# On the Exponentiality of Stochastic Linear Systems under the Max-Plus Algebra

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(July 1994; revision: Sep. 1995)

**Abstract**

In this paper, we consider stochastic linear systems under the max-plus algebra. For such a system, the states are governed by the recursive equation $X_n = A_n \otimes X_{n-1} \oplus U_n$ with the initial condition $X_0 = x_0$. By transforming the linear system under the max-plus algebra into a sublinear system under the usual algebra, we establish various exponential upper bounds for the tail distributions of the states $X_n$ under the i.i.d. assumption on $\{(A_n, U_n), n \geq 1\}$ and a couple of regularity conditions on $(A_1, U_1)$ and the initial condition $x_0$. These upper bounds are related to the spectral radius (or the Perron-Frobenius eigenvalue) of the nonnegative matrix in which each element is the moment generating function of the corresponding element in the state-feedback matrix $A_1$. In particular, we have Kingman's upper bound for $GI/GI/1$ queue when the system is one-dimensional. We also show that some of these upper bounds can be achieved if $A_1$ is lower triangular. These bounds are applied to some commonly used systems to derive new results or strengthen known results.

Keywords: max-plus algebra, tandem queue, parallel processing, large deviations

# 1 Introduction

Because of the development of high speed digital networks, especially ATM networks, the problem of finding or bounding the tail distributions of queue lengths and delays in networks has raised a great deal of interest (see e.g., [21] for historical remarks and [8] for a review).

In this paper, we extend previous development in queueing networks [2, 5] to stochastic linear systems under the max-plus algebra. For such a system, the states are characterized by the recursive equation $X_n = A_n \otimes X_{n-1} \oplus U_n$ with the initial condition $X_0 = x_0$ for some constant $x_0$, where $X_n$ is a $d$-dimensional vector and $A_n$ is a $d \times d$ matrix. Both the state-feedback matrices $\{A_n, n \geq 1\}$ and the inputs $\{U_n, n \geq 1\}$ are random. Also, the operators $\otimes$ and $\oplus$ replace the usual matrix multiplication and matrix addition. (We will describe the system more in Section 2.) Our objectives include establishing upper bounds of the tail distributions of the states $X_n$, deriving conditions when these upper bounds can be achieved, and applying these bounds to systems that are of interests.

As discussed in the recent book by Baccelli et al. [2], the advantage of using such a system as a modelling tool is not only its many useful algebraic properties, but also its broad representations of systems, such as queueing systems, Petri nets, and parallel processing systems. Accordingly, our paper is organized mainly in two parts: bounds in Section 2 and applications in Section 3. Further extensions are addressed in Section 4.

In Section 2.1, we first show that by an exponential transformation, a stochastic linear system under the max-plus algebra induces a stochastic sublinear system under the usual algebra. Along with the i.i.d. assumption of $\{(A_n, U_n), n \geq 1\}$ and a couple of regularity conditions on $(A_1, U_1)$ and the initial condition $x_0$, we establish various upper bounds for the tail distributions of the states $X_n$ as $n \to \infty$ (see Theorem 2.2). These upper bounds are related to the spectral radius (or the Perron-Frobenius eigenvalue) of the nonnegative matrix in which each element is the moment generating function of the corresponding element in the state-feedback matrix $A_1$. Also, we strengthen some of the results in Baccelli et al. [2] that are derived by the notion of association and the results of the multitype branching process in Biggins [4].

In Section 2.2, we derive a supermartingale representation of the induced sublinear system. This supermartingale is related to the martingale in the branching process [18, 4] and the supermartingale in the partial synchronization problem [7]. The supermartingale representation enables us to apply the optional sampling theorem to the induced sublinear system. By considering appropriate stopping times, we strengthen some of the upper bounds in Section 2.1 when the matrices $\{A_n, n \geq 1\}$ are *irreducible* (see Theorem 2.7). In particular, when the system is one-dimensional, we have Kingman's upper bound for $GI/GI/1$ queues.

In Section 2.3, we show that some of the upper bounds in Section 2.1 can be achieved if the matrices $\{A_n, n \geq 1\}$ are *lower triangular* with a "reachability" condition (see Theorem 2.9). The intuition behind this is that the spectral radius of a lower triangular matrix can be obtained directly from its diagonal elements and thus the system can be decomposed into several one-dimensional subsystems that are weakly connected by the "reachability" condition.

That the upper bound is achievable for each one-dimensional system follows from the large deviation principle for i.i.d. random variables (Cramer's theorem).

In Section 3, we apply the results in Section 2 to some commonly used models. In Section 3.1, we consider a tandem queue with infinite buffer at each stage. We show that the decay rate of the tail distribution of the sojourn time in such a system is determined by a nonlinear equation that involves the moment generating function of the service times at the "bottleneck" stage and the Laplace transform of the interarrival times. This nonlinear equation is an extension of the known result for a single queue [5, 16, 13] to tandem queues when both the interarrival times and service times are i.i.d. random variables. Upper bounds for the cycle times in the tandem queue with communication blocking [2] are strengthened in Section 3.2.

# 2 Exponential bounds

Let $\{(A_n, U_n), n \geq 1\}$ be a sequence of $d \times d$ random matrices and $d \times 1$ random vectors. Let $x_0 \in \mathcal{R}^d$. Consider a stochastic linear system under the max-plus algebra as follows:

$$X_n = A_n \otimes X_{n-1} \oplus U_n, \quad X_0 = x_0, \tag{1}$$

where $\{X_n, n \geq 1\}$ are $d \times 1$ random vectors and the operators $\otimes$ and $\oplus$ replace the usual matrix multiplication and matrix addition in (1). To illustrate this, let $X_{n,i}$ (resp. $U_{n,i}$) be the $i^{th}$ element of $X_n$ (resp. $U_n$) and $A_{n,i,j}$ be the element of $A_n$ at the $i^{th}$ row and the $j^{th}$ column. The matrix equation in (1) is equivalent to

$$X_{n,i} = \max\Big( \max_{1 \leq j \leq d} (A_{n,i,j} + X_{n-1,j}), U_{n,i} \Big), \;\; i = 1, \ldots, d. \tag{2}$$

For other properties of linear systems under the max-plus algebra, we refer to Baccelli et al. [2].

## 2.1 General upper bounds

In this section, we derive upper bounds for $X_{n,i}$, $i = 1, \ldots, d$. Consider the family of random vectors $\{X_n(\theta), \theta \geq 0\}$ such that the $i^{th}$ element of $X_n(\theta)$, denoted by $X_{n,i}(\theta)$, is equal to $\exp(\theta X_{n,i})$. Define the vectors $\{\bar{X}_n(\theta), \theta \geq 0\}$ such that the $i^{th}$ element of $\bar{X}_n(\theta)$, denoted by $\bar{X}_{n,i}(\theta)$, is the expectation of $X_{n,i}(\theta)$, i.e., $\bar{X}_{n,i}(\theta) = E(X_{n,i}(\theta))$. Define $A_n(\theta)$, $\bar{A}_n(\theta)$, $U_n(\theta)$ and $\bar{U}_n(\theta)$ accordingly.

