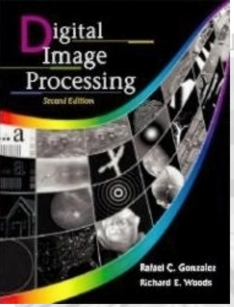

Chapter 7
Wavelets and Multiresolution Processing

- Wavelet transform vs Fourier transform
- Basis functions are small waves called *wavelet* with *different frequency* and *limited duration*
- Multiresolution theory: representation and analysis of signals/images at more than one resolutions.
  - Subband coding
  - Quadrature mirror filtering
  - Pyramid image processing

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7.1 Background

- Small objects or low contrast images are examined at high resolutions; whereas large object and high contrast images are examined in low resolution.
- Images are 2-D array of intensity with locally varying statistics

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7.1 Background

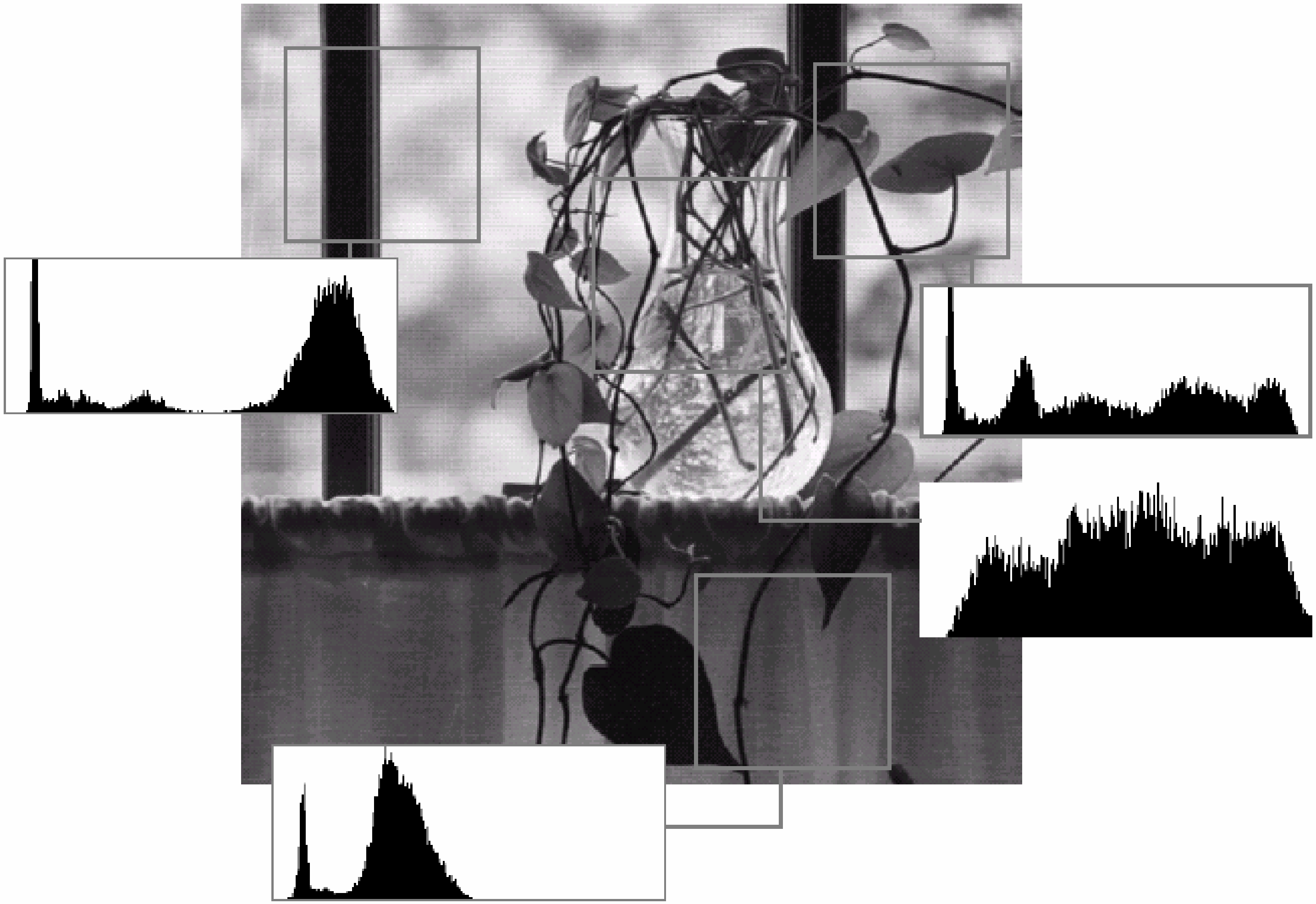

FIGURE 7.1 A natural image and its local histogram variations.

