On the Input-Output Map of a $G/G/1$ Queue

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Nov., 1993

Abstract

In this note, we consider $G/G/1$ queues with stationary and ergodic inputs. We show that if the service times are independent and identical distributed with unbounded supports, then for a given mean of interarrival times, the number of sequences (distributions) of interarrival times that induce identical distributions on interdeparture times is at most one. As a direct consequence, among all the $G/M/1$ queues with stationary and ergodic inputs, the $M/M/1$ queue is the only queue whose departure process is identically distributed as the input process.

1 Introduction

In the literature, the departure processes from queueing systems have been extensively studied, e.g. Daley [6], Whitt [12], Berman and Westcott [5], Bambos and Walrand [4], Anantharam [1, 2] and many others. Daley [6] examined departure processes from $GI/M/1$ queue. Berman and Westcott [5] analyzed those from $GI/GI/1$ queues and showed that if the departure processes from a $GI/GI/1$ queue is a renewal process, then its interrenewal times must be the same as the arrival processes. Bambos and Walrand [4] considered the $G/G/1$ queue as an operator that maps interarrival times to interdeparture times of points, given the service times. They showed the existence of interarrival times that induce identical distribution on interdeparture times if the mean of interarrival times is the same as the mean of service times. They also conjectured that the existence of such interarrival times even if the condition on the same means of interarrival times and service times is removed. Using a metric on stochastic processes, Anantharam [1, 2] recently proved that the only stationary and ergodic interarrival times that induce identical distribution on the interdeparture times of a $G/M/c$ queue is the Poisson process. He also showed similar results for discrete-time queues.

In this note, we consider $G/G/1$ queues with stationary and ergodic inputs as in [4]. We show that the number of sequences (distributions) of interarrival times that induce identical distributions on interdeparture times is at most one for a given mean of interarrival times if the service times are independent and identical distributed (i.i.d.) with unbounded support. As a direct consequence, the Poisson process is the only process among the stationary and ergodic processes that induces identical distributions on interdeparture times when the service times are exponentially distributed. As in [1, 2], our approach is to show that the metric used in [1, 2] is contractive. Instead of using a construction approach in [1, 2], we will show that the contractiveness of the metric follows from well known monotone properties of $G/G/1$ queues in Whitt [12], Stoyan [10], and Ross [9].

2 Contractiveness of input-output map

Consider a $G/G/1$ queue. Let $\tau_{1,n}$, $\tau_{2,n}$, S_n and W_n be the arrival time, departure time, service time and waiting time of the n-th customer. Let $T_{i,n} = \tau_{i,n+1} - \tau_{i,n}$, $i = 1$ and 2, be the interarrival and interdeparture times between the n-th and $n+1$-st customers. In this note, we make the following assumptions on the interarrival times and service times.

(A1) The sequence of the interarrival times $T_1 \equiv \{T_{1,n}, n \geq 1\}$ is stationary and ergodic and it is independent of the system (including the service times).

(A2) The sequence of the service times $S \equiv \{S_n, n \geq 1\}$ is stationary and ergodic and it is independent of the system.

First, we note that if the interarrival times in a $G/G/1$ queue are increased, then the waiting times are decreased and the interdeparture times are increased.

Proposition 2.1 *If $T^1_{1,n} \leq T^2_{1,n}$ for all n and $W^1_1 \geq W^2_1$, then (i) $W^1_n \geq W^2_n$ (ii) $T^1_{2,n} \leq T^2_{2,n}$ for all n.*

The proof can be found in Whitt [12] (Theorem 12(a)), Stoyan [10], and Ross [9]. The first part follows from the Lindley's equation

$$W_{n+1} = max(0, W_n + S_n - T_{1,n}). \tag{1}$$

The second part is a direct consequence of the following recursive equations ([12], Theorem 12), which we will use in the proof of our main result (Theorem 2.4 below).

$$\begin{aligned} T_{2,n} &= T_{1,n} + max(0, W_n + S_n - T_{1,n}) - W_n + S_{n+1} - S_n & (2)\\ &= max(0, T_{1,n} - W_n - S_n) + S_{n+1}. & (3)\end{aligned}$$

As commented in [12], the monotone property in Proposition (ii) is *not* true for $G/G/c$ queues in general.

Remark 2.2 Note that generally the sequence of interdeparture times may not be stationary. However, if the stability condition of a $G/G/1$ queue is satisfied (see [3]), i.e., $ET_{1,n} > ES_n$, then the interdeparture times, as $n \to \infty$, form a stationary and ergodic sequence with the same mean as that of the interarrival times. Thus, Proposition 2.1 (ii) implies that there exists a *jointly stationary* and *ergodic* process (T^1_2, T^2_2) with $T^1_{2,n} \leq T^2_{2,n}$ for all n if (T^1_1, T^2_1) is also jointly stationary and ergodic with $T^1_{1,n} \leq T^2_{1,n}$ for all n and $ET^1_{1,n} > ES_n$. From now on, we will work on the stationary representation of the interdeparture times T_2 if the interarrival times satisfies the stability condition.