We make the following assumptions:

**(A1)** The sequence $\{(A_n, U_n), n \geq 1\}$ is *i.i.d.*.

**(A2)** The initial condition $X_0 = x_0$ for some constant vector $-\infty < x_0 < \infty$.

**(A3)** There exists a $\theta^u > 0$ such that $\bar{A}_1(\theta^u) < \infty$ and $\bar{U}_1(\theta^u) < \infty$.

Note that (A3) implies $\bar{A}_1(\theta) < \infty$ and $\bar{U}_1(\theta) < \infty$ for all $\theta \in [0, \theta^u]$. This is due to the fact that every element in $\bar{A}_1(\theta)$ and $\bar{U}_1(\theta)$ is convex in $\theta$ and that every element in $\bar{A}_1(0)$ and $\bar{U}_1(0)$ is at most 1.

In the following lemma, we transform the linear system under the max-plus algebra into a sublinear system under the usual algebra.

**Lemma 2.1** *(i)* $X_n(\theta) \le A_n(\theta)X_{n-1}(\theta) + U_n(\theta)$ *for all* $\theta \ge 0$.
*(ii) Under (A1),*

$$\bar{X}_n(\theta) \le (\bar{A}_1(\theta))^n \bar{X}_0(\theta) + \sum_{i=1}^{n} (\bar{A}_1(\theta))^{n-i} \bar{U}_1(\theta) \tag{3}$$

*for all* $\theta \ge 0$ *and for all* $n \ge 1$.

**Proof.** (i) Note that $e^{\theta \max_i(a_i)} = \max_i(e^{\theta a_i}) \le \sum_{i=1}^{d} e^{\theta a_i}$ for all $\theta \ge 0$. Thus, we have from (2) that

$$e^{\theta X_{n,i}} \le \sum_{j=1}^{d} e^{\theta A_{n,i,j}} e^{\theta X_{n-1,j}} + e^{\theta U_{n,i}}, \quad i = 1, \ldots, d. \tag{4}$$

Writing (4) in the matrix form yields the desired result in (i).

(ii) Note that $X_{n-1}$ only depends on $\{(A_m, U_m), m \le n-1\}$. Thus, from (A1), it follows that $X_{n-1}(\theta)$ is independent of $A_n(\theta)$ and $U_n(\theta)$. Taking the expectation on both sides of (i) and using the *i.i.d.* property yields

$$\bar{X}_n(\theta) \le \bar{A}_1(\theta)\bar{X}_{n-1}(\theta) + \bar{U}_1(\theta) \tag{5}$$

for all $\theta \ge 0$ and for all $n \ge 1$. Summing (5) for all $n$ completes the argument. ■

Lemma 2.1 not only provides a crucial link between these two algebras but also allows us to analyze systems under the max-plus algebra with tools under the usual algebra.

In the following theorem, we derive various exponential upper bounds.

**Theorem 2.2** *Suppose (A1-3) hold. Let* $\rho(\theta)$ *be the spectral radius (or the Perron-Frobenius eigenvalue) of* $\bar{A}_1(\theta)$.

**(i)** *If* $\rho(\theta) > 1$ *for some* $\theta \in (0, \theta^u]$, *then for all* $i = 1, \ldots, d$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \bar{X}_{n,i}(\theta) \le \log \rho(\theta). \tag{6}$$

*Moreover, for all* $t$

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathit{Prob}(X_{n,i} \ge nt) \le -\theta t + \log \rho(\theta). \tag{7}$$

**(ii)** *If* $1 < \rho(\theta) < e^{\theta t}$ *for some* $t$ *and some* $\theta \in (0, \theta^u]$, *then for all* $i = 1, \ldots, d$,

$$\limsup_{x \to \infty} \frac{1}{x} \log Prob(\max_{1 \le n < \infty} [X_{n,i} - nt] \ge x) \le -\theta. \tag{8}$$

**(iii)** *If* $\rho(\theta) < 1$ *for some* $\theta \in [0, \theta^u]$, *then for all* $i = 1, \ldots, d$,

$$\limsup_{x \to \infty} \frac{1}{x} \log \sup_n Prob(X_{n,i} \ge x) \le -\theta. \tag{9}$$

**Proof.** (i) Since $\rho(\theta)$ is the spectral radius of the nonnegative matrix $\bar{A}_1(\theta)$, for every $\epsilon > 0$ there exists a constant $\sigma_\epsilon(\theta) < \infty$ such that every element of $(\bar{A}_1(\theta))^n$ is bounded above by $\sigma_\epsilon(\theta)(\rho(\theta) + \epsilon)^n$ ([14], Corollary 5.6.13). In conjunction with (3) in Lemma 2.1 and the assumptions (A2-3), one can easily show there exist $\hat{\sigma}_1(\theta), \hat{\sigma}_2(\theta) < \infty$ such that

$$\bar{X}_{n,i}(\theta) \le \hat{\sigma}_1(\theta)(\rho(\theta) + \epsilon)^n + \hat{\sigma}_2(\theta). \tag{10}$$

Since $\epsilon$ is arbitrary and we assume $\rho(\theta) > 1$, taking log on both sides of (10) and letting $\epsilon \to 0$ yields (6). To show (7), apply Chernoff's bound, i.e., $\text{Prob}(X_{n,i} \ge nt) \le e^{-\theta nt} \bar{X}_{n,i}(\theta)$.

(ii) Using the inequality in the proof of Lemma 2.1(i) and Markov's inequality, one has for any $t > 0$ and $\theta \ge 0$

$$\begin{aligned} &\text{Prob}(\max_{1 \le n < \infty} [X_{n,i} - nt] \ge x) = \text{Prob}(\max_{1 \le n < \infty} [X_{n,i}(\theta) e^{-\theta nt}] \ge e^{\theta x}) \\ &\le \text{Prob}(\sum_{n=1}^{\infty} X_{n,i}(\theta) e^{-\theta nt} \ge e^{\theta x}) \le e^{-\theta x} \hat{\sigma}_3(\theta), \end{aligned} \tag{11}$$

where

$$\hat{\sigma}_3(\theta) = \sum_{n=1}^{\infty} \bar{X}_{n,i}(\theta) e^{-\theta nt}. \tag{12}$$

Since we assume that $1 < \rho(\theta) < e^{\theta t}$, we can choose $\epsilon > 0$ such that $\rho(\theta) + \epsilon < e^{\theta t}$. In conjunction with (10), we have $\hat{\sigma}_3(\theta) < \infty$. Taking log on both sides of (11) and letting $x \to \infty$ completes the argument.