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7.1 Background

- ***Image Pyramids***
- Base level J with size $2^J \times 2^J$ or $N \times N$ where $J = log_2 N$
- Intermediate level j with size $2^j \times 2^j$ with $0 \le j \le J$
- Most pyramids are truncated to $P+1$ levels where $j = J-P, \ldots J-1, J$
- The total number of pixel elements in a $P+1$ level pyramid is

$$N^2\left(1+\frac{1}{4^1}+\frac{1}{4^1}+\ldots..+\frac{1}{4^P}\right) \le \frac{4}{3}N^2$$

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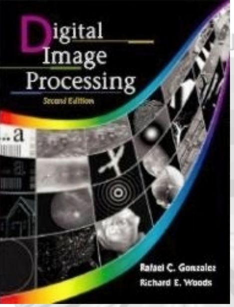
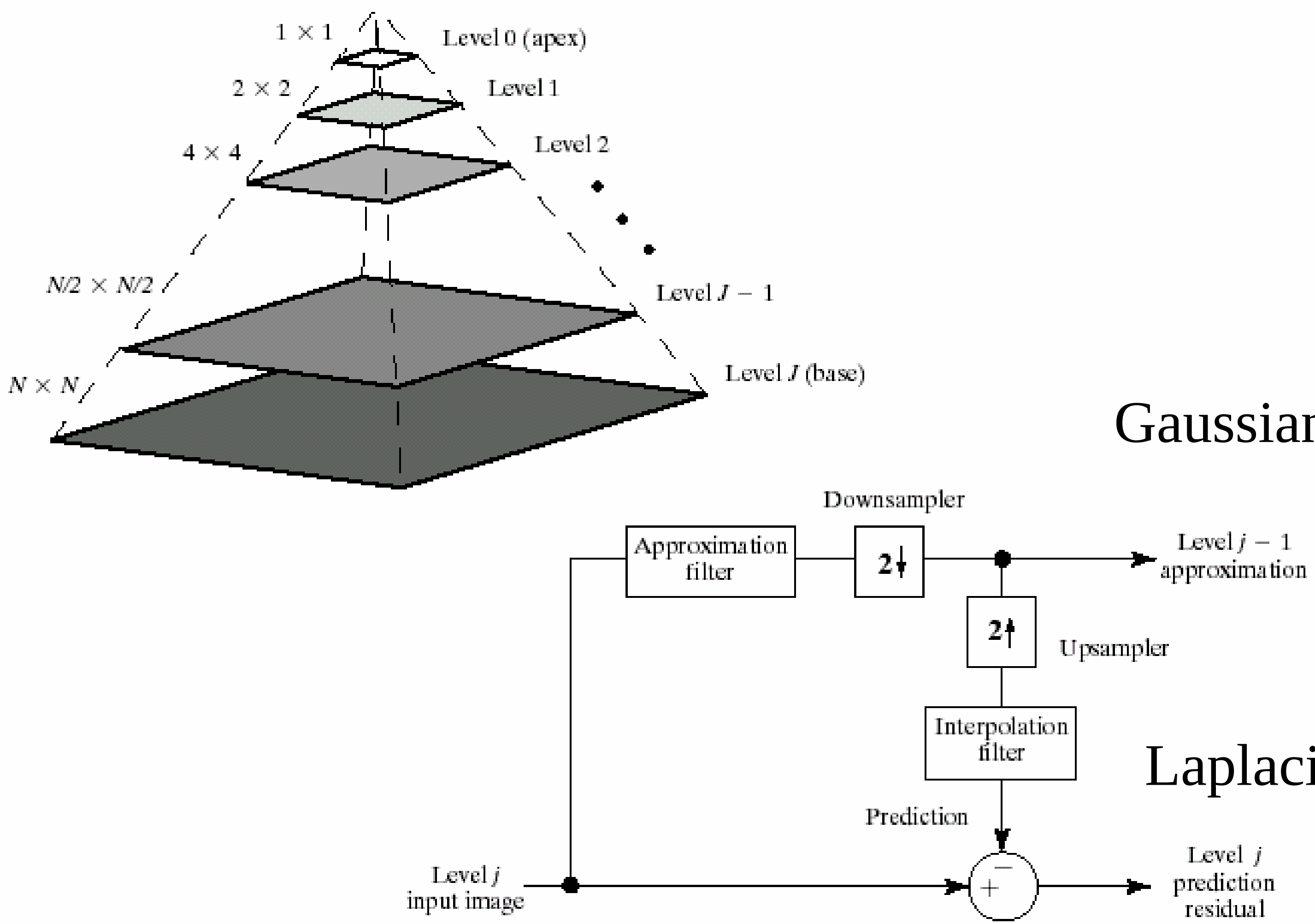


7.1 Background



a
b

FIGURE 7.2 (a) A pyramidal image structure and (b) system block diagram for creating it.

