\boldsymbol{X}$ is PQD if for all ϕ supermodular

$$E\phi(X_1, X_2) \ge E\phi(Y_1, Y_2),$$

where $Y_i =_{st} X_i$, $i = 1$ and 2, and Y_1 is independent of Y_2. One of our objectives in this paper is to provide a unified characterization by functional inequalities for these three notions defined in Definition 2.1.

It is well known that

$$\mathrm{TP}_2 \Rightarrow \mathrm{SI}(X_2|X_1) \text{ or } \mathrm{SI}(X_1|X_2) \Rightarrow \text{ Association } \Rightarrow \mathrm{PQD}.$$

Thus, one expects that the classes of functions for characterizing the TP_2 property should be a superset of that for $\mathrm{SI}(X_2|X_1)$ which in turn is a superset of that for association. These classes of functions are introduced in the next section.

3 Functions that characterizes positive dependence of bivariate random vectors

In the following, we briefly review the definitions of the classes of functions that are used in this paper.

Definition 3.1 *A function $g : \mathcal{R}^2 \to \mathcal{R}$ is said to be (i) increasing if $g(x_1, x_2) \leq g(y_1, y_2)$ for all $(x_1, x_2) \leq (y_1, y_2)$, (ii) supermodular (resp. submodular) if $g(x_1, x_2) + g(y_1, y_2) \geq$ (resp. $\leq$) $g(x_1, y_2) + g(y_1, x_2)$ for all $(x_1, x_2) \leq (y_1, y_2)$, (iii) convex in the first variable if $g(\alpha x_1' + (1-\alpha)x_1'', x_2) \leq \alpha g(x_1', x_2) + (1-\alpha)g(x_1'', x_2)$ for all $0 \leq \alpha \leq 1$ (convex in the second variable and convex in each variable are defined accordingly) and (iv) arrangement-increasing (resp. arrangement-decreasing) if $g(x_1, x_2) \geq g(x_2, x_1)$ for $x_1 \geq x_2$. A function is symmetric if it is both arrangement-increasing and arrangement-decreasing.*

Now we define some new classes of arrangement-increasing functions that are used for characterizing the notions of positive dependence, TP_2, $\mathrm{SI}(X_2|X_1)$ and $\mathrm{SI}(X_1|X_2)$, and PQD.

Definition 3.2 *Let $\boldsymbol{x} = (x_1, x_2)$ and $\boldsymbol{y} = (y_1, y_2)$ be two 2-vectors in $\mathcal{R}^2$. A function $\psi : \mathcal{R}^2 \times \mathcal{R}^2 \to \mathcal{R}$ is said to be* symmetric *if $\psi(\boldsymbol{x}, \boldsymbol{y}) = \psi(\boldsymbol{y}, \boldsymbol{x})$. We say a* symmetric *function is*

(i) *in the class of functions AI_1 if for all $\boldsymbol{x} \leq \boldsymbol{y}$,*

$$\psi\Big((y_1, y_2), (x_1, x_2)\Big) \geq \psi\Big((y_1, x_2), (x_1, y_2)\Big), \tag{5}$$

(ii) *in the class of functions AI_5 if for all $\boldsymbol{x}^i \leq \boldsymbol{y}^i$, $i = 1$ and 2,*

$$\psi\Big((y_1^1, y_2^1), (x_1^2, x_2^2)\Big) + \psi\Big((x_1^1, x_2^1), (y_1^2, y_2^2)\Big) \geq \psi\Big((y_1^1, x_2^1), (x_1^2, y_2^2)\Big) + \psi\Big((x_1^1, y_2^1), (y_1^2, x_2^2)\Big), \tag{6}$$

(iii) *in the class of functions $AI_{5,1}$ (resp. $AI_{5,2}$) if for all $\boldsymbol{x}^i \le \boldsymbol{y}^i$, $i = 1$ and 2, with their first (resp. second) variables being the same, i.e., $y_1^1 = y_1^2$, $x_1^1 = x_1^2$ (resp. $y_2^1 = y_2^2$, $x_2^1 = x_2^2$), (6) is satisfied,*

As Kendall's τ introduced in §1, comparison of concordant vectors and discordant vectors could be used for measuring positive dependence. For a function in AI_1, the function value for two *concordant* vectors is not less than that of the corresponding *discordant* vectors. The class of function AI_5 can be viewed as a natural extension of AI_1 from the notion of concordance for two vectors to that for two pairs of two vectors. In $\mathrm{AI}_{5,1}$ (resp. $\mathrm{AI}_{5,2}$), we put the constraints back by restricting the first (resp. second) variables to be the same. Clearly, we have the inclusion lists: $\mathrm{AI}_5 \subset \mathrm{AI}_{5,1} \subset \mathrm{AI}_1$ and $\mathrm{AI}_5 \subset \mathrm{AI}_{5,2} \subset \mathrm{AI}_1$.

To link up with the arrangement-increasing condition in Marshall and Olkin [19], we use the transformation

$$\psi((x_1, x_2), (y_1, y_2)) = g(x_1, y_1; x_2, y_2).$$

One can verify that our definition of AI_1 functions is equivalent to the arrangement-increasing condition in Marshall and Olkin [19], Section 6.F,

$$g(x_1, y_1; x_2, y_2) = g(y_1, x_1; y_2, x_2) \ge g(x_1, y_1; y_2, x_2) = g(y_1, x_1; x_2, y_2)$$

for $x_1 \le y_1$ and $x_2 \le y_2$.

Let $\boldsymbol{x} \vee \boldsymbol{y} = (\max(x_1, y_1), \max(x_2, y_2))$ and $\boldsymbol{x} \wedge \boldsymbol{y} = (\min(x_1, y_1), \min(x_2, y_2))$. One can easily verify that (5) is equivalent to

$$\psi(\boldsymbol{x} \vee \boldsymbol{y}, \boldsymbol{x} \wedge \boldsymbol{y}) \ge \psi(\boldsymbol{x}, \boldsymbol{y}), \tag{7}$$

for any two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$. Moreover, a sufficient condition for (6) is

$$\psi(\boldsymbol{x}^1 \vee \boldsymbol{y}^1, \boldsymbol{x}^2 \wedge \boldsymbol{y}^2) + \psi(\boldsymbol{x}^1 \wedge \boldsymbol{y}^1, \boldsymbol{x}^2 \vee \boldsymbol{y}^2) \ge \psi(\boldsymbol{x}^1, \boldsymbol{x}^2) + \psi(\boldsymbol{y}^1, \boldsymbol{y}^2), \tag{8}$$

for any four vectors $\boldsymbol{x}^i, \boldsymbol{y}^i$, $i = 1$ and 2.

The reason that we name these three classes of functions by the indices 1 and 5 is that they are closely related to the stochastic majorization orderings, E_1 and E_5 in [19, 6]. These connections will be further explored in §5.

Now we present examples that are in these classes of functions. Example 3.3 below will be used in the proofs of the main results of functional characterizations for TP_2, $\mathrm{SI}(X_2|X_1)$ and PQD properties in the next section. The verification of this example is direct.

Example 3.3 Let ϕ be a supermodular function. Consider the class of functions

$$\psi\Big((x_1,x_2),(y_1,y_2)\Big) = \phi\Big(g_1(x_1,y_1), g_2(x_2,y_2)\Big).$$

(i) If g_1 and g_2 are arrangement-increasing, then $\psi(\boldsymbol{x},\boldsymbol{y})+\psi(\boldsymbol{y},\boldsymbol{x}) \in \mathrm{AI}_1$.

(ii) If g_1 is arrangement-increasing and $g_2(x_2,y_2)=x_2$, then $\psi(\boldsymbol{x},\boldsymbol{y})+\psi(\boldsymbol{y},\boldsymbol{x}) \in \mathrm{AI}_{5,1}$.