(iii) Since $\rho(\theta) < 1$, we can choose $\epsilon$ small enough such that $\rho(\theta) + \epsilon < 1$. In conjunction with (10) and Chernoff's bound, one has

$$\text{Prob}(X_{n,i} \ge x) \le (\hat{\sigma}_1(\theta) + \hat{\sigma}_2(\theta)) e^{-\theta x}. \tag{13}$$

Note that the bound in (13) is independent of $n$. Analogous to the argument in (ii), taking sup and log on both sides of (13) and letting $x \to \infty$ yields (9). ■

From Theorem 2.2 (i), we have the following corollary.

**Corollary 2.3** *If (A1-3) hold and* $\rho(\theta) > 1$ *for* all $\theta \in (0, \theta^u]$, *then for all* $i = 1, \ldots, d$,

**(i)**

$$\limsup_{n\to\infty} \frac{EX_{n,i}}{n} \leq \inf_{0<\theta\leq\theta^u} \frac{\log \rho(\theta)}{\theta}. \tag{14}$$

**(ii)**

$$\limsup_{n\to\infty} \frac{X_{n,i}}{n} \leq t^*, \ \ a.s. \tag{15}$$

*where* $t^* = \inf[t : I(t) > 0]$ *and* $I(t) = \sup_{0<\theta\leq\theta^u}[\theta t - \log \rho(\theta)]$.

**Proof.** (i) Note that $\bar{X}_{n,i}(\theta) \geq exp(\theta E X_{n,i})$ from Jensen's inequality. The result follows from (6) in Theorem 2.2 (i) by taking the best bound for $\theta \in (0, \theta^u]$.

(ii) Taking the best bound in (7) for $\theta \in (0, \theta^u]$ yields

$$\limsup_{n\to\infty} \frac{1}{n} \log \text{Prob}(X_{n,i} \geq nt) \leq -I(t). \tag{16}$$

If $I(t) > 0$ for some $t$, choose $\epsilon$ small enough such that $I(t) - \epsilon > 0$. In conjunction with (16), there exists $n_\epsilon < \infty$ such that for all $n \geq n_\epsilon$,

$$\text{Prob}(X_{n,i} \geq nt) \leq e^{-n(I(t)-\epsilon)} \tag{17}$$

if $I(t) > 0$. Thus, $\sum_n \text{Prob}(X_{n,i} \geq nt) < \infty$ if $I(t) > 0$. Applying the Borel-Cantelli lemma yields

$$\limsup_{n\to\infty} \frac{X_{n,i}}{n} \leq t, \ \ a.s. \tag{18}$$

if $I(t) > 0$. Taking the best bound completes the argument. ■

**Remark 2.4** Note that $I(t)$ defined in Corollary 2.3 is not exactly the same as the usual Legendre transform, where the supremum is usually taken over for $-\infty < \theta < \infty$. Also, one can easily verify that $I(t) = 0$ for all $t \leq \inf_{0<\theta\leq\theta^u} \frac{\log \rho(\theta)}{\theta}$ since

$$I(t) \leq \sup_{0<\theta\leq\theta^u} [\theta(t - \inf_{0<\theta\leq\theta^u} \frac{\log \rho(\theta)}{\theta})] \leq 0. \tag{19}$$

Thus, $t^*$ in (15) is not less than $\inf_{0<\theta\leq\theta^u} \frac{\log \rho(\theta)}{\theta}$ in (14).

If $\log \rho(\theta)/\theta$ is increasing (or nondecreasing) and continuous for all $\theta \in (0, \theta^u]$, then $t^* = \inf_{0<\theta\leq\theta^u} \frac{\log \rho(\theta)}{\theta}$. This can be argued as follows: let $\mu = \lim_{\theta\downarrow 0} \frac{\log \rho(\theta)}{\theta}$. By continuity, for every $\epsilon > 0$ there exists a $\tilde{\theta} \in (0, \theta^u]$ such that $\frac{\log \rho(\tilde{\theta})}{\tilde{\theta}} < \mu + \epsilon$. Thus,

$$I(\mu + \epsilon) \geq \tilde{\theta}(\mu + \epsilon) - \log \rho(\tilde{\theta}) > 0. \tag{20}$$

This implies $t^* \geq \mu$. Since $\log \rho(\theta)/\theta$ is increasing, $\mu = \inf_{0<\theta\leq\theta^u} \frac{\log\rho(\theta)}{\theta}$ and thus $t^* \geq \inf_{0<\theta\leq\theta^u} \frac{\log\rho(\theta)}{\theta}$.

**Remark 2.5** If the system in (1) is without inputs, i.e., $U_{n,i} = -\infty$ for all $i$ and $n$, then the condition $\rho(\theta) > 1$ in Theorem 2.2(i)-(ii) and Corollary 2.3 can be removed. To see this, consider an auxiliary system $\tilde{X}_n = \tilde{A}_n \otimes \tilde{X}_{n-1}$ with $\tilde{X}_0 = x_0$ and $\tilde{A}_{n,i,j} = A_{n,i,j} + \delta$. Let $\tilde{\rho}(\theta)$ be the the corresponding spectral radius in the auxiliary system. It is easy to see that $\tilde{X}_{n,i} = X_{n,i} + n\delta$ for all $i$ and $n$, and $\tilde{\rho}(\theta) = e^{\theta\delta}\rho(\theta)$. Since the spectral radius of a nonnegative matrix is nonnegative, one can find $\delta > 0$ such that $\tilde{\rho}(\theta) > 1$ (except the trivial case that $\rho(\theta) = 0$). Thus, Theorem 2.2(i)-(ii) and Corollary 2.3 can be applied to the auxiliary system and thus derive the corresponding results by substituting $\tilde{X}_{n,i} = X_{n,i} + n\delta$. An alternative proof is to follow the same proof as in Theorem 2.2(i)-(ii) and Corollary 2.3. The only difference is that $\hat{\sigma}_2(\theta) = 0$ in (10).

In the literature, Corollary 2.3 (ii) without inputs was previously obtained by Baccelli et al. ([2], Theorem 8.40) using the property of associated random variables and the results in Biggins [4]. For other related results, see [18, 4, 7, 2] and references therein.

## 2.2 Supermartingale upper bounds

The algebraic approach in the previous section allows us to obtain various exponential upper bounds for $X_{n,i}$ in certain sets. To probe further, we are interested in the stopping times for $X_{n,i}$ in these sets. Our approach in the section is to identify a supermartingale associated with $X_{n,i}$ and apply the optional sampling theorem for these stopping times. For other related martingale approaches, see e.g. [18, 4, 20, 7, 11, 12].

Continue the development from the previous section. Let $v(\theta)$ be the left eigenvector of $\bar{A}_1(\theta)$ associated with its spectral radius $\rho(\theta)$. Since the matrix $\bar{A}_1(\theta)$ is nonnegative, $v(\theta)$ is also nonnegative. Let

$$r(\theta) = \frac{v(\theta)\bar{U}_1(\theta)}{\rho(\theta) - 1} \tag{21}$$

and $\mathcal{F}_m$ be the $\sigma$-algebra generated by $\{(A_n, U_n), n = 1, \ldots, m\}$ and $x_0$. Note that the representation in (21) holds only when $\rho(\theta) \neq 1$.