Gaussian pyramid

Laplacian Pyramid

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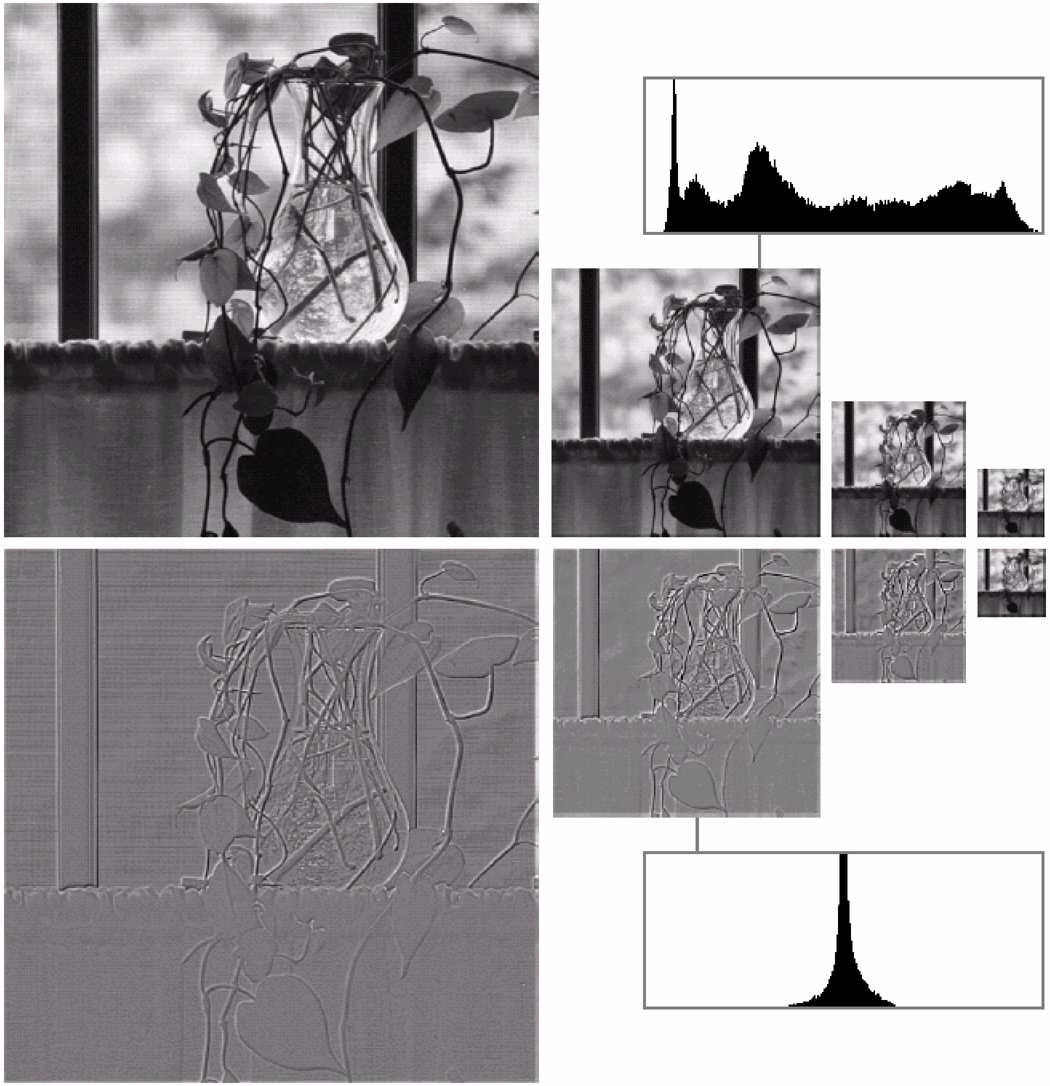
7.1 Background



a
b

FIGURE 7.3 Two image pyramids and their statistics: (a) a Gaussian (approximation) pyramid and (b) a Laplacian (prediction residual) pyramid.