(iii) If g_2 is arrangement-increasing and $g_1(x_1,y_1)=x_1$, then $\psi(\boldsymbol{x},\boldsymbol{y})+\psi(\boldsymbol{y},\boldsymbol{x}) \in \mathrm{AI}_{5,2}$.

(iv) If $g_1(x_1,y_1)=x_1$ and $g_2(x_2,y_2)=x_2$, then $\psi(\boldsymbol{x},\boldsymbol{y})+\psi(\boldsymbol{y},\boldsymbol{x}) \in \mathrm{AI}_5$.

As an extension of Example 3.3, we note that the condition $g(x,y)=x$ in (ii)(iii) and (iv) can be replaced by the weaker condition that $g(x,y)$ is increasing in x and decreasing in y.

In the following, we introduce three new classes of functions that can also be used to characterize the TP_2, $\mathrm{SI}(X_2|X_1)$ and $\mathrm{SI}(X_1|X_2)$ properties. In Lemma 3.5 below, we will show the connections between the three classes of functions defined below and their corresponding counterparts of arrangement-increasing functions.

Definition 3.4 *A* symmetric *function* $\psi : \mathcal{R}^2 \times \mathcal{R}^2 \to \mathcal{R}$ *is said to be*

(i) *in the class of functions* PD_1 *(with PD standing for positive dependence) if*

$$\psi(\boldsymbol{x}\vee\boldsymbol{y},\boldsymbol{x}\wedge\boldsymbol{y})+\psi(\boldsymbol{x},\boldsymbol{y}) \geq 0. \tag{9}$$

(ii) *in the class of functions* $PD_{5,1}$ *(resp.* $PD_{5,2}$*) if for all* $\boldsymbol{x}^i \leq \boldsymbol{y}^i$, $i=1$ *and 2, with their first (resp. second) variables being the same, i.e.,* $y_1^1=y_1^2$, $x_1^1=x_1^2$ *(resp.* $y_2^1=y_2^2$, $x_2^1=x_2^2$*) such that*

$$\psi\Big(\boldsymbol{x}^1,\boldsymbol{y}^2\Big)+\psi\Big(\boldsymbol{y}^1,\boldsymbol{x}^2\Big) \geq 0. \tag{10}$$

For a symmetric function ψ, equation (9) is equivalent to the following two conditions:

$$\psi((y_1,y_2),(x_1,x_2))+\psi((y_1,x_2),(x_1,y_2)) \geq 0 \tag{11}$$
$$\psi((y_1,y_2),(x_1,x_2)) \geq 0 \tag{12}$$

for all $\boldsymbol{x}\leq\boldsymbol{y}$. A sufficient condition for $\psi \in \mathrm{PD}_1$ is $\psi(\boldsymbol{x},\boldsymbol{y})=\psi(\boldsymbol{y},\boldsymbol{x}) \geq 0$. Using (11) and (12), one can easily verify that $\mathrm{PD}_{5,1}\cup\mathrm{PD}_{5,2}\subset\mathrm{PD}_1$.

In the following lemma, we state the relations between the two classes of functions AI_1 and PD_1 and their corresponding counterparts. The proof is direct.

Lemma 3.5 *Let*

$$\psi_2((x_1,x_2),(y_1,y_2)) = \psi\Big((x_1,x_2),(y_1,y_2)\Big) - \psi\Big((x_1,y_2),(y_1,x_2)\Big).$$

Then $\psi_2 \in PD_1$ (resp. $PD_{5,1}, PD_{5,2}$) if $\psi(\boldsymbol{x},\boldsymbol{y})$ is AI_1 (resp. $AI_{5,1}$, $AI_{5,2}$).

One can also define a class of function PD_5 based on (10). However, there does not exist a similar relation to that in Lemma 3.5 between the two classes of functions AI_5 and PD_5.

One can easily verify the following closure properties for these classes of functions (since concordance and discordance are preserved under the transformation).

Lemma 3.6 *Suppose*

$$\psi_2\Big((x_1,x_2),(y_1,y_2)\Big) = \psi\Big((h_1(x_1),h_2(x_2)),(h_3(y_1),h_4(y_2)\Big) + \psi\Big((h_1(y_1),h_2(y_2)),(h_3(x_1),h_4(x_2)\Big) \tag{13}$$

for some $\psi : \mathcal{R}^2 \to \mathcal{R}$ and h_i, $i = 1,2,3$ and 4, monotone in the same direction. Then

(i) *$\psi_2 \in AI_1$ (resp. PD_1) if $h_1 = h_3$, $h_2 = h_4$ and $\psi \in AI_1$ (resp. PD_1),*

(ii) *$\psi_2 \in AI_{5,1}$ (resp. $PD_{5,1}$) if $h_1 = h_3$ and $\psi \in AI_{5,1}$ (resp. $PD_{5,1}$),*

(iii) *$\psi_2 \in AI_{5,2}$ (resp. $PD_{5,2}$) if $h_2 = h_4$ and $\psi \in AI_{5,2}$ (resp. $PD_{5,2}$),*

(iv) *$\psi_2 \in AI_5$ if $\psi \in AI_5$.*

We note the closure property for AI_1 in Lemma 3.6 is reported in Marshall and Olkin, Proposition 6.F.6. The reason that we have two terms in (13) is to preserve symmetry. We will show that the symmetric requirement can be relaxed in Lemma 4.4.

Example 3.7 below is the key in the proof for the FKG [10] inequality in the next section. The verification of this example is straightforward.

Example 3.7 Consider the class of functions

$$\psi\Big((x_1,x_2),(y_1,y_2)\Big) = \Big(h_1(x_1,x_2) - h_1(y_1,y_2)\Big)\Big(h_2(x_1,x_2) - h_2(y_1,y_2)\Big),$$

where h_1 and h_2 are increasing. Then $\psi \in \mathrm{PD}_{5,1} \cup \mathrm{PD}_{5,2}$ which in turns implies $\psi \in \mathrm{PD}_1$.

4 Functional characterizations of positively dependent bivariate random vectors

In the following theorems, we provide functional characterizations for bivariate random vectors with the TP_2, $\mathrm{SI}(X_2|X_1)$, $\mathrm{SI}(X_1|X_2)$ and PQD properties.

Theorem 4.1 *Let $\boldsymbol{X}'$ and $\boldsymbol{X}''$ be two independent copies of a bivariate random vector $\boldsymbol{X}$. Then the following statements are equivalent.*

(i) *The density of $\boldsymbol{X}$ is TP_2.*

(ii) *For all $\psi \in PD_1$,*

$$E\psi\Big((X_1', X_2'), (X_1'', X_2'')\Big) \geq 0. \tag{14}$$

(iii) *For all $\psi \in AI_1$,*

$$E\psi\Big((X_1', X_2'), (X_1'', X_2'')\Big) \geq E\psi\Big((X_1', X_2''), (X_1'', X_2')\Big). \tag{15}$$

(iv) *For all ϕ supermodular and g_1 and g_2 arrangement-increasing,*

$$E\phi\Big(g_1(X_1', X_1''), g_2(X_2', X_2'')\Big) \geq E\phi\Big(g_1(X_1', X_1''), g_2(X_2'', X_2')\Big). \tag{16}$$

If one views PD_1 as a class of positively dependent measures, then (ii) of this theorem means that these measures are nonnegative for TP_2 densities. This might lead to a new definition of positively dependent ordering. Intuitively, the two independent copies of $\boldsymbol{X}$ in (iii) and (iv) are more likely to be concordant than the two vectors with X_2' and X_2'' interchanged in (15) and (16). We will show that similar properties hold for $\mathrm{SI}(X_2|X_1)$ and PQD properties in Theorems 4.2 and 4.3 below.