**Lemma 2.6** *Under (A1-3),* $\{(Z_n(\theta), \mathcal{F}_n), n \geq 0\}$ *forms a supermartingale, where*

$$Z_n(\theta) = \frac{v(\theta)X_n(\theta) - r(\theta)}{\rho(\theta)^n}. \tag{22}$$

**Proof.** One needs to verify that $E(Z_n(\theta)|\mathcal{F}_{n-1}) \leq Z_{n-1}(\theta)$ a.s. Note that $X_{n-1}(\theta)$ only depends on $x_0$ and $\{(A_l, U_l), 1 \leq l \leq n-1\}$, and thus $\mathcal{F}_{n-1}$-measurable. Applying Lemma

2.1(i) yields

$$E(Z_n(\theta)|\mathcal{F}_{n-1}) \leq \frac{1}{\rho(\theta)^n} E\Big(v(\theta)A_n(\theta)X_{n-1}(\theta) + v(\theta)U_n(\theta) - r(\theta)|\mathcal{F}_{n-1}\Big) \ \ a.s. \tag{23}$$

Since $X_{n-1}(\theta)$ is $\mathcal{F}_{n-1}$-measurable and both $A_n(\theta)$ and $U_n(\theta)$ are independent of $\mathcal{F}_{n-1}$, one has

$$E\Big(v(\theta)A_n(\theta)X_{n-1}(\theta) + v(\theta)U_n(\theta) - r(\theta)|\mathcal{F}_{n-1}\Big) = v(\theta)\bar{A}_1(\theta)X_{n-1}(\theta) + v(\theta)\bar{U}_1(\theta) - r(\theta) \ \ a.s. \tag{24}$$

From the assumption that $v(\theta)$ is the left eigenvector of $\bar{A}_1(\theta)$ associated with its spectral radius $\rho(\theta)$, it follows that

$$v(\theta)\bar{A}_1(\theta) = \rho(\theta)v(\theta). \tag{25}$$

In conjunction with (21),(24) and (23), one derives $E(Z_n(\theta)|\mathcal{F}_{n-1}) \leq Z_{n-1}(\theta)$ a.s. ■

**Theorem 2.7** *Suppose (A1-3) hold. If $\theta_t \in (0, \theta^u]$ is a solution of $\log \rho(\theta)/\theta = t$ for some $t > 0$ and $v_i(\theta_t) > 0$, then*

$$Prob(\max_{1 \leq n < \infty} [X_{n,i} - nt] \geq x) \leq \frac{v(\theta_t)X_0(\theta_t)}{v_i(\theta_t)} e^{-\theta_t x}. \tag{26}$$

**Proof.** Consider the stopping time $N = \inf\{n : X_{n,i} \geq nt + x\}$. Clearly,

$$\text{Prob}(\max_{1 \leq n < \infty} [X_{n,i} - nt] \geq x) = \text{Prob}(N < \infty). \tag{27}$$

Let $N_n = \min(N, n)$. Then $N_n$ is a bounded stopping time. Since $\theta_t$ is a solution of $\log \rho(\theta)/\theta = t$ for some $t > 0$ and $\theta_t \in (0, \theta^u]$, $\rho(\theta_t) = e^{\theta_t t} > 1$. Thus, the supermartingale $\{(Z_n(\theta_t), \mathcal{F}_n), n \geq 1\}$ is bounded below by $-r(\theta_t)$. Apply the optional sampling theorem (e.g. see Karlin and Taylor [15] Theorem 6.4.1) yields $EZ_{N_n}(\theta_t) \leq EZ_0(\theta_t)$. Thus,

$$E\Big(\frac{v(\theta_t)X_{N_n}(\theta_t) - r(\theta_t)}{\rho(\theta_t)^{N_n}}\Big) \leq v(\theta_t)X_0(\theta_t) - r(\theta_t). \tag{28}$$

Since $\rho(\theta_t) > 1$, $\rho(\theta_t)^{N_n} > 1$. Simplifying (28) yields

$$E\Big(\frac{v(\theta_t)X_{N_n}(\theta_t)}{\rho(\theta_t)^{N_n}}\Big) \leq v(\theta_t)X_0(\theta_t). \tag{29}$$

Note that both $v(\theta_t)$ and $X_{N_n}(\theta_t)$ are nonnegative. Thus, from the assumption $v_i(\theta_t) > 0$, it follows that

$$E\Big(\frac{X_{N_n,i}(\theta_t)}{\rho(\theta_t)^{N_n}}\Big) \leq \frac{v(\theta_t)X_0(\theta_t)}{v_i(\theta_t)}. \tag{30}$$

Let $\mathbf{1}_{\{X_{N_n,i} \geq N_n t+x\}}$ be the indicator random variable such that it equals 1 if $X_{N_n,i} \geq N_n t + x$ and 0 otherwise. Again, from the nonnegativity of $X_{N_n,i}(\theta_t)$ and $\rho(\theta_t)$, it follows that

$$E\Big(\frac{X_{N_n,i}(\theta_t)}{\rho(\theta_t)^{N_n}}\Big) \geq E\Big(\frac{X_{N_n,i}(\theta_t)}{\rho(\theta_t)^{N_n}}\mathbf{1}_{\{X_{N_n,i} \geq N_n t+x\}}\Big) \geq E\Big(\frac{e^{\theta_t(N_n t+x)}}{\rho(\theta_t)^{N_n}}\mathbf{1}_{\{X_{N_n,i} \geq N_n t+x\}}\Big)$$
$$\geq e^{\theta_t x}\text{Prob}(X_{N_n,i} \geq N_n t + x), \tag{31}$$

where we use $\log \rho(\theta_t)/\theta_t = t$ in the last inequality. Note that

$$\text{Prob}(X_{N_n,i} \geq N_n t + x) \geq \text{Prob}(N_n < n) = \text{Prob}(N < n). \tag{32}$$

In conjunction with (30) and (31), we have

$$\text{Prob}(N < n) \leq \frac{v(\theta_t)X_0(\theta_t)}{v_i(\theta_t)}e^{-\theta_t x}. \tag{33}$$

Letting $n \to \infty$ completes the argument. ■

**Remark 2.8** **(i)** The exponential upper bound obtained in Theorem 2.7 is tighter than (8) in Theorem 2.2(ii). This is at the cost of the assumption $v_i(\theta_t) > 0$. A sufficient condition for $v_i(\theta_t) > 0$ is $\bar{A}_1(\theta_t)$ is *irreducible*.