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7.1 Background-Subband Coding

- In subband coding: an image is decomposed into a set of bandlimited components, called subbands.
- The subbands can be downsampled without loss of information.
- Reconstrunction of the original image is accomplished by upsampling, filtering, and summing the individual subbands

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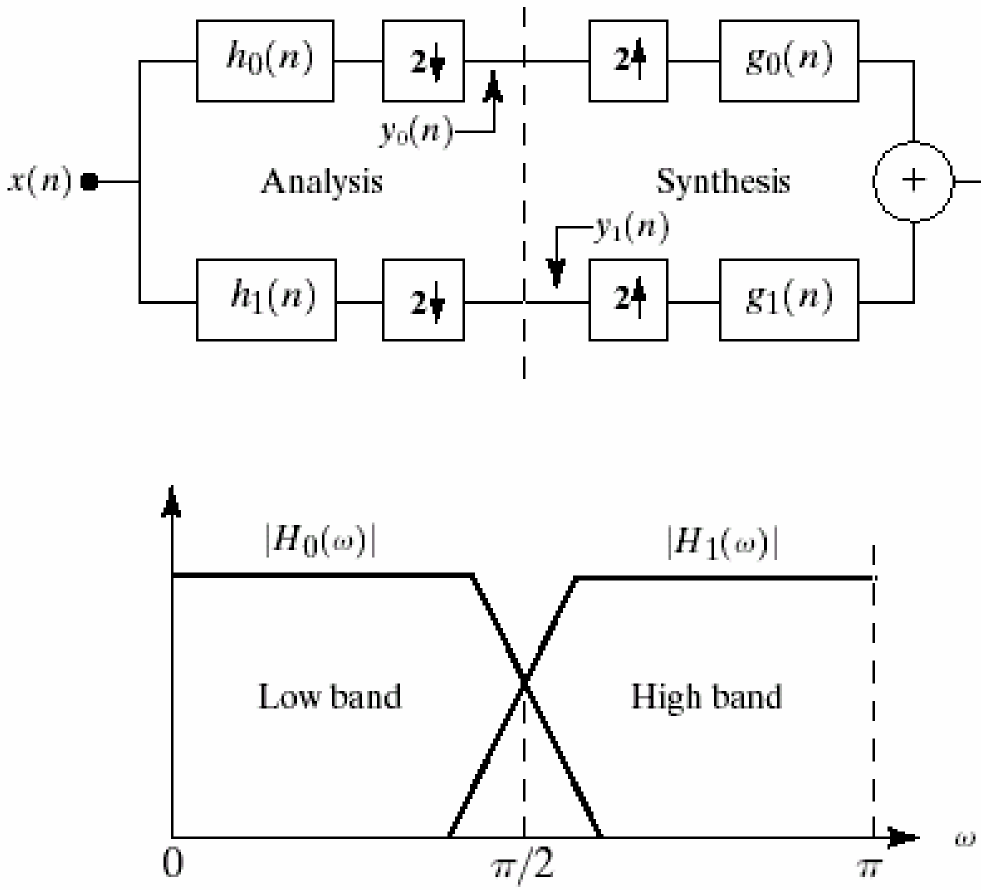
7.1 Background-Subband Coding



a
b

FIGURE 7.4 (a) A two-band filter bank for one-dimensional subband coding and decoding, and (b) its spectrum splitting properties.

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7.1 Background-Subband Coding

- The input is 1-D band-limited discrete time signals *$x(n)$, $n=0,1,2,\ldots$*
- *The output $\hat{x}(n)$ is formed through the decomposition of $x(n)$ into $y_0(n)$ and $y_1(n)$ via analysis filter $h_0(n)$ and $h_1(n)$, and subsequent recombination via synthesis filters $g_0(n)$ and $g_1(n)$.*
- *The Z-transform of $x(n)$ is* $X(z)=\sum_{-\infty}^{\infty} x(n)z^{-n}$
- $x_{down}(n)=x(2n)$ $\quad X_{down}(z)=1/2[X(z^{1/2})+X(-z^{1/2})]$
- $x_{up}(n)=x(n/2)$ *for* $n=0,2,4,\ldots$ $\quad X_{up}(z)=X(z^2)$

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7.1 Background-Subband Coding

- $\hat{x}(n)$ is obtained from the downsampled and upsampled *x(n)*, therefore
- $$\hat{X}(z) = \frac{1}{2}\left[X(z) + X(-z)\right]$$
- *X(-z)* is the modulated version of *x(n)*
- $Z^{-1}\left[X(-z)\right] = (-1)^n x(n)$
- *The system output is*

$$\hat{X}(z) = \frac{1}{2}G_0(z)\left[H_0(z)X(z) + H_0(-z)X(-z)\right]$$

$$+\frac{1}{2}G_1(z)\left[H_1(z)X(z) + H_1(-z)X(-z)\right]$$

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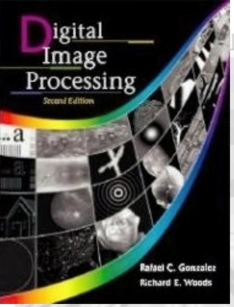