Theorem 4.2 *Let $\boldsymbol{X}'$ and $\boldsymbol{X}''$ be two independent copies of a bivariate random vector $\boldsymbol{X}$. Then the following statements are equivalent.*

(i) *The random vector is $SI(X_2|X_1)$ (resp. $SI(X_1|X_2)$).*

(ii) *For all $\psi \in PD_{5,1}$ (resp. $PD_{5,2}$),*

$$E\psi\Big((X_1', X_2'), (X_1'', X_2'')\Big) \geq 0.$$

(iii) *For all* $\psi \in AI_{5,1}$ *(resp.* $AI_{5,2}$*),*

$$E\psi\Big((X_1', X_2'), (X_1'', X_2'')\Big) \geq E\psi\Big((X_1', X_2''), (X_1'', X_2')\Big).$$

(iv) *For all* ϕ *supermodular and* g_1 *(resp.* g_2*) arrangement-increasing,*

$$E\phi\Big(g_1(X_1', X_1''), X_2'\Big) \geq E\phi\Big(g_1(X_1', X_1''), X_2''\Big)$$
$$(\textit{resp. } E\phi\Big(X_1', g_2(X_2', X_2'')\Big) \geq E\phi\Big(X_1', g_2(X_2'', X_2')\Big)).$$

Theorem 4.3 *Let* $\boldsymbol{X}'$ *and* $\boldsymbol{X}''$ *be two independent copies of a bivariate random vector* $\boldsymbol{X}$*. Then the following statements are equivalent.*

(i) *The random vector is PQD.*

(ii) *For all* $\psi \in AI_5$*,*

$$E\psi\Big((X_1', X_2'), (X_1'', X_2'')\Big) \geq E\psi\Big((X_1', X_2''), (X_1'', X_2')\Big).$$

(iii) *(Tchen [29]) For all* ϕ *supermodular,*

$$E\phi(X_1', X_2') \geq E\phi(X_1', X_2'').$$

These three theorems are summarized in Table 1.

Table 1. Summaries of Theorem 4.1-4.3

	Eq. (14)	Eq. (15)	Eq. (16)
TP_2	$\psi \in \mathrm{PD}_1$	$\psi \in \mathrm{AI}_1$	$g_1, g_2 \in \mathrm{AI}$
$\mathrm{SI}(X_2 \mid X_1)$	$\psi \in \mathrm{PD}_{5,1}$	$\psi \in \mathrm{AI}_{5,1}$	$g_1 \in \mathrm{AI}, g_2(x_2, y_2) = x_2$
PQD		$\psi \in \mathrm{AI}_5$	$g_1(x_1, y_1) = x_1, g_2(x_2, y_2) = x_2$

In the following, we use the property of independent copies to extend the results in these three theorems to nonsymmetric functions, as discussed in Lemma 3.6 and Example 3.3.

Lemma 4.4 *Suppose* $\psi_2(\boldsymbol{x}, \boldsymbol{y}) = \psi_1(\boldsymbol{x}, \boldsymbol{y}) + \psi_1(\boldsymbol{y}, \boldsymbol{x})$ *for some function* ψ_1*. Then for two independent copies of* $\boldsymbol{X}$*,* $\boldsymbol{X}'$ *and* $\boldsymbol{X}''$*,*

$$E\psi_1\Big((X_1', X_2'), (X_1'', X_2'')\Big) \geq 0 \quad \textit{if and only if} \quad E\psi_2\Big((X_1', X_2'), (X_1'', X_2'')\Big) \geq 0.$$

Proof. The *only if* part is trivial. Since $\boldsymbol{X}'$ and $\boldsymbol{X}''$ are independent copies of $\boldsymbol{X}$,

$$E\psi_1\Big((X_1', X_2'), (X_1'', X_2'')\Big) = E\psi_1\Big((X_1'', X_2''), (X_1', X_2')\Big).$$

This proves the *if* part. $\square$

As an application of Theorem 4.2 (ii) and Example 3.7, we provide another proof for $\mathrm{SI}(X_2|X_1)$ or $\mathrm{SI}(X_1|X_2) \Rightarrow$ *Association.* Since $\mathrm{TP}_2 \Rightarrow \mathrm{SI}(X_2|X_1)$, this also proves the FKG inequality (Fortuin, Kasteleyn and Ginibre [10]) for bivariate random vectors.

Theorem 4.5 *If a bivariate random vector $\boldsymbol{X}$ is either $SI(X_2|X_1)$ or $SI(X_1|X_2)$, then for all h_1 and h_2 increasing,*

$$Eh_1(\boldsymbol{X})h_2(\boldsymbol{X}) \geq Eh_1(\boldsymbol{X})Eh_2(\boldsymbol{X}).$$

Proof. Since the function

$$\psi\Big((x_1, x_2), (y_1, y_2)\Big) = \Big(h_1(x_1, x_2) - h_1(y_1, y_2)\Big)\Big(h_2(x_1, x_2) - h_2(y_1, y_2)\Big)$$

is shown to be in $\mathrm{PD}_{5,1} \cup \mathrm{PD}_{5,2}$ in Example 3.7, applying Theorem 4.2 (ii) yields

$$E\Big(h_1(X_1', X_2') - h_1(X_1'', X_2'')\Big)\Big(h_2(X_1', X_2') - h_2(X_1'', X_2'')\Big) \geq 0 \tag{17}$$

for two independent copies of $\boldsymbol{X}$, $\boldsymbol{X}'$ and $\boldsymbol{X}''$. The result of this theorem then follows from (17). $\square$

As another application of our functional characterizations, we provide another proof to identify the following TP_2 density in Lehmann [17] and Tong [30], Theorem 5.1.3.

Theorem 4.6 *Consider a family of bivariate random vectors parameterized by θ, $\{\boldsymbol{X}(\theta) = (X_1(\theta), X_2(\theta))\}$. If (i) $X_1(\theta)$ is independent of $X_2(\theta)$ for all θ and (ii) $X_i(\theta_1) \leq_{lr} X_i(\theta_2)$ for all $\theta_1 \leq \theta_2$, $i = 1$ and 2, then the random vector $\boldsymbol{X}(\Theta)$, where Θ is a random variable, has a TP_2 density.*

Proof. According to Theorem 4.1 (iv), it suffices to show that for two independent copies of $\boldsymbol{X}(\Theta)$, $\boldsymbol{X}(\Theta')$ and $\boldsymbol{X}(\Theta'')$,

$$E\phi\Big(g_1(X_1(\Theta'), X_1(\Theta'')), g_2(X_2(\Theta'), X_2(\Theta''))\Big) \geq E\phi\Big(g_1(X_1(\Theta'), X_1(\Theta'')), g_2(X_2(\Theta''), X_2(\Theta'))\Big) \tag{18}$$

for g_1 and g_2 arrangement-increasing and ϕ supermodular. Consider the event $\{\Theta' = \theta'$ and $\Theta'' = \theta''\}$. First, we assume that $\theta' \leq \theta''$. On this event, $X_1(\theta') \leq_{lr} X_1(\theta'')$ and $X_2(\theta') \leq_{lr} X_2(\theta'')$. Thus, we have from the rearrangement inequalities in (Brown and Solomon [5], Ross [23], Proposition 8.4.2, Shanthikumar and Yao [27]) that $g_1(X_1(\theta'), X_1(\theta'')) \leq_{st} g_1(X_1(\theta''), X_1(\theta'))$ and that $g_2(X_2(\theta'), X_2(\theta'')) \leq_{st} g_2(X_2(\theta''), X_2(\theta'))$. Using the stochastic version of the Hardy-Littlewood-Polya (HLP) inequality in [7], Theorem 3.2 (see Theorem 6.2(ii) in this paper),

$$\begin{aligned} &E\phi\Big(g_1(X_1(\theta'), X_1(\theta'')), g_2(X_2(\theta'), X_2(\theta''))\Big) + E\phi\Big(g_1(X_1(\theta''), X_1(\theta')), g_2(X_2(\theta''), X_2(\theta'))\Big) \\ &\geq E\phi\Big(g_1(X_1(\theta'), X_1(\theta'')), g_2(X_2(\theta''), X_2(\theta'))\Big) + E\phi\Big(g_1(X_1(\theta''), X_1(\theta')), g_2(X_2(\theta'), X_2(\theta''))\Big). \quad (19) \end{aligned}$$

Using the same argument, one can verify that (19) also holds when $\theta' \geq \theta''$. Thus, (19) holds when Θ' and Θ'' are independent. Since the first two terms in (19) have the same expected value and the last two terms in (19) also have the same expected value, the inequality in (18) follows. □

Analogous to Theorem 4.6, we also provide another example for PQD by replacing the likelihood ratio ordering with the stochastic ordering in the assumption (ii).