**(ii)** Analogous to 2.5, for the systems without inputs, the condition $t > 0$ can be removed. As a special case, consider the scalar case ($d = 1$). In this case $\rho(\theta) = A_{1,1,1}(\theta)$ and $v(\theta) = 1$. Let $x_0 = 0$ and $\theta_0$ be a solution of $\log \rho(\theta)/\theta = 0$. Then Theorem 2.7 implies Kingman's upper bound for $GI/GI/1$ queues [17], i.e.,

$$\text{Prob}(\max_{1\leq n<\infty}[\sum_{j=1}^{n} A_{j,1,1}] \geq x) = \text{Prob}(\max_{1\leq n<\infty}[X_{n,1}] \geq x) \leq e^{-\theta_0 x}. \tag{34}$$

## 2.3 Systems that achieve upper bounds

In the section, we consider systems that achieve the upper bounds derived in the previous sections. The matrix $A_1$ is said to be lower triangular if $A_{1,i,j} = -\infty$ for all $j > i$. Beside the assumptions (A1-3), we add a "reachability" condition.

**(A4)** $\text{Prob}(A_{1,d,i} > -\infty) > 0, i = 1, \ldots, d.$

Intuitively speaking, Assumption (A4) implies there exists a "path" from any state with a lower index to reach state $d$. Also, it implies $\bar{A}_{1,d,i}(\theta) > 0$ for all $\theta \geq 0$ and $i$. To see this, note that (A4) implies there exist $\epsilon > 0$ and $-\infty < \gamma_{d,i} < \infty$ such that $\text{Prob}(A_{1,d,i} > -\gamma_{d,i}) \geq \epsilon$. It then follows from Chernoff's bound that $\bar{A}_{1,d,i}(\theta) \geq e^{-\theta\gamma_{d,i}}\text{Prob}(A_{1,d,i} > -\gamma_{d,i}) > 0$ for all $\theta \geq 0$.

**Theorem 2.9** *Suppose (A1-4) hold and $A_1$ is a lower triangular matrix. Then $\rho(\theta) = \max_{1\le i\le d}[\bar{A}_{1,i,i}(\theta)]$.*

**(i)** *If $\rho(\theta) > 1$ for some $\theta \in (0, \theta^u]$, then*

$$\lim_{n\to\infty} \frac{1}{n} \log \bar{X}_{n,d}(\theta) = \log \rho(\theta). \tag{35}$$

**(ii)** *Let $\mu_{max} = \max_{1\le i\le d} E(A_{1,i.i})$. If $\rho(\theta) > 1$ for all $\theta \in (0, \theta^u]$, then*

$$\lim_{n\to\infty} \frac{X_{n,d}}{n} = \lim_{n\to\infty} \frac{E(X_{n,d})}{n} = \mu_{max} \ \ a.s. \tag{36}$$

**(iii)** *If $\theta_t \in (0, \theta^u]$ is the unique solution of $\log \rho(\theta)/\theta = t$ for some $t > 0$, then*

$$\lim_{x\to\infty} \frac{1}{x} \log Prob(\max_{1\le n<\infty} [X_{n,d} - nt] \ge x) = -\theta_t. \tag{37}$$

**(iv)** *If $\theta_0 \in (0, \theta^u]$ is the unique solution of $\log \rho(\theta)/\theta = 0$ and $\rho(\theta_0) < 1$, then*

$$\lim_{x\to\infty} \frac{1}{x} \log \sup_n Prob(X_{n,d} \ge x) = -\theta_0. \tag{38}$$

As in Remark 2.5, for the systems without inputs, the condition $\rho(\theta) > 1$ in Theorem 2.9(i)(ii) and (iii) can be removed. The proof of Theorem 2.9 is based on the upper bounds derived in the previous sections and the lemmas below. In the following lemma, we derive some properties of the spectral radius $\rho(\theta)$ considered in this section.

**Lemma 2.10** *Suppose (A1-4) hold and $A_1$ is a lower triangular matrix.*

**(i)** $\rho(\theta) = \max_{1\le i\le d} \bar{A}_{1,i.i}(\theta)$.

**(ii)** $\rho(0) = 1$.

**(iii)** $\log \rho(\theta)$ *is convex and continuous in $\theta$ for all $\theta \in (0, \theta^u]$.*

**(iv)** $\log \rho(\theta)/\theta$ *is increasing and continuous in $\theta$ for all $\theta \in (0, \theta^u]$.*

**(v)** $\mu_{max} = \lim_{\theta\downarrow 0} \log \rho(\theta)/\theta = \inf_{0<\theta\le\theta} \log \rho(\theta)/\theta$.

**(vi)** *If $\theta_t \in (0, \theta^u]$ is the* unique *solution of $\log \rho(\theta)/\theta = t$ for some $t$, then $\theta_t$ is the* unique *solution of $\log \bar{A}_{1,i,i}(\theta)/\theta = t$ for some $i$.*

**Proof.** (i) Since $A_1$ is lower triangular, $\bar{A}_1(\theta)$ is also lower triangular (in the classical sense). Thus, $\rho(\theta) = \max_{1\le i\le d} \bar{A}_{1,i,i}(\theta)$.

(ii) Recall that $\bar{A}_{1,i,i}(\theta) = Ee^{\theta A_{1,i,i}}$. Thus, $\bar{A}_{1,i,i}(0) = 1$ for all $i$ and $\rho(0) = 1$.

(iii) Note that $\log \bar{A}_{1,i,i}(\theta)$, $i = 1,\ldots,d$, are convex for all $\theta \in (0,\theta^u]$. Since convexity is preserved under the maximum operator, $\rho(\theta)$ is convex for all $\theta \in (0,\theta^u]$. Also, the differentiability of $\log \bar{A}_{1,i,i}(\theta)$, $i = 1,\ldots,d$, implies the continuity of $\rho(\theta)$. (Note that $\rho(\theta)$ might not be differentiable.)

(iv) Note that $\log \bar{A}_{1,i,i}(\theta)/\theta$ is increasing and continuous in $\theta$ for all $\theta \in (0,\theta^u]$. Again, these two properties are preserved under the maximum operator.

(v) The second equality follows from (iv). Since the maximum operator is a continuous function,

$$\lim_{\theta\downarrow 0} \log \rho(\theta)/\theta = max_{1\le i\le d}[\lim_{\theta\downarrow 0} \log \bar{A}_{1,i,i}(\theta)/\theta] = \mu\max. \tag{39}$$

(vi) Clearly, $\theta_t$ is also a solution of $\log \bar{A}_{1,i,i}(\theta)/\theta = t$ for some $i$. Let $i^*$ be one of the indices that $\log \bar{A}_{1,i^*,i^*}(\theta_t)/\theta_t = t$.