7.1 Background-Subband Coding

- Rearrange $$\hat{X}(z) = \frac{1}{2}\left[H_0(z)G_0(z) + H_1(z)G_1(z)\right]X(z)$$ $$+\frac{1}{2}\left[H_0(-z)G_0(z) + H_1(-z)G_1(z)\right]X(-z)$$
- The second component contain the –z dependence, represents the aliasing that is introduced by downsampling-upsampling process.
- For error-free reconstruction of the input, $x(n) = \hat{x}(n)$
- The second component is zero.

$$H_0(-z)G_0(z)+H_1(-z)G_1(z)=0$$

$$H_0(z)G_0(z)+H_1(z)G_1(z)=2$$

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7.1 Background-Subband Coding

- Reduce to matrix expression

$$[G_0(z) \quad G_1(z)]\mathbf{H}_m(z) = [2 \quad 0]$$

- Where

$$\mathbf{H}_m(z) = \begin{bmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{bmatrix}$$

- Assume that $\boldsymbol{H}_m(z)$ is nonsingular then

$$\begin{bmatrix} G_0(z) \\ G_1(z) \end{bmatrix} = \frac{2}{\det(\mathbf{H}_m(z))} \begin{bmatrix} H_1(-z) \\ -H_0(-z) \end{bmatrix} \qquad (7\text{-}12)$$

- The analysis and synthesis filtered are cross-modulated.

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7.1 Background-Subband Coding

- For FIR filters, the determinate of the modulation matrix is a pure delay, i.e.,

$$det(\boldsymbol{H}_m(z)) = \alpha z^{-(2k+1)}$$

- Ignoring the delay and let α=2, by taking inverse Z transform we have

$$g_0(n)=(-1)^n h_1(n) \quad and \; g_1(n)=(-1)^{n+1} h_0(n)$$

- If α=-2 then

$$g_0(n)=(-1)^{n+1} h_1(n) \quad and \; g_1(n)=(-1)^n h_0(n) :$$

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7.1 Background-Subband Coding

- *The biorthogonality of the analysis and synthesis filters, let P(z) be the product of he lowpass analysis and synthesis filter*

  from (7.12)

$$P(z)=G_0(z)H_0(z)=2H_0(z)H_1(-z)/\det(\mathbf{H}_m(z))$$

also $\det(\mathbf{H}_m(z))=-\det(\mathbf{H}_m(-z))$

and $G_1(z)H_1(z)=-2H_0(-z)H_1(z)/\det(\mathbf{H}_m(z))=P(-z)$

Thus $G_1(z)H_1(z)=P(-z)=G_0(-z)H_0(-z)$

and $G_0(-z)H_0(-z)+G_0(-z)H_0(-z)=2$

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7.1 Background-Subband Coding

- Inverse z-transform

$$\sum g_0(k)h_0(n-k)+(-1)^n\sum_k g_0(k)h_0(n-k)=2\delta(n)$$

- Odd index terms cancel, it is simplified as

$$\sum g_0(k)h_0(2n-k)=\langle g_0(k),h_0(2n-k)\rangle=\delta(n)$$

- Express G_0 and H_0 as function of G_1 and H_1

$$\langle g_1(k),\quad h_1(2n-k)\rangle=\delta(n),\quad \langle g_0(k),\quad h_1(2n-k)\rangle=0$$

$$\langle g_1(k),\quad h_0(2n-k)\rangle=0$$

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7.1 Background-Subband Coding

- More general expression:

$$\left\langle g_j(k),\quad h_i(2n-k)\right\rangle = \delta(i-j)\delta(n) \quad i, j = \{0, \quad 1\} \quad (7.21)$$

- Filter banks satisfying this condition are ***bi-orthogonal***
- Quadrature mirror filter(QMF) (Table 7.1)
- Conjugate quadrature filter (CQF) (Table 7.1)
- Orthonormal filter (Table 7.1) for *fast wavelet transform*, it requires that (7.22)

$$\left\langle g_i(n),\quad g_i(n+2m)\right\rangle = \delta(i-j)\delta(m) \quad i, j = \{0, \quad 1\}$$

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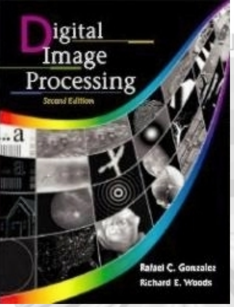

7.1 Background-Subband Coding

Filter	QMF	CQF	Orthonormal
$H_0(z)$	$H_0^2(z) - H_0^2(-z) = 2$	$H_0(z)H_0(z^{-1}) +$ $H_0^2(-z)H_0(-z^{-1}) = 2$	$G_0(z^{-1})$
$H_1(z)$	$H_0(-z)$	$z^{-1}H_0(-z^{-1})$	$G_1(z^{-1})$
$G_0(z)$	$H_0(z)$	$H_0(z^{-1})$	$G_0(z)G_0(z^{-1}) +$ $G_0(-z)G_0(-z^{-1}) = 2$
$G_1(z)$	$-H_0(-z)$	$zH_0(-z)$	$-z^{-2K+1}G_0(-z^{-1})$

TABLE 7.1 Perfect reconstruction filter families.