Theorem 4.7 *Consider a family of bivariate random vectors parameterized by θ, $\{\boldsymbol{X}(\theta) = (X_1(\theta), X_2(\theta))\}$. If (i) $X_1(\theta)$ is independent of $X_2(\theta)$ for all θ and (ii) $X_i(\theta_1) \leq_{st} X_i(\theta_2)$ for all $\theta_1 \leq \theta_2$, $i = 1$ and 2, then the random vector $\boldsymbol{X}(\Theta)$, where Θ is a random variable, is PQD.*

As a corollary of this theorem, the joint density of $(X_1 + \Theta, X_2 + \Theta)$ is PQD if X_1, X_2 and Θ are independent.

Proof. Let $(X_1(\Theta'), X_2(\Theta'))$ and $(X_1(\Theta''), X_2(\Theta''))$ be two independent copies of $\boldsymbol{X}(\Theta)$. From Theorem 4.3 (iii), it suffices to show that $E\phi(X_1(\Theta'), X_2(\Theta')) \geq E\phi(X_1(\Theta'), X_2(\Theta''))$ for all ϕ supermodular. Analogous to the proof of Theorem 4.6, we consider the event $\{\Theta' = \theta'$ and $\Theta'' = \theta''\}$. Assume that $\theta' \leq \theta''$. On the event, $X_1(\theta') \leq_{st} X_1(\theta'')$ and $X_2(\theta') \leq_{st} X_2(\theta'')$. Moreover, $X_1(\theta')$ is independent of $X_2(\theta')$ and $X_1(\theta'')$ is independent of $X_2(\theta'')$. The rest of proof follows from the HLP inequality as in the proof of Theorem 4.6. □

We note that $X_1(\Theta)$ and $X_2(\Theta)$ in Theorem 4.7 are also associated random variables. The argument is similar to that in [2], pp. 144 for proving SI$(X_2|X_1) \Rightarrow$ association. Let U_1 and U_2 be two independent random variables that are uniformly distributed over [0,1] and independent of Θ. Then the assumptions (i) and (ii) in Theorem 4.7 implies that

$$(X_1(\Theta), X_2(\Theta)) =_{st} (h_1(U_1, \Theta), h_2(U_2, \Theta))$$

for some increasing functions h_i, $i = 1$ and 2. Since independent random variables are associated and increasing functions of associated random variables are associated, the result follows.

The rest of this section is devoted to the proofs of Theorems 4.1, 4.2 and 4.3.

Proof.(Theorem 4.1) We first show that (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i). From Lemma 3.5, it follows that (ii) $\Rightarrow$ (iii). That (iii) $\Rightarrow$ (iv) follows from Example 3.3 (i) and Lemma 4.4. To show that (iv) $\Rightarrow$ (i), we consider the arrangement-increasing functions $g_1(u, v) = \mathbf{1}\{u = y_1, v = x_1\}$, $x_1 \le y_1$ and $g_2(u, v) = \mathbf{1}\{u = y_2, v = x_2\}$, $x_2 \le y_2$ and the supermodular function $\phi(u, v) = uv$. Since $\boldsymbol{X}'$ and $\boldsymbol{X}''$ are two independent copies of $\boldsymbol{X}$, we have from (iv) that

$$P(X_1' = y_1, X_2' = y_2)P(X_1'' = x_1, X_2'' = x_2) \ge P(X_1' = y_1, X_2' = x_2)P(X_1'' = x_1, X_2'' = y_2).$$

We will use a constructive proof to show that (i) $\Rightarrow$ (ii). Our construction is an extension of that in Brown and Solomon [5], but different from those in Holley [12] and Preston [22]. In view of the results, our construction is also more general than those in [12, 22] for bivariate random vectors. Beside the pair of independent copies, $\boldsymbol{X}'$ and $\boldsymbol{X}''$, we consider another pair of independent copies of $\boldsymbol{X}$, $\tilde{X}'$ and $\tilde{X}''$. Our objective is to construct these two pairs of independent copies on the same probability space such that

$$\psi(\boldsymbol{Z}^1) + \psi(\boldsymbol{Z}^2) \ge 0 \tag{20}$$

for all $\psi \in \mathrm{PD}_1$, where

$$\boldsymbol{Z}^1 = \Big((X_1', X_2'), (X_1'', X_2'')\Big) \quad \text{and} \quad \boldsymbol{Z}^2 = \Big((\tilde{X}_1'', \tilde{X}_2''), (\tilde{X}_1', \tilde{X}_2')\Big). \tag{21}$$

Since $\boldsymbol{X}'$, $\boldsymbol{X}''$, $\tilde{X}'$ and $\tilde{X}''$ are independent copies of $\boldsymbol{X}$, $\psi(\boldsymbol{Z}^1) =_{st} \psi(\boldsymbol{Z}^2)$. Thus, by taking expectations in (20), one can easily verify that (i) $\Rightarrow$ (ii).

Consider the event $E = \{X_1' = y_1, X_1'' = x_1, \max(X_2', X_2'') = y_2, \min(X_2', X_2'') = x_2\}$ and the event $\tilde{E} = \{\tilde{X}_1' = y_1, \tilde{X}_1'' = x_1, \max(\tilde{X}_2', \tilde{X}_2'') = y_2, \min(\tilde{X}_2', \tilde{X}_2'') = x_2\}$. Since $\boldsymbol{X}'$ and $\boldsymbol{X}''$ (resp. $\tilde{X}'$ and $\tilde{X}''$) are independent copies of $\boldsymbol{X}$, without loss of generality we may assume that $(x_1, x_2) \le (y_1, y_2)$. Analogous to Brown and Solomon [5], let p be the conditional probability on the event E that $X_2' = \min(X_2', X_2'')$, i.e.,

$$p = P(X_2' = x_2, X_2'' = y_2 | E) = \frac{f(x_1, y_2)f(y_1, x_2)}{f(x_1, x_2)f(y_1, y_2) + f(x_1, y_2)f(y_1, x_2)}.$$

Since $f(x_1, x_2)f(y_1, y_2) \ge f(x_1, y_2)f(y_1, x_2)$, we have $p \le 1/2$. Conditioning on the events E and $\tilde{E}$, we generate $X_2', X_2'', \tilde{X}_2'$ and $\tilde{X}_2''$ by the following rule:

Case 1: $X_2' = x_2, X_2'' = y_2, \tilde{X}_2' = y_2, \tilde{X}_2'' = x_2$ with probability p,

Case 2: $X_2' = y_2, X_2'' = x_2, \tilde{X}_2' = x_2, \tilde{X}_2'' = y_2$ with probability p,

Case 3: $X_2' = y_2, X_2'' = x_2, \tilde{X}_2' = y_2, \tilde{X}_2'' = x_2$ with probability $1 - 2p$.