Now we argue that $\theta_t$ is also a *unique* solution of $\log \bar{A}_{1,i^*,i^*}(\theta)/\theta = t$. If it is not, then there are at least two solutions. This implies $A_{1,i^*,i^*}$ is a constant since (from Jensen's inequality) $\log \bar{A}_{1,i^*,i^*}(\theta)/\theta$ is *strictly increasing* for all $\theta > 0$ if $A_{1,i^*,i^*}$ is not a constant. Thus, any $\theta < \theta_t$ is also a solution of $\log \bar{A}_{1,i^*,i^*}(\theta)/\theta = t$. From the monotonicity of $\log \bar{A}_{1,i,i}(\theta)/\theta$, $i = 1,\ldots,d$, it follows that $\log \rho(\theta)/\theta = \log \rho(\theta_t)/\theta_t = t$ for all $\theta < \theta_t$. Thus, any $\theta < \theta_t$ is also a solution of $\log \rho(\theta)/\theta = t$, which violates our assumption of the uniqueness for the solution of $\log \rho(\theta)/\theta = t$. ■

Now we derive a lower bound for $X_{n,d}$.

**Lemma 2.11** *Consider a linear system in (1). For all n,*

$$X_{n,d} \ge \max_{1\le i\le d}[A_{n,d,i} + S_{n-1,i} + x_{0,i}], \tag{40}$$

*where* $S_{n,i} = \sum_{j=1}^{n} A_{j,i,i}$ *for all* $n$ *and* $i$.

**Proof.** Note that $S_{n,i} = A_{n,i,i} \otimes \ldots \otimes A_{1,i,i}$. From (1) and (2), it is easy to see that $X_{n,i} \ge S_{n,i} + x_{0,i}$, $i = 1,\ldots,d$. Since $X_{n,d} = \max[\max_{1\le i\le d}[A_{n,d,i} + X_{n-1,i}], U_{n,d}]$, one has

$$X_{n,d} \ge \max_{1\le i\le d}[A_{n,d,i} + X_{n-1,i}] \ge \max_{1\le i\le d}[A_{n,d,i} + S_{n-1,i} + x_{0,i}]. \tag{41}$$

■

**Proof.** (Theorem 2.9) (i) The upper bound is shown in (6). We need to show the lower bound,

$$\liminf_{n\to\infty} \frac{1}{n} \log \bar{X}_{n,d}(\theta) \geq \log \rho(\theta). \tag{42}$$

According to Lemma 2.10(i), it suffices to show for all $i = 1, \ldots, d$,

$$\liminf_{n\to\infty} \frac{1}{n} \log \bar{X}_{n,d}(\theta) \geq \log \bar{A}_{1,i,i}(\theta). \tag{43}$$

Apply Lemma 2.11. and the i.i.d. property in (A1) yields

$$\bar{X}_{n,d}(\theta) \geq e^{\theta x_{0,i}} \bar{A}_{1,d,i}(\theta)(\bar{A}_{1,i,i}(\theta))^{n-1} \tag{44}$$

for $\theta > 0$ and $i = 1, \ldots, d$. From (A4), $\bar{A}_{1,d,i}(\theta) > 0$ for all $\theta > 0$ and $i = 1, \ldots, d$. In view of (A2-3), the argument for the lower bound is completed by taking log and letting $n \to \infty$ in (44).

(ii) The upper bound is shown in Corollary 2.3, 2.4 and Lemma 2.10(v). The lower bound follows trivially from Lemma 2.11.

(iii) Note that Theorem 2.2(ii) implies that

$$\limsup_{x\to\infty} \frac{1}{x} \log \text{Prob}(\max_{1\leq n<\infty} [X_{n,d} - nt] \geq x) \leq -\hat{\theta}_t, \tag{45}$$

where $\hat{\theta}_t = \sup\{\theta \in (0, \theta^u] : \log \rho(\theta)/\theta < t\}$. Since we assume that $\theta_t \in (0, \theta^u]$ is the unique solution of $\log \rho(\theta)/\theta = t$ for some $t > 0$, it then follows from the monotonicity and the continuity of $\log \rho(\theta)/\theta$ in Lemma 2.10(iv) that $\hat{\theta}_t = \theta_t$. This complete the argument for the upper bound.

To show the lower bound, it suffices to show that

$$\liminf_{x\to\infty} \frac{1}{x} \sup_n \log \text{Prob}(X_{n,d} \geq nt + x) \geq -\theta_t \tag{46}$$

since $\text{Prob}(\max_{1\leq n<\infty}[X_{n,d} - nt] \geq x) \geq \sup_n \text{Prob}(X_{n,d} \geq nt + x)$. From (A4) , there exist $\epsilon > 0$ and $-\infty < \gamma_{d,i} < \infty$ such that $\text{Prob}(A_{1,d,i} > -\gamma_{d,i}) \geq \epsilon$. Using Lemma 2.11 and the independent property of $\{A_n, n \geq 1\}$, one has for all $i = 1, \ldots, d$ and for all $\beta$

$$\begin{aligned} \text{Prob}(X_{n,d} \geq n\beta) &\geq \text{Prob}(A_{n,d,i} + S_{n-1,i} + x_{0,i} \geq n\beta) \\ &\geq \text{Prob}(A_{n,d,i} \geq -\gamma_{d,i}, S_{n-1,i} + x_{0,i} \geq n\beta + \gamma_{d,i}) \\ &\geq \epsilon \text{Prob}(S_{n-1,i} \geq n\beta + \gamma_{d,i} - x_{0,i}). \end{aligned} \tag{47}$$

Applying the large deviation lower bound for sum of i.i.d. random variables (see Cramer's theorem in [10], Corollary 2.2.19) yields

$$\liminf_{n\to\infty} \frac{1}{n} \log \text{Prob}(X_{n,d} \geq n\beta) \geq -I_i(\beta), \quad i = 1, \ldots, d, \quad \beta \geq E(A_{1,i,i}), \tag{48}$$

where $I_i(\beta) = \sup_\theta[\theta\beta - \log \bar{A}_{1,i,i}(\theta)]$. Letting $\beta = t + x/n$ $(\beta > t)$ in (48) yields

$$\begin{aligned} &\liminf_{x\to\infty} \frac{1}{x} \log \sup_n \text{Prob}(X_{n,d} \geq nt + x) \\ &\geq \liminf_{n\to\infty} \frac{1}{(\beta - t)n} \log \text{Prob}(X_{n,d} \geq n\beta) \geq -\frac{I_i(\beta)}{\beta - t}, \end{aligned} \tag{49}$$

for all $\beta > t$ and $i = 1, \ldots, d$. From Lemma 2.10(vi), there exists an index $i^*$ such that $\theta_t$ is the unique solution of

$$\log \bar{A}_{1,i^*,i^*}(\theta)/\theta = t. \tag{50}$$

Let $\beta_t$ be the derivative of $\log \bar{A}_{1,i^*,i^*}(\theta)$ at $\theta_t$, i.e.,

$$\beta_t = (\log \bar{A}_{1,i^*,i^*}(\theta_t))'. \tag{51}$$

Since $I_{i^*}(t)$ and $\log \bar{A}_{1,i^*,i^*}(\theta)$ form a Legendre transform pair ( see e.g., [1], Theorem 5.3)

$$\log \bar{A}_{1,i^*,i^*}(\theta_t) = \theta_t\beta_t - I_{i^*}(\beta_t). \tag{52}$$

From (50) and (52), it follows that

$$I_{i^*}(\beta_t) = \theta_t(\beta_t - t). \tag{53}$$

Thus, if $\beta_t > t$ we can replace $\beta$ in (49) by $\beta_t$. This results in

$$\liminf_{x\to\infty} \frac{1}{x} \log \sup_n \text{Prob}(X_{n,d} \geq nt + x) \geq -\frac{I_{i^*}(\beta_t)}{\beta_t - t}. \tag{54}$$

In conjunction with (53), we derive (46) if $\beta_t > t$.