In the following, we briefly review the "Rho bar distance" between two *stationary* stochastic sequences (see Gray [7] for details).

Definition 2.3 *The $\bar{\rho}$ (Rho bar) distance between two* stationary *stochastic sequences (distributions) $X \equiv \{X_n, n \geq 1\}$ and $Y \equiv \{Y_n, n \geq 1\}$ is*

$$\bar{\rho}(X, Y) = \inf_{(\hat{X}, \hat{Y}) \in \mathcal{M}} E|\hat{X}_1 - \hat{Y}_1|,$$

where $\mathcal{M}$ is the set of jointly stationary *stochastic processes* $(\hat{X}, \hat{Y})$ *such that* $\hat{X}$ *and* $\hat{Y}$ *have the same distributions as* X *and* Y.

In words, the $\bar{\rho}$ distance of two distributions is the difference of the maximum coupled *jointly stationary* process. It is shown in [7], Theorem 8.3.1, that the $\bar{\rho}$ distance is a metric and the infimum is a minimum, i.e., the maximum coupling is achievable. In particular, if X and Y are two sequences of i.i.d. random variables with distributions $F(\cdot)$ and $G(\cdot)$, then the maximum coupled random process $(\hat{X}, \hat{Y})$ to minimize $\bar{\rho}(X, Y)$ is generated by $\hat{X}_n = F^{-1}(U_n)$ and $\hat{Y}_n = G^{-1}(U_n)$, where $F^{-1}(\cdot)$ and $G^{-1}(\cdot)$ are inverse functions of $F(\cdot)$ and $G(\cdot)$, and U_n is a sequence of i.i.d. random variables uniformly distributed over [0,1] (see Tchen [11]).

In the following theorem, we show that the $\bar{\rho}$ distance of the interdeparture times is not larger than that of the interarrival times if the stability condition is satisfied. The inequality becomes strict if an additional assumption on the service times is added.

Theorem 2.4 *[Contractiveness of input-output map] (i) If* $ET^i_{1,n} > ES_n$ *for* $i = 1$ *and 2, then*

$$\bar{\rho}(T^1_2, T^2_2) \le \bar{\rho}(T^1_1, T^2_1). \tag{4}$$

(ii) If T^1_1 *and* T^2_1 *are not identically distributed but have the same mean, i.e.,* $ET^1_{1,n} = ET^2_{1,n}$, *and if* $\{S_n, n \ge 1\}$ *are i.i.d. with unbounded support, i.e.,* $P(S_n > c) > 0$ *for all* $c \ge 0$, *then the inequality in (i) is* strict.

Proof. In order to prove (i), it suffices to show that

$$E|T^1_{1,n} - T^2_{1,n}| \ge E|T^1_{2,n} - T^2_{2,n}|$$

for any two jointly stationary and ergodic interarrival times, T^1_1 and T^2_1. We use an argument similar to that in Kamae, Krengel and O'Brien [8], Theorem 8. Let $T^3_{1,n} = \max(T^1_{1,n}, T^2_{1,n})$ and queue i, $i = 1, 2$ and 3, be the queue subject to the input T^i_1. Thus,

$$|T^1_{1,n} - T^2_{1,n}| = 2T^3_{1,n} - T^1_{1,n} - T^2_{1,n}. \tag{5}$$

Clearly, $\{T^i_{1,n}, n \ge 1\}$, $i = 1, 2$ and 3 are also jointly stationary and ergodic. Moreover, $ET^3_{1,n} \ge ET^2_{1,n} > ES_n$. Since the stability conditions are all satisfied, the interdeparture times $\{T^i_{2,n}, n \ge 1\}$, i=1,2, and 3, are also jointly stationary and ergodic with

$$ET^i_{2,n} = ET^i_{1,n}, i = 1, 2, \text{ and } 3. \tag{6}$$

Using the monotone property in Proposition 2.1 (ii) and Remark 2.2 yields that $T^3_{2,n} \geq T^i_{2,n}$, i=1 and 2. Thus,

$$|T^1_{2,n} - T^2_{2,n}| = 2\max(T^1_{2,n}, T^2_{2,n}) - T^1_{2,n} - T^2_{2,n} \leq 2T^3_{2,n} - T^1_{2,n} - T^2_{2,n}. \tag{7}$$

Taking expectations on both sides and using (6)-(5) completes the argument for (i).