Since $p \leq 1/2$, the coupling is feasible. The constructions are shown in Table 2.

Table 2. Constructions of $\boldsymbol{Z}^i$, $i = 1, 2$

	Case 1	Case 2	Case 3
$\boldsymbol{Z}^1$	$((y_1, x_2), (x_1, y_2))$	$((y_1, y_2), (x_1, x_2))$	$((y_1, y_2), (x_1, x_2))$
$\boldsymbol{Z}^2$	$((x_1, x_2), (y_1, y_2))$	$((x_1, y_2), (y_1, x_2))$	$((x_1, x_2), (y_1, y_2))$

Thus, for any function $\psi \in \mathrm{PD}_1$, we have for all these three cases

$$\psi(\boldsymbol{Z}^1) + \psi(\boldsymbol{Z}^2) \geq 0$$

taking account of $(x_1, x_2) \leq (y_1, y_2)$. □

Proof.(Theorem 4.2) We first show that (i) $\Rightarrow$ (ii) by coupling. Analogous to the proof of Theorem 4.1, we consider another pair of independent copies of $\boldsymbol{X}$, $\tilde{X}'$ and $\tilde{X}''$ beside the pair of independent copies, $\boldsymbol{X}'$ and $\boldsymbol{X}''$. Our objective is to construct these two pairs of independent copies on the same probability space such that

$$\psi(\boldsymbol{Z}^1) + \psi(\boldsymbol{Z}^2) \geq 0 \tag{22}$$

for all $\psi \in \mathrm{PD}_{5,1}$, where

$$\boldsymbol{Z}^1 = \Big((X_1', X_2'), (X_1'', X_2'')\Big) \quad \text{and} \quad \boldsymbol{Z}^2 = \Big((\tilde{X}_1', \tilde{X}_2'), (\tilde{X}_1'', \tilde{X}_2'')\Big). \tag{23}$$

Since $\boldsymbol{X}'$, $\boldsymbol{X}''$, $\tilde{X}'$ and $\tilde{X}''$ are independent copies of $\boldsymbol{X}$, $\psi(\boldsymbol{Z}^1) =_{st} \psi(\boldsymbol{Z}^2)$. Again, by taking expectations in (22), one can easily verify that (i) $\Rightarrow$ (ii).

Couple these two pairs of independent copies on the event $E = \{X_1' = y_1, X_1'' = x_1\}$ and the event $\tilde{E} = \{\tilde{X}_1' = x_1, \tilde{X}_1'' = y_1\}$. Since $\boldsymbol{X}'$ and $\boldsymbol{X}''$ (resp. $\tilde{X}'$ and $\tilde{X}''$) are independent copies of $\boldsymbol{X}$, without loss of generality we may assume that $x_1 \leq y_1$. On these two events E and $\tilde{E}$, the $\mathrm{SI}(X_2|X_1)$ property implies that $X_2' =_{st} \tilde{X}_2'' \geq_{st} X_2'' =_{st} \tilde{X}_2'$. Thus, we can construct these four random variables on the same probability space such that $X_2' \geq \tilde{X}_2'$ and $\tilde{X}_2'' \geq X_2''$ (see [23], Proposition 8.2.2). Since $\psi \in \mathrm{PD}_{5,1}$, (22) is derived.

That (ii) $\Rightarrow$ (iii) follows from Lemma 3.5. From Example 3.3 (ii) and Lemma 4.4, it follows that (iii) $\Rightarrow$ (iv). To show (iv) $\Rightarrow$ (i), we consider the following function:

$$\psi\Big((u_1, u_2), (v_1, v_2)\Big) = \mathbf{1}\{u_1 = y_1, v_1 = x_1, u_2 \geq x_2\} \tag{24}$$

Clearly, the class of functions ψ is in the class of functions described in Example 3.3 (ii) if $x_1 \leq y_1$. Thus, it follows from (iv) and (24) that for $x_1 \leq y_1$

$$P(X_1 = y_1, X_2 \geq x_2)P(X_1 = x_1) \geq P(X_1 = x_1, X_2 \geq x_2)P(X_1 = y_1),$$

or equivalently,

$$P(X_2 \geq x_2 | X_1 = y_1) \geq P(X_2 \geq x_2 | X_1 = x_1).$$

□

Proof.(Theorem 4.3) That (i) and (iii) are equivalent is shown in [29]. Now we show that (ii) and (iii) are equivalent. From Lemma 4.4 and Example 3.3 (iv), it follows that (ii) $\Rightarrow$ (iii).

To show (iii) $\Rightarrow$ (ii), we consider the function

$$g(x_1, x_2) = \psi\Big((x_1, x_2), (y_1, y_2)\Big) - \psi\Big((x_1, y_2), (y_1, x_2)\Big). \tag{25}$$

One can easily verify that $g(x_1, x_2)$ is supermodular for any y_1, y_2 if $\psi \in \mathrm{AI}_5$. Let $\boldsymbol{X}^i$, $i = 1, 2, 3$ and 4, be four independent copies of $\boldsymbol{X}$. From (iii), it follows that

$$\begin{aligned} &E\psi\Big((X_1^1, X_2^1), (y_1, y_2)\Big) - E\psi\Big((X_1^1, y_2), (y_1, X_2^1)\Big) \\ &\geq E\psi\Big((X_1^1, X_2^2), (y_1, y_2)\Big) - E\psi\Big((X_1^1, y_2), (y_1, X_2^2)\Big) \end{aligned} \tag{26}$$

Replacing y_1 and y_2 in (26) by the two independent random variables X_1^3 and X_2^4 yields

$$\begin{aligned} &E\psi\Big((X_1^1, X_2^1), (X_1^3, X_2^4)\Big) - E\psi\Big((X_1^1, X_2^4), (X_1^3, X_2^1)\Big) \\ &\geq E\psi\Big((X_1^1, X_2^2), (X_1^3, X_2^4)\Big) - E\psi\Big((X_1^1, X_2^4), (X_1^3, X_2^2)\Big) \end{aligned} \tag{27}$$

Since X_2^2 and X_2^4 are i.i.d. and they are independent of X_1^1 and X_1^3, the difference of the last two terms in (27) is 0. Thus,

$$E\psi\Big((X_1^1, X_2^1), (X_1^3, X_2^4)\Big) - E\psi\Big((X_1^1, X_2^4), (X_1^3, X_2^1)\Big) \geq 0 \tag{28}$$

Replacing y_1 and y_2 in (26) by the $X_1^3$and X_2^3, we have

$$\begin{aligned} &E\psi\Big((X_1^1, X_2^1), (X_1^3, X_2^3)\Big) - E\psi\Big((X_1^1, X_2^3), (X_1^3, X_2^1)\Big) \\ &\geq E\psi\Big((X_1^1, X_2^2), (X_1^3, X_2^3)\Big) - E\psi\Big((X_1^1, X_2^3), (X_1^3, X_2^2)\Big). \end{aligned} \tag{29}$$

Since ψ is symmetric and $\boldsymbol{X}^i$, i=1,2,3 and 4 are independent copies of $\boldsymbol{X}$, the third term in (29) is the same as the first term in (28) and the fourth term in (29) is the same as the second term in (28). Thus, we have from (28) and (29) that

$$E\psi\Big((X_1^1, X_2^1), (X_1^3, X_2^3)\Big) - E\psi\Big((X_1^1, X_2^3), (X_1^3, X_2^1)\Big) \geq 0.$$

□

5 Positive dependence and stochastic majorization

In this section, we show the relation between positive dependence and stochastic majorization. We first briefly review the definition of majorization and stochastic majorization. Since our interests are in bivariate random vectors, we will concentrate on the two-dimensional case. For generalization to n-dimension, we refer to [19, 6, 7].