It remains to show that $\beta_t > t$. First we argue that $A_{1,i^*,i^*}$ cannot be a constant. If it is, then the solution of (50) is not *unique*. Since $A_{1,i^*,i^*}$ is not a constant, one has (from Jensen's inequality) that $\log \bar{A}_{1,i^*,i^*}(\theta)$ is *strictly convex* for all $\theta$. This implies

$$\log \bar{A}_{1,i^*,i^*}(0) > \log \bar{A}_{1,i^*,i^*}(\theta_t) + (0 - \theta_t)\beta_t. \tag{55}$$

Since $\log \bar{A}_{1,i^*,i^*}(0) = 0$ and $\log \bar{A}_{1,i^*,i^*}(\theta_t)/\theta_t = t$, it then follows from (50) $\beta_t > t$. (Note from Lemma 2.10(v) that $\beta_t > \mu_{\max}$.)

(iv) For the upper bound, apply Theorem 2.2(iii) and the continuity argument in (iii). The lower bound is shown in (46) when $t = 0$. ■

# 3 Applications

In this section, we apply the results derived in the previous section to some commonly used models in queueing and parallel processing systems. These systems include (i) tandem queues with infinite buffers, and (ii) tandem queues with communication blocking. For other applications in parallel processing systems, see the full report [6].

## 3.1 Tandem queues with infinite buffers

Consider a $K$-stage tandem queue. Assume the tandem queue is empty at time 0 and the buffer size at each stage is infinite. Let $\alpha_{n,0}$ be the interarrival time between the $n-1^{th}$ customer and the $n^{th}$ customer. Let $\alpha_{n,k}, k = 1, \ldots, K$, be the service time of the $n^{th}$ customer at the $k^{th}$ stage. Assume that $\{(\alpha_{n,k}, k = 0, \ldots, K), n \geq 1\}$ form a sequence of i.i.d. random vectors and that $Ee^{\theta^u \alpha_{1,k}} < \infty$ for some $\theta^u > 0$, $k = 1, \ldots, K$. (To avoid triviality, we also assume that $\alpha_{1,k} \neq 0$ a.s. for all $k = 0, 1, \ldots, K$.) Let $\tau_{n,0}$ be the arrival epoch of the $n^{th}$ customer and $\tau_{n,k}, k = 1, \ldots, K$ be the departure epoch of the $n^{th}$ customer at the $k^{th}$ stage. Clearly, one has the following recursive equations

$$\tau_{n,0} = \tau_{n-1,0} + \alpha_{n,0} \tag{56}$$

and

$$\tau_{n,k} = \max[\tau_{n-1,k}, \tau_{n,k-1}] + \alpha_{n,k}, \;\; k = 1, \ldots, K. \tag{57}$$

Let $\tau_n$ be the column vector with its $k^{th}$ element being $\tau_{n,k}$, $k = 0, 1, \ldots, K$. Writing these in the matrix form yields

$$\tau_n = B_n \otimes \tau_{n-1}, \tag{58}$$

where $B_n$ is a $(K+1) \times (K+1)$ lower triangular random matrix with $B_{n,i,j} = -\infty$, $j > i$, and $B_{n,i,j} = \sum_{k=j}^{i} \alpha_{n,k}$, $0 \leq j \leq i$.

Let $\bar{B}_1(\theta)$ be the matrix such that its element at the $i^{th}$ row and the $j^{th}$ column, denoted by $\bar{B}_{1,i,j}(\theta)$, is $Ee^{\theta B_{1,i,j}}$. For the linear system in (58), the spectral radius of $\bar{B}_1(\theta)$ is $\max_{0 \leq k \leq K} Ee^{\theta \alpha_{1,k}} > 1$ for all $\theta \in (0, \theta^u]$ (since $\alpha_{1,k}$, $k = 0, 1, \ldots, K$, are nonnegative and not identical to 0). It is easy to see that (A1-4) in Theorem 2.9 are satisfied with $\tau_{0,k} = 0$, $k = 0, 1, \ldots, K$. Applying Theorem 2.9(i) and (ii) yields the following results for the departure processes.

**Theorem 3.1** *For the tandem queue considered in this section,*

$$\lim_{n \to \infty} \frac{1}{n} \log Ee^{\theta \tau_{n,K}} = \max_{0 \leq k \leq K} (\log Ee^{\theta \alpha_{1,k}}), \tag{59}$$

*for all $\theta \in (0, \theta^u]$. Moreover,*

$$\lim_{n \to \infty} \frac{\tau_{n,K}}{n} = \lim_{n \to \infty} \frac{E\tau_{n,K}}{n} = \max_{0 \leq k \leq K} E\alpha_{1,k} \;\; a.s. \tag{60}$$

Now let $R_{n,k} = \tau_{n,k} - \tau_{n,0}, k = 1, \ldots, K$, be the sojourn time of the $n^{th}$ customer until it leaves the $k^{th}$ stage. Let $R_n$ be the column vector with its $k^{th}$ element being $R_{n,k}, k = 1, \ldots, K$. From (58), it follows that

$$R_n = A_n \otimes R_{n-1} \oplus U_n, \tag{61}$$

where $A_n$ is a $K \times K$ lower triangular matrix with $A_{n,i,j} = -\infty, j > i$, and $A_{n,i,j} = B_{n,i,j} - \alpha_{n,0} = \sum_{k=j}^{i} \alpha_{n,k} - \alpha_{n,0}$, $1 \le j \le i$, and $U_n$ is a $K \times 1$ column vector with $U_{n,i} = B_{n,i,0} - \alpha_{n,0} = \sum_{k=1}^{i} \alpha_{n,k}$, $1 \le i \le K$.

The spectral radius, $\rho(\theta)$, of $\bar{A}_1(\theta)$ in (61) is simply $\max_{1 \le k \le K} E e^{\theta(\alpha_{1,k} - \alpha_{1,0})}$. The following theorem is a direct application of Theorem 2.9(iv).