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7.1 Background-Subband Coding

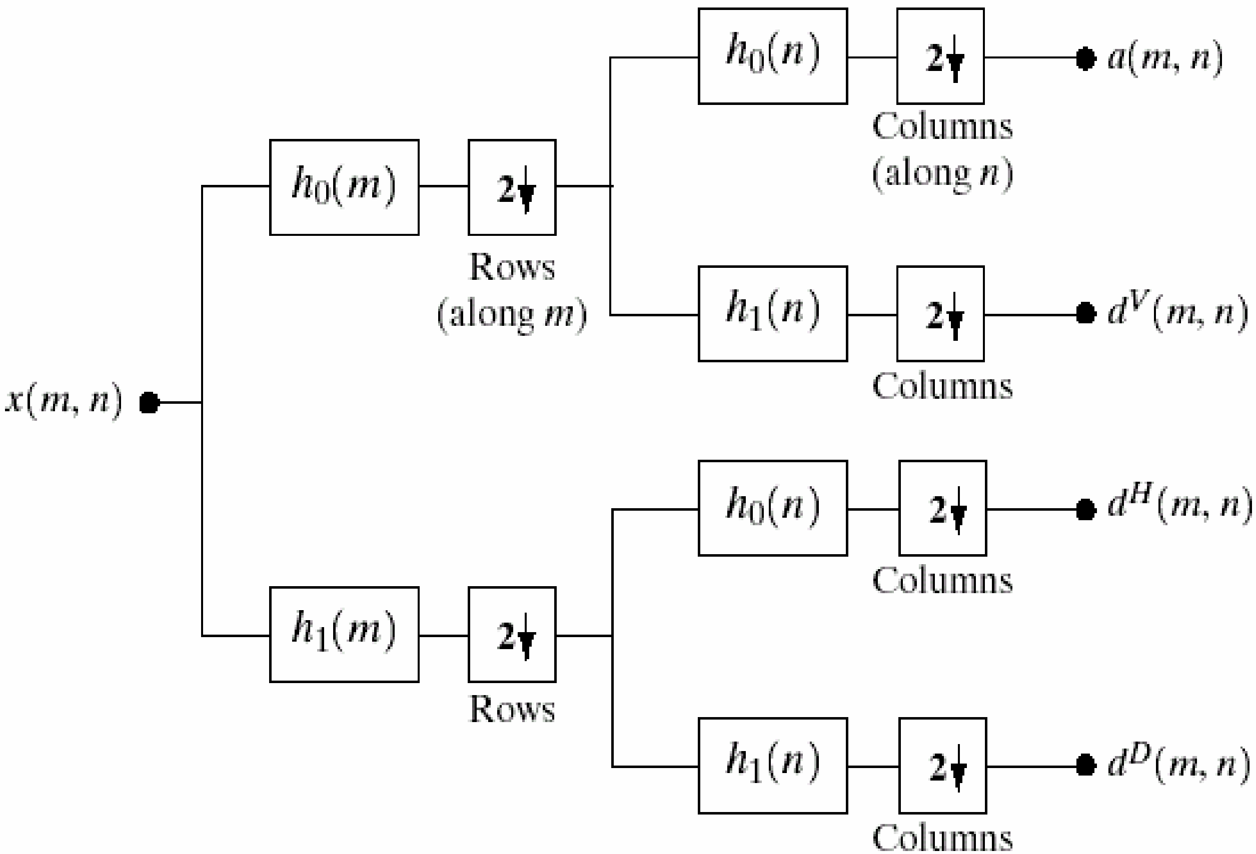

FIGURE 7.5 A two-dimensional, four-band filter bank for subband image coding.