Definition 5.1 *A 2-vector $\boldsymbol{x}$ is majorized by $\boldsymbol{y}$, denoted $\boldsymbol{x} \prec \boldsymbol{y}$, if (i) $\max(x_1, x_2) \leq \max(y_1, y_2)$ and (ii) $x_1 + x_2 = y_1 + y_2$.*

The vector $\boldsymbol{x}$ is weakly submajorized by $\boldsymbol{y}$, denoted $\boldsymbol{x} \prec_w \boldsymbol{y}$, if (i) $\max(x_1, x_2) \leq \max(y_1, y_2)$ and (ii) $x_1 + x_2 \leq y_1 + y_2$.

Definition 5.2 *A function $g : \mathcal{R}^2 \to \mathcal{R}$ is called Schur convex if $g(\boldsymbol{x}) \leq g(\boldsymbol{y})$ for all $\boldsymbol{x} \prec \boldsymbol{y}$.*

The following two classes of functions are considered in [19], Chapter 11 and [6]:

$\mathbf{C}_1 = \{g : g$ is Schur convex $\}$.

$\mathbf{C}_5 = \{g : g$ is symmetric, submodular and convex in each variable $\}$.

In addition, for $i = 1$ and 5, let $\mathbf{C}_i^{\uparrow} = \{g : g \in \mathbf{C}_i$ and g is increasing $\}$.

It is known that $\mathbf{C}_5 \subset \mathbf{C}_1$ and that these two classes of functions are equally effective in characterizing (weak) majorization, i.e., $\boldsymbol{x} \prec (\prec_w)\boldsymbol{y}$ if and only if $f(\boldsymbol{x}) \leq f(\boldsymbol{y})$ for all $f \in \mathbf{C}_i$ ($\mathbf{C}_i^{\uparrow}$), $i = 1$ and 5.

Based on the two classes of functions, one can define different versions of *stochastic* majorization:

Definition 5.3 *For $i = 1$ and 5, define E_i: $\boldsymbol{X} \prec_{E_i} \boldsymbol{Y}$ if $Eg(\boldsymbol{X}) \leq Eg(\boldsymbol{Y})$ for all $g \in \mathbf{C}_i$; and similarly define $E_i^{\uparrow}$: $\boldsymbol{X} \prec_{wE_i} \boldsymbol{Y}$ if $Eg(\boldsymbol{X}) \leq Eg(\boldsymbol{Y})$ for all $g \in \mathbf{C}_i^{\uparrow}$.*

In contrast to the deterministic case, each class $\mathbf{C}_i$ gives rise to a distinct version of stochastic majorization E_i. Since $\mathbf{C}_5 \subset \mathbf{C}_1$, $E_1 \Rightarrow E_5$. This implication is strict. In the following theorem, we show how various positive dependence notions are related to these stochastic majorizations.

Theorem 5.4 *Let $\boldsymbol{X}'$ and $\boldsymbol{X}''$ be two independent copies of a bivariate random vector $\boldsymbol{X}$ and ϕ be a monotone supermodular function.*

(i) *If the density is TP$_2$, then $(\phi(X_1', X_2''), \phi(X_1'', X_2')) \prec_{wE_1} (\phi(X_1', X_2'), \phi(X_1'', X_2''))$.*

(ii) *If the density is PQD, then $(\phi(X_1', X_2''), \phi(X_1'', X_2')) \prec_{wE_5} (\phi(X_1', X_2'), \phi(X_1'', X_2''))$.*

If $\phi(x_1, x_2) = x_1 + x_2$, then (i) and (ii) hold for the stochastic majorization orderings E_1 and E_5 respectively.

The proof of Theorem 5.4 relies on the following lemma.

Lemma 5.5 *Consider the class of functions*

$$\psi\Big((x_1, x_2), (y_1, y_2)\Big) = h\Big(\phi(x_1, x_2), \phi(y_1, y_2)\Big)$$

where ϕ is monotone and supermodular.

(i) *If h is in $\mathbf{C}_1^{\uparrow}$, then $\psi \in AI_1$.*

(ii) *If h is in $\mathbf{C}_5^{\uparrow}$, then $\psi \in AI_5$.*

If $\phi(x_1, x_2) = x_1 + x_2$, then (i) and (ii) hold with $\mathbf{C}_i^{\uparrow}$ replaced by $\mathbf{C}_i$, $i = 1$ and 2.

Proof. Since the proofs for ϕ increasing and supermodular and for ϕ decreasing supermodular are similar, we only show the first case. It is known that functions in $\mathbf{C}_i^{\uparrow}$, $i = 1$ and 5, are symmetric. Thus, ψ is also symmetric. To show (i), we apply Day's inequality ([8] or [19], Proposition 6.C.3.). For all ϕ increasing and supermodular and $\boldsymbol{x} \le \boldsymbol{y}$,

$$\Big(\phi(x_1, y_2), \phi(y_1, x_2)\Big) \prec_w \Big(\phi(x_1, x_2), \phi(y_1, y_2)\Big).$$

Thus, $\psi \in \mathrm{AI}_1$ if $h \in \mathbf{C}_1^{\uparrow}$.

To show (ii), we consider two pairs of vectors, $\boldsymbol{x}^i \leq \boldsymbol{y}^i$, $i = 1$ and 2. From the supermodularity of ϕ, it follows that

$$\phi(y_1^i, y_2^i) + \phi(x_1^i, x_2^i) \geq \phi(y_1^i, x_2^i) + \phi(x_1^i, y_2^i), \quad i = 1 \text{ and } 2. \tag{30}$$

Since ϕ is also increasing, we also have that

$$\phi(y_1^i, y_2^i) \geq \max\Big(\phi(x_1^i, x_2^i), \phi(y_1^i, x_2^i), \phi(x_1^i, y_2^i)\Big), \quad i = 1 \text{ and } 2.$$

It then follows from Lemma 3.3 in [7] that for all $h \in \mathbf{C}_5^{\uparrow}$

$$h\Big(\phi(y_1^1, y_2^1), \phi(x_1^2, x_2^2)\Big) + h\Big(\phi(x_1^1, x_2^1), \phi(y_1^2, y_2^2)\Big) \geq h\Big(\phi(y_1^1, x_2^1), \phi(x_1^2, y_2^2)\Big) + h\Big(\phi(x_1^1, y_2^1), \phi(y_1^2, x_2^2)\Big).$$

Thus, $\psi \in \mathrm{AI}_5$ if $h \in \mathbf{C}_5^{\uparrow}$

The proof for the last statement in this lemma is similar taking into account that the inequalities in (30) are equalities. □

Proof.(Theorem 5.4) The statement (i) in this theorem follows from Lemma 5.5 (i) and Theorem 4.1 (iii). In conjunction with Lemma 5.5(ii) and Theorem 4.3 (ii), we obtain (ii). □

6 Functional characterizations for bivariate random vectors with independent components

As discussed in §3, for these classes of arrangement-increasing functions, the function value for two "concordant" vectors is not less than that of two corresponding "discordant" vectors. To be precise, one can consider two partial orderings, a and b, for random variables and generalize the notion of concordance to random vectors with *independent* components. Two independent random vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ with independent components are said to be concordant with respect to (a, b) if either $X_1 \leq_a Y_1$ and $X_2 \leq_b Y_2$ or $X_1 \geq_a Y_1$ and $X_2 \geq_b Y_2$. In the following theorem, we show the relation between the generalized notion of concordance and the arrangement-increasing functions considered in this section. The proof will be presented at the end of this section.