**Theorem 3.2** *For the tandem queue considered in this section, if $\theta_0 > 0$ is the unique solution of*

$$\max_{1 \le k \le K} E e^{\theta(\alpha_{1,k} - \alpha_{1,0})} = 1, \tag{62}$$

*then*

$$\lim_{x \to \infty} \frac{1}{x} \sup_n \log Prob(R_{n,K} \ge x) = -\theta_0. \tag{63}$$

From Jensen's inequality, $\theta_0 > 0$ and the uniqueness of $\theta_0$, one has the traffic conditions, i.e., $E(\alpha_{1,0}) > E(\alpha_{1,k})$, $k = 1, \ldots, K$. Thus, $R_{n,K}$ converges in distribution to a steady state random variable as $n \to \infty$ (see .e.g. Baccelli and Bremaud [3]). In view of (63), $\theta_0$ is the decay rate of the tail distribution of the sojourn time in steady state.

### 3.2 Tandem queues with communication blocking

Consider the tandem queue model with communication blocking in Baccelli et al. [2] Example 8.37. Let $K$ denote the number of processors. Processor 1 has an infinite number of customers to serve. Between two processors, the buffer capacity is 1, i.e., at most two customers can be in front of processor $k$, $k = 2, \ldots, K$. Assume that the service times at all stages are i.i.d. exponential random variables with parameter 1 and that processor $k$, $k = 2, \ldots, K$, has exactly one customer at time 0. Let $\tau_{n,k}$ be the time that $n$ customers are completed by processor $k$ and $\tau_{0,k} = 0$, $k = 1, \ldots, K$. Clearly, one has the following recursive equations:

$$\tau_{n,1} = \alpha_{n,1} + \max(\tau_{n-1,1}, \tau_{n-1,2}), \tag{64}$$

$$\tau_{n,k} = \alpha_{n,k} + \max(\tau_{n-1,k-1}, \tau_{n-1,k}, \tau_{n,k+1}), \quad k = 2, \ldots, K-1, \tag{65}$$

$$\tau_{n,K} = \alpha_{n,K} + \max(\tau_{n-1,K-1}, \tau_{n-1,K}), \tag{66}$$

where $\alpha_{n,k}$ are i.i.d. exponential random variables with parameter 1.

Let $\tau_n$ be the column vector with its $k^{th}$ element being $\tau_{n,k}$, $k = 1, \ldots, K$. Writing the above equations in the matrix form yields

$$\tau_n = A_n \otimes \tau_{n-1}, \tag{67}$$

where $A_n$ is a $K \times K$ random matrix with $A_{n,1,1} = A_{n,1,2} = \alpha_{n,1}$, $A_{n,i,i-1} = A_{n,i,i} = A_{n,i,i+1} = \alpha_{n,i}$, $i = 2, \ldots K-1$, $A_{n,K,K-1} = A_{n,K,K} = \alpha_{n,K}$, and $A_{n,i,j} = -\infty$ otherwise.

One has the following upper bounds for the completion times.

**Theorem 3.3** *Consider the tandem queue with communication blocking as described above.*

**(i)** *(Baccelli et al. [2]) For $k = 1, \ldots, K$,*

$$\lim_{n\to\infty} \frac{\tau_{n,k}}{n} = \lim_{n\to\infty} \frac{E\tau_{n,k}}{n} \leq t^*, \ \ a.s. \tag{68}$$

*where $t^* = \inf[t : 1 - t + \log t < -\log(1 + 2\cos(\frac{\pi}{K+1}))]$.*

**(ii)** *If $\theta_t \in (0, \theta^u]$ is a solution of*

$$\log\Big(\frac{1 + 2\cos(\frac{\pi}{K+1})}{(1-\theta)}\Big)/\theta = t \tag{69}$$

*for some $t$, then for $k = 1, \ldots, K$,*

$$Prob(\max_{1\leq n<\infty}[\tau_{n,k} - nt] \geq x) \leq \frac{\sin(\frac{\pi}{K+1})}{(1 - \cos(\frac{\pi}{K+1}))\sin(\frac{k\pi}{K+1})} e^{-\theta_t x}. \tag{70}$$

**Proof.** (i) Note that $\bar{A}_1(\theta)$ can be written as $\frac{1}{1-\theta}\tilde{A}$, where $\tilde{A}$ is a $K \times K$ matrix with $\tilde{A}_{1,1} = \tilde{A}_{1,2} = \tilde{A}_{K,K-1} = \tilde{A}_{K,K} = \tilde{A}_{i,i-1} = \tilde{A}_{i,i} = \tilde{A}_{i,i+1} = 1$, $i = 2, \ldots K-1$, and $\tilde{A}_{i,j} = 0$ otherwise.

The spectral radius of $\tilde{A}$, denoted by $\lambda$, is known to be $(1 + 2\cos(\frac{\pi}{K+1}))$ (see Baccelli et al. [2]). Thus, the spectral radius $\rho(\theta)$ of $\bar{A}_1(\theta)$ is $(1 + 2\cos(\frac{\pi}{K+1}))/(1-\theta)$. Applying Corollary 2.3(ii) completes the argument for (i). (Note that the limits in (68) exist follows from subadditivity [19].)

(ii) In order to apply Theorem 2.7, we need to find $v(\theta)$, the left eigenvector of $\bar{A}_1(\theta)$ corresponding to $\rho(\theta)$. Let $\tilde{v}$ be the left eigenvector of $\tilde{A}$ corresponding to $\lambda$. Clearly, $v(\theta) = \tilde{v}$ (up to a constant factor). Let $\tilde{v}_k$, $k = 1, \ldots, K$, be the $k^{th}$ element of $\tilde{v}$. Simple algebra shows

$$\tilde{v}_k = \frac{\sin(\frac{k\pi}{K+1})}{\sin(\frac{\pi}{K+1})}\tilde{v}_1, \ \ k = 1, 2, \ldots, K. \tag{71}$$

According to Theorem 2.7, the constant in the right hand side of (70) is obtained by computing $\sum_{k=1}^{K} \tilde{v}_k/\tilde{v}_i$, $i = 1, \ldots, K$.

■

# 4 Extensions

Possible extensions along this line include (i) relaxation of the i.i.d. assumptions of the state-feedback matrices and inputs, and (ii) tight exponential lower bounds. The first part seems possible for certain systems. For instance, recently Cruz and Liu [9] applied the technique in [5] to tandem queues where service times between two successive customers are allowed to *dependent*. However, their approach required service times between two successive stages to be independent, which is not needed in our setting. For the second part, it is possible to construct martingales as used in the branching processes [18, 4]. The main difficulty arises in proving the uniform integrability for such martingales though the corresponding martingales in the branching processes are uniformly integrable.

**Acknowledgments:**
The author would like to thank Randy Nelson and David Yao for their valuable discussion on the max-plus algebra, which motivated this research. This research is also supported in part by the National Science Council, Taiwan, R.O.C., under Grants NSC 83-0208-M007-091 and NSC 83-0404-E007-052.

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