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7.1 Background-Subband Coding

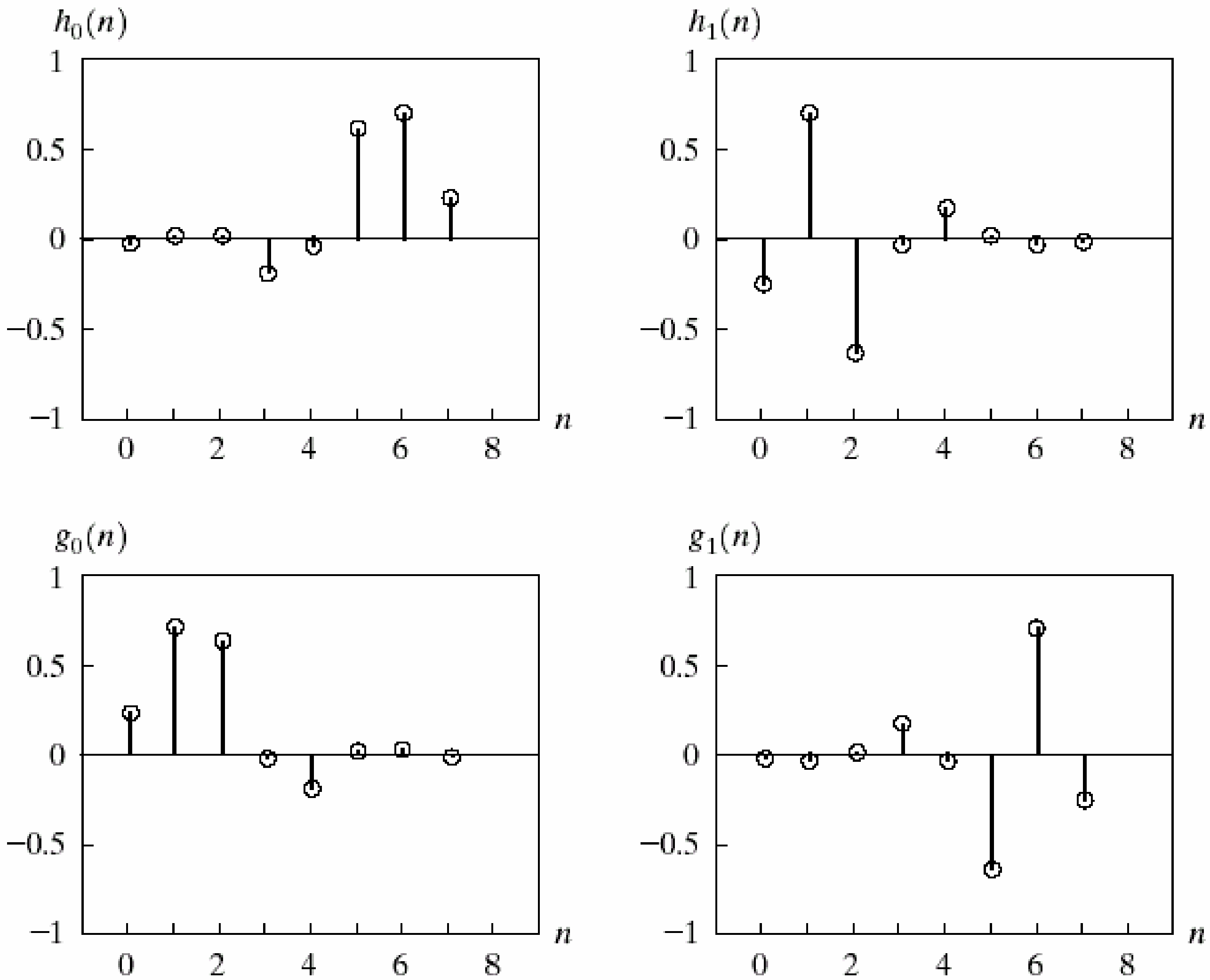

FIGURE 7.6 The impulse responses of four 8-tap Daubechies orthonormal filters.

They satisfy perfect reconstruction and (7.21) and (7.22)

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7.1 Background-Subband Coding