Theorem 6.1 *Let $\boldsymbol{X} = (X_1, X_2)$ and $\boldsymbol{Y} = (Y_1, Y_2)$ be two* independent *random vectors with independent components. The following two statements are equivalent for the four cases: (i) $\mathbf{C} = AI_1$, $(a, b) = (lr, lr)$, (ii) $\mathbf{C} = AI_{5,1}$, $(a, b) = (lr, st)$, (iii) $\mathbf{C} = AI_{5,2}$, $(a, b) = (st, lr)$, and (iv) $\mathbf{C} = AI_5$, $(a, b) = (st, st)$.*

(S1) *The two random vectors* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *are concordant with respect to* (a, b).

(S2) *For all* $\psi \in \mathbf{C}$,

$$E\psi\Big((Y_1, Y_2), (X_1, X_2)\Big) \geq E\psi\Big((Y_1, X_2), (X_1, Y_2)\Big). \tag{31}$$

These characterizations in Theorem 6.1 can be viewed as generalizations of the rearrangement inequalities below [7] which follow as a direct consequence of Theorems 6.1(i) and 6.1(iv) and Lemma 5.5.

Theorem 6.2 *([7], Theorem 3.2 (i) and (iii)) Let* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *be two independent random vectors with independent components as in Theorem 6.1.*

(i) *If* $X_1 \leq_{lr} Y_1$ *and* $X_2 \leq_{lr} Y_2$, *then for all* ϕ *monotone and supermodular and* $h \in \mathbf{C}_1^{\uparrow}$,

$$Eh\Big(\phi(Y_1, Y_2), \phi(X_1, X_2)\Big) \geq Eh\Big(\phi(Y_1, X_2), \phi(X_1, Y_2)\Big).$$

(ii) *If* $X_1 \leq_{st} Y_1$ *and* $X_2 \leq_{st} Y_2$, *then for all* ϕ *monotone and supermodular and* $h \in \mathbf{C}_5^{\uparrow}$,

$$Eh\Big(\phi(Y_1, Y_2), \phi(X_1, X_2)\Big) \geq Eh\Big(\phi(Y_1, X_2), \phi(X_1, Y_2)\Big).$$

Now we prove Theorem 6.1.

Proof.(Theorem 6.1) (i) (S1) $\Rightarrow$ (S2). We only show the case $X_1 \leq_{lr} Y_1$ and $X_2 \leq_{lr} Y_2$. The other case can be shown similarly. Our argument is similar to the proof of Theorem 3.2 (i) in [7]. Let $h(y_2, x_2) = \psi\Big((y_1, y_2), (x_1, x_2)\Big)$. Since ψ is AI_1, h is arrangement-increasing if $x_1 \leq y_1$. Thus, it follows from the assumption $X_2 \leq_{lr} Y_2$ and the rearrangement inequality in ([5, 23, 27]) that for $x_1 \leq y_1$, $Eh(Y_2, X_2) \geq Eh(X_2, Y_2)$. Thus, for $x_1 \leq y_1$,

$$E\Big[\psi\Big((Y_1, Y_2), (X_1, X_2)\Big)|X_1 = x_1, Y_1 = y_1\Big] \geq E\Big[\psi\Big((Y_1, X_2), (X_1, Y_2)\Big)|X_1 = x_1, Y_1 = y_1\Big].$$

Since $\psi(\boldsymbol{x}, \boldsymbol{y}) = \psi(\boldsymbol{y}, \boldsymbol{x})$,

$$E\Big[\psi\Big((Y_1, Y_2), (X_1, X_2)\Big)|X_1 = x_1, Y_1 = y_1\Big] \geq E\Big[\psi\Big((X_1, Y_2), (Y_1, X_2)\Big)|X_1 = x_1, Y_1 = y_1\Big]. \tag{32}$$

Write the left hand side of (32) as $g(y_1, x_1)$. Then the right hand side is $g(x_1, y_1)$ since these four random variables are independent. Thus, g is also arrangement-increasing. Again, it follows from the assumption $X_1 \leq_{lr} Y_1$ and the rearrangement inequality in [5, 23, 27] that (31) holds.

(i) (S2) $\Rightarrow$ (S1). Let $f_i(\cdot)$, $i = 1, 2, 3, 4$ be the density of X_1, X_2, Y_1, and Y_2, respectively. Consider the following function:

$$\begin{aligned}
&\psi\Big((u_1, u_2), (v_1, v_2)\Big) \\
&= \mathbf{1}\{u_1 = x_1, u_2 = x_2, v_1 = y_1, v_2 = y_2\} + \mathbf{1}\{u_1 = y_1, u_2 = y_2, v_1 = x_1, v_2 = x_2\}. \quad (33)
\end{aligned}$$

Clearly, $\psi \in \mathrm{AI}_1$ for all $(x_1, x_2) \le (y_1, y_2)$ (see Example 3.3(i)). It follows from (31) and (33) that

$$\Big(f_4(x_2)f_2(y_2) - f_4(y_2)f_2(x_2)\Big)\Big(f_3(x_1)f_1(y_1) - f_3(y_1)f_1(x_1)\Big) \ge 0$$

for *all* $(x_1, x_2) \le (y_1, y_2)$. This implies that either $X_1 \le_{lr} Y_1$, $X_2 \le_{lr} Y_2$ or $X_1 \ge_{lr} Y_1$, $X_2 \ge_{lr} Y_2$.

(iv) (S1) $\Rightarrow$ (S2). Again, we only show the case that $X_1 \le_{st} Y_1$ and $X_2 \le_{st} Y_2$. Using the coupling property of stochastically ordered random variables (Ross [23], Proposition 8.2.2), we can construct four independent pairs of random variables (X_1^1, Y_1^1), (X_2^2, Y_2^2), (X_1^3, Y_1^3), and (X_2^4, Y_2^4) on the same probability space such that

$$\begin{aligned}
&X_1^1 =_{st} X_1^3 =_{st} X_1,\ X_2^2 =_{st} X_2^4 =_{st} X_2,\ Y_1^1 =_{st} Y_1^3 =_{st} Y_1,\ Y_2^2 =_{st} Y_2^4 =_{st} Y_2, \\
&X_1^1 \le Y_1^1,\ X_2^2 \le Y_2^2,\ X_1^3 \le Y_1^3,\ X_2^4 \le Y_2^4.
\end{aligned}$$

Form the four random vectors,

$$\begin{aligned}
\boldsymbol{Z}_1 &= \Big((X_1^1, X_2^2), (Y_1^3, Y_2^4)\Big), \quad \boldsymbol{Z}_2 = \Big((Y_1^1, Y_2^2), (X_1^3, X_2^4)\Big) \\
\boldsymbol{Z}_3 &= \Big((X_1^1, Y_2^2), (Y_1^3, X_2^4)\Big), \quad \boldsymbol{Z}_4 = \Big((Y_1^1, X_2^2), (X_1^3, Y_2^4)\Big).
\end{aligned}$$

From these constructions, it follows that

$$\psi(\boldsymbol{Z}_1) + \psi(\boldsymbol{Z}_2) \ge \psi(\boldsymbol{Z}_3) + \psi(\boldsymbol{Z}_4) \tag{34}$$

for all $\psi \in \mathrm{AI}_5$. Since ψ is symmetric, we have that

$$\psi(\boldsymbol{Z}_1) =_{st} \psi(\boldsymbol{Z}_2) =_{st} \psi\Big((X_1, X_2), (Y_1, Y_2)\Big)$$

and that

$$\psi(\boldsymbol{Z}_3) =_{st} \psi(\boldsymbol{Z}_4) =_{st} \psi\Big((X_1, Y_2), (Y_1, X_2)\Big).$$

From (34), it follows that for $\psi \in \mathrm{AI}_5$

$$E\psi\Big((X_1, X_2), (Y_1, Y_2)\Big) \ge E\psi\Big((X_1, Y_2), (Y_1, X_2)\Big). \tag{35}$$