To prove the second part, it suffices to show that $E\max(T^1_{2,n}, T^2_{2,n}) < ET^3_{2,n}$ in (7). Since we assume that T^1_1 and T^2_1 are not identically distributed but with the same mean $ET^1_{1,n} = ET^2_{1,n}$, then it follows from ergodicity that T^1_1 and T^2_1 must cross each other. Moreover, we have from the stationarity of $T^i_1, i = 1, 2, 3$, that these three sequences are tight. Thus, there must exist some $1 \leq N < \infty$ and some $c < \infty$ such that

$$P\Big(T^1_{1,n} < T^2_{1,n}, T^1_{1,n+j} \leq T^2_{1,n+j}, 1 \leq j \leq N-1, T^1_{1,n+N} > T^2_{1,n+N}, T^3_{n+j} \leq c, 0 \leq j \leq N\Big) > 0. \tag{8}$$

Denote the event in (8) by event B. This implies that

$$P\Big(\sum_{j=0}^{N} T^3_{1,n+j} > \sum_{j=0}^{N} T^i_{1,n+j}, i = 1, 2, T^3_{n+j} \leq c, 0 \leq j \leq N\Big) > 0. \tag{9}$$

We consider the event $E \equiv \{S_j > c, j = n, \ldots, n+N, i = 1, 2\}$. Under this event, all the queues are not empty and we have that

$$W^i_{j+1} = W^i_j + S_j - T^i_{1,j}, \quad j = n, \ldots, n+N, i = 1, 2, 3. \tag{10}$$

Since we assume that S_n's are i.i.d. and that $P(S_n > c) > 0$ for all $c > 0$, $P(E) > 0$. Observe that under events B and E, T^1_1 and T^2_1 cross each other and all the queues are not empty. Thus, we have from (10), (9) and Proposition 2.1(i) that

$$P\Big(W^3_{n+N+1} < W^i_{n+N+1}, i = 1, 2\Big) > 0. \tag{11}$$

Let M be the first time that $W^3_{n+N+1+M} = 0$. In words, we are looking for the first time that queue 3 becomes empty or the busy period ends. At that moment, we will argue that the interdeparture time of queue 3 is strictly larger than those of queue 1 and queue 2. Since $ET^3_{1,n} > ES_n$, M is finite, i.e., $1 \leq M < \infty$. Before the busy period of queue 3 ends, we have

$$W^i_{j+1} = W^i_j + S_j - T^i_{1,j}, \quad j = n+N+1, \ldots, n+N+M-1, i = 1, 2, 3. \tag{12}$$

In conjunction with (11), we have that

$$P\Big(W^3_{n+N+M} < W^i_{n+N+M}, i = 1, 2\Big) > 0. \tag{13}$$

Also from the definition of M and (3)-(2), it follows that under the event E

$$T^3_{2,j} = S_{j+1}, \quad j = n, \ldots, n+N+M-1 \tag{14}$$

and that

$$T^3_{2,n+N+M} = T^3_{1,n+N+M} - W^3_{n+N+M} - S_{n+N+M} + S_{n+N+M+1}. \tag{15}$$

Thus, we have from (15) and (13) that

$$P\Big(T^3_{2,n+N+M} > T^i_{2,n+N+M}, i = 1, 2\Big) > 0. \tag{16}$$

Since both N and M are finite,

$$P\Big(T^3_{2,1} > T^i_{2,1}, i = 1, 2\Big) > 0. \tag{17}$$

Thus, The inequality in (4) is strict. □

Theorem 2.5 *For a $G/GI/1$ queue with service times satisfying the assumptions in Theorem 2.4 (ii), the number of sequences of interarrival times that induce identical distributions on interdeparture times is* at most one *for a given mean of interarrival times.*

Proof. We will use the same argument as in [1], Theorem 9. Suppose $T^i \equiv \{T^i_{1,n}, n \geq 1\}$, $i = 1$ and 2, be two such sequences with the same mean but different distributions. Since the distributions of the input and output sequences are the same,

$$\bar{\rho}(T^1_1, T^2_1) = \bar{\rho}(T^1_2, T^2_2).$$

But this contradicts the result in Theorem 2.4 (ii). □

It is clear that $G/M/1$ queues satisfy the assumptions in Theorem 2.4 (ii). Since the departure process of an $M/M/1$ queue is also a Poisson process, it then follows from Theorem 2.5 that the Poisson process is the only process that induces an identical distribution on the departure process of a $G/M/1$ queue. This result was obtained in Anantharam [2].

For $G/GI/1$ queues with service times bounded above by a constant c, it is easy to see that any sequences of interarrival times that are larger than c would induce identical distributions on the sequences of interdeparture times. Thus, the input processes that induce identical distributions on the departure process of $G/GI/1$ queues with bounded service times are not unique. As an illustrating example, every departure process of a $G/D/1$ queue is one of such processes for $G/D/1$ queues.

Acknowledgements:

The author would like to thank Venkat Anantharam and Randy Nelson for their careful comments which enhanced the presentation of this work. This work was partially completed while the author was at IBM T.J. Watson Research Center.

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