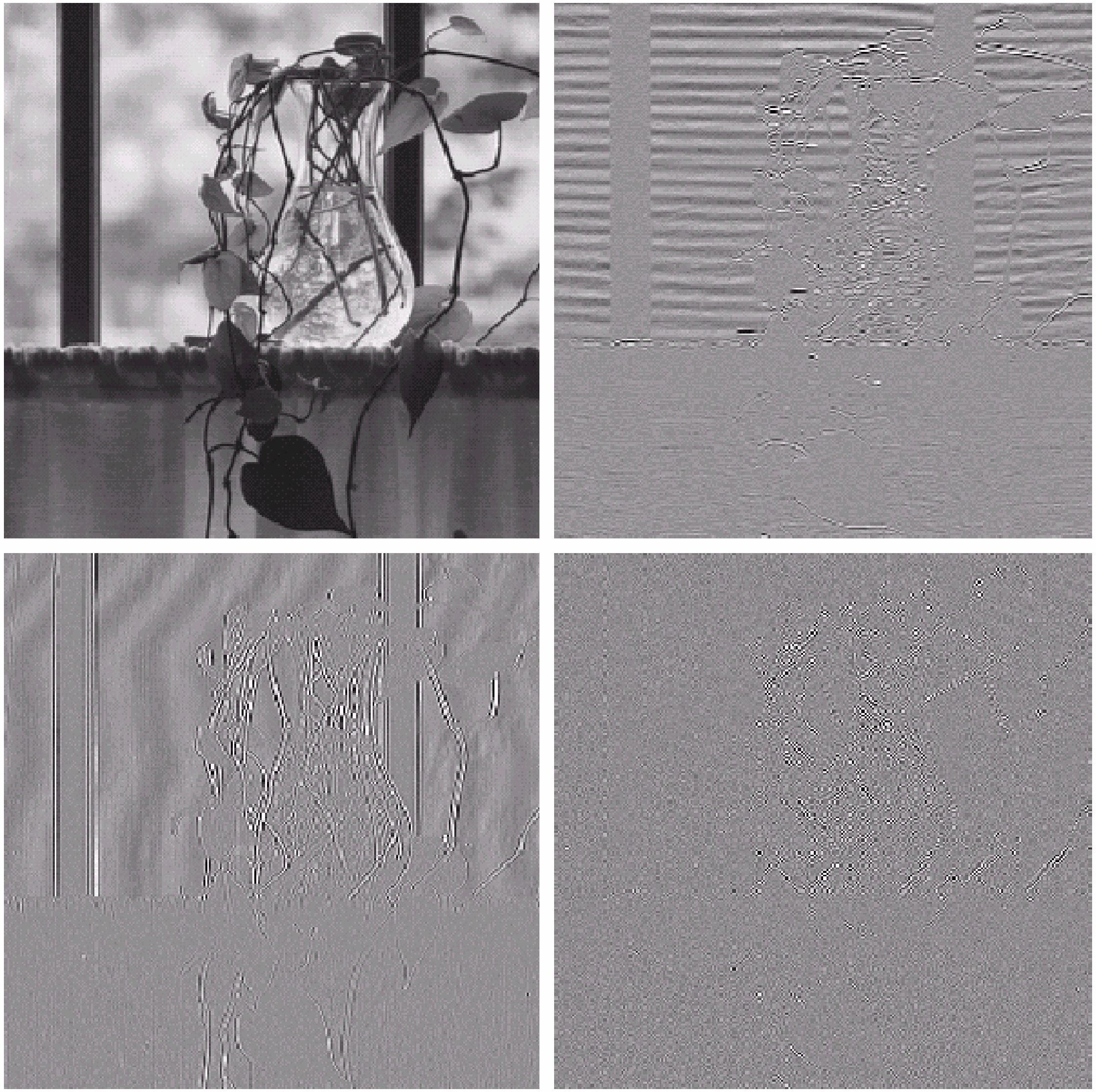

FIGURE 7.7 A four-band split of the vase in Fig. 7.1 using the subband coding system of Fig. 7.5.

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7.1 Background-Harr Transform

- The oldest and simplest known orthonormal wavelets
- The Harr transform is both separable and symmetric as

$$\mathbf{T}=\mathbf{HFH}$$

where **F** is $N \times N$ image matrix and **H** is $N \times N$ transform matrix and **T** is the resulting $N \times N$ transform.

- For Harr transform, the transformation matrix **H** contains the Harr basis function $h_k(z)$ defined over the continuous closed interval [0,1] for k=0,1,….N-1 where $N=2^n$.

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7.1 Background-Harr Transform

- To generate **H**, we define $k=2^p+q-1$ where $0\le p\le n-1$, $q=0$ or 1 for $p=0$, and $1\le q\le 2^p$ for $p\neq 0$
- The Harr basis functions are

$$h_0(z)=h_{00}(z)=1/N^{1/2}, \quad z\in[0,1]$$

and

$$h_k(z)=h_{pq}(z)=\frac{1}{\sqrt{N}}\begin{cases} 2^{p/2} & (q-1)/2^p \le z < (q-0.5)/2^p \\ -2^{p/2} & (q-0.5)/2^p \le z < q/2^p \\ 0 & otherwise,\ z\in[0,1] \end{cases}$$

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7.1 Background-Harr Transform

- The *i*th row of an $N \times N$ Harr transformation matrix contains the elements of $h_i(z)$, $z=0/N, 1/N,\ldots.(N-1)/N$
- If $N=4$, k, q, and p are assumed as the following values:

k	p	q
0	0	0
1	0	1
2	1	1
3	1	2

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7.1 Background-Harr Transform

- The *4×4* transformation matrix is

$$H_4 = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}$$

- The *2×2* transformation matrix is

$$H_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

- The basis functions satisfy the QMF prototype filter in Table 7.1, the $h_0(n)$ *and* $h_1(n)$ are the elements of the first and the second rows of $\mathbf{H}_2$.

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7.1 Background-Harr Transform