(iv) (S2) $\Rightarrow$ (S1). Let $\bar{F}_i(\cdot)$, $i = 1, 2, 3, 4$ be the survival function of X_1, X_2, Y_1, and Y_2, respectively. We consider the classes of functions

$$\psi\Big((u_1, u_2), (v_1, v_2)\Big) = \mathbf{1}\{u_1 \geq x_1, u_2 \geq x_2\} + \mathbf{1}\{v_1 \geq x_1, v_2 \geq x_2\}. \tag{36}$$

Since $\phi(u_1, u_2) \stackrel{\text{def}}{=} \mathbf{1}\{u_1 \geq x_1, u_2 \geq x_2\}$ is supermodular, we have from Example 3.3 (iv) that $\psi \in \mathrm{AI}_5$ for all (x_1, x_2). It follows from (31) in (ii) and (36) that

$$\Big(\bar{F}_4(x_2) - \bar{F}_2(x_2)\Big)\Big(\bar{F}_3(x_1) - \bar{F}_1(x_1)\Big) \geq 0$$

for *all* (x_1, x_2). This implies that either $X_1 \leq_{st} Y_1$, $X_2 \leq_{st} Y_2$ or $X_1 \geq_{st} Y_1$, $X_2 \geq_{st} Y_2$.

The proofs for (ii) and (iii) are similar. We only show (ii). Our proof for (ii) is based on the arguments for (iv) and (i).

(ii) (S1) $\Rightarrow$ (S2). Again, we only show the case that $X_1 \leq_{lr} Y_1$ and $X_2 \leq_{st} Y_2$. Since $X_2 \leq_{st} Y_2$, we can construct two independent pairs of random variables (X_2^1, Y_2^1) and (X_2^2, Y_2^2) on the same probability space such that $X_2^1 =_{st} X_2^2 =_{st} X_2$, $Y_2^1 =_{st} Y_2^2 =_{st} Y_2$, and $X_2^1 \leq Y_2^1$, $X_2^2 \leq Y_2^2$. Form the four random vectors,

$$\boldsymbol{Z}_1 = \Big((x_1, X_2^1), (y_1, Y_2^2)\Big), \quad \boldsymbol{Z}_2 = \Big((y_1, Y_2^1), (x_1, X_2^2)\Big)$$
$$\boldsymbol{Z}_3 = \Big((x_1, Y_2^1), (y_1, X_2^2)\Big), \quad \boldsymbol{Z}_4 = \Big((y_1, X_2^1), (x_1, Y_2^2)\Big).$$

Analogous to the argument for (iv), one can show that for $\psi \in \mathrm{AI}_{5,1}$ and $x_1 \leq y_1$

$$E\psi\Big((y_1, Y_2), (x_1, X_2)\Big) \geq E\psi\Big((x_1, Y_2), (y_1, X_2)\Big). \tag{37}$$

Since $X_1 \leq_{lr} Y_1$, one can apply the rearrangement inequality as in the proof for (i) to complete the argument.

(ii) (S2) $\Rightarrow$ (S1). Consider the following class of functions.

$$\psi\Big((u_1, u_2), (v_1, v_2)\Big) = \mathbf{1}\{u_1 = y_1, u_2 \geq x_2, v_1 = x_1\} + \mathbf{1}\{v_1 = y_1, v_2 \geq x_2, u_1 = x_1\}. \tag{38}$$

Clearly, $\psi \in \mathrm{AI}_1$ for all $x_1 \leq y_1$ and all x_2 (see Example 3.3(ii)). The rest of proof is direct. $\square$

7 Conclusions and future research

In this paper, we obtained functional characterizations for three notions of positive dependence for a bivariate random vector: (i) total positivity of order 2, (ii) stochastic increasingness and

(iii) positive quadrant dependence. We note that similar results can be extended easily to negatively dependent bivariate random vectors that are (i) reverse rule of order 2 (RR_2 in [14]), (ii) stochastically decreasing ($\mathrm{SD}(X_2|X_1)$) and (iii) negatively quadrant dependent (NQD). This is due to the fact that a bivariate random vector (X_1, X_2) is TP_2, $\mathrm{SI}(X_2|X_1)$, and PQD, if and only if $(X_1, -X_2)$ is RR_2, $\mathrm{SD}(X_2|X_1)$, and NQD, respectively.

Another interesting question is whether our functional characterizations can be extended to multivariate random vectors with dimension larger than 2. So far, we do not have a satisfactory answer for this question.

First, note that the positive dependence notions for multivariate random vectors are more complicated than those for bivariate random vectors. For instance, the notion of positive upper orthant dependence (PUOD), i.e.,

$$P(X_i \geq x_i, i = 1, \ldots, n) \geq \prod_{i=1}^{n} P(X_i \geq x_i),$$

and the notion of positive lower orthant dependence (PLOD), i.e.,

$$P(X_i \leq x_i, i = 1, \ldots, n) \geq \prod_{i=1}^{n} P(X_i \leq x_i),$$

are different for $n \geq 3$, but they are equivalent for $n = 2$, i.e., PQD.

Second, it is not clear how one extends the definitions of arrangement-increasing functions to dimensions larger than 2. For two 2-vectors, (x_1, x_2) and (y_1, y_2), they are either concordant or discordant. This is the property that does not exist for vectors with dimensions larger than 2. A naive extension is to use (7) as the definition of AI_1 and then conjecture that a result similar to that in (15) in Theorem 4.1 holds under the assumption of a MTP_2 joint density (see [13]), i.e.,

$$E\psi\Big((X_1', \ldots, X_n'), (X_1'', \ldots, X_n'')\Big) \geq E\psi\Big((X_1', \ldots, X_{n-1}', X_n''), (X_1'', \ldots, X_{n-1}'', X_n')\Big).$$

As we just discussed, the inequality in (7) only describes the property between a pair of vectors and the corresponding pair of perfectly concordant vectors. No other comparisons between two pairs of partially concordant vectors are given. This leads to the following counterexample for the conjecture of the MTP_2 joint density. Consider a random vector (X_1, X_2, X_3). Assume that (X_1, X_3) and X_2 are independent. With probability 1/2, $(X_1, X_3) = (2, 2) = (3, 3)$. Also with probability 1/2, $X_2 = 0 = 1$. Thus, with probability 1/4, $(X_1, X_2, X_3) = (2, 0, 2) = (2, 1, 2) = (3, 0, 3) = (3, 1, 3)$. It is easy to verify that (X_1, X_2, X_3) is MTP_2. Now consider the function

$$\psi\Big((x_1, x_2, x_3), (y_1, y_2, y_3)\Big) = \mathbf{1}\{x_1 x_2 x_3 + y_1 y_2 y_3 \geq 6\}.$$

Clearly, ψ satisfies (7) for $x_i, y_i \geq 0$, $i = 1, 2, 3$. However, a direct evaluation shows that

$$\frac{1}{2} = P(X_1' X_2' X_3' + X_1'' X_2'' X_3'' \geq 6) < P(X_1' X_2' X_3'' + X_1'' X_2'' X_3' \geq 6) = \frac{5}{8}.$$

Thus, the conjecture for the MTP$_2$ density is false.

From the discussion above, it seems difficult to extend our functional characterizations directly to multivariate random vectors with dimension larger than 2. A possible alternative is to map the problems with dimension larger than 2 to the problems with dimension equal to 2. For instance, a multivariate random vector $\boldsymbol{X}$ is associated if and only if for any increasing functions h_1 and h_2, the bivariate random vector $(h_1(\boldsymbol{X}), h_2(\boldsymbol{X}))$ is PQD. Thus, a positively dependent notion for a multivariate random vector could be defined by a class of functions that define h_1 and h_2, and by a positively dependent notion for a bivariate random vector. This idea was used by Shaked [25], Lemma 4.1 to derive functional characterizations for various notions of positive dependence weaker than association.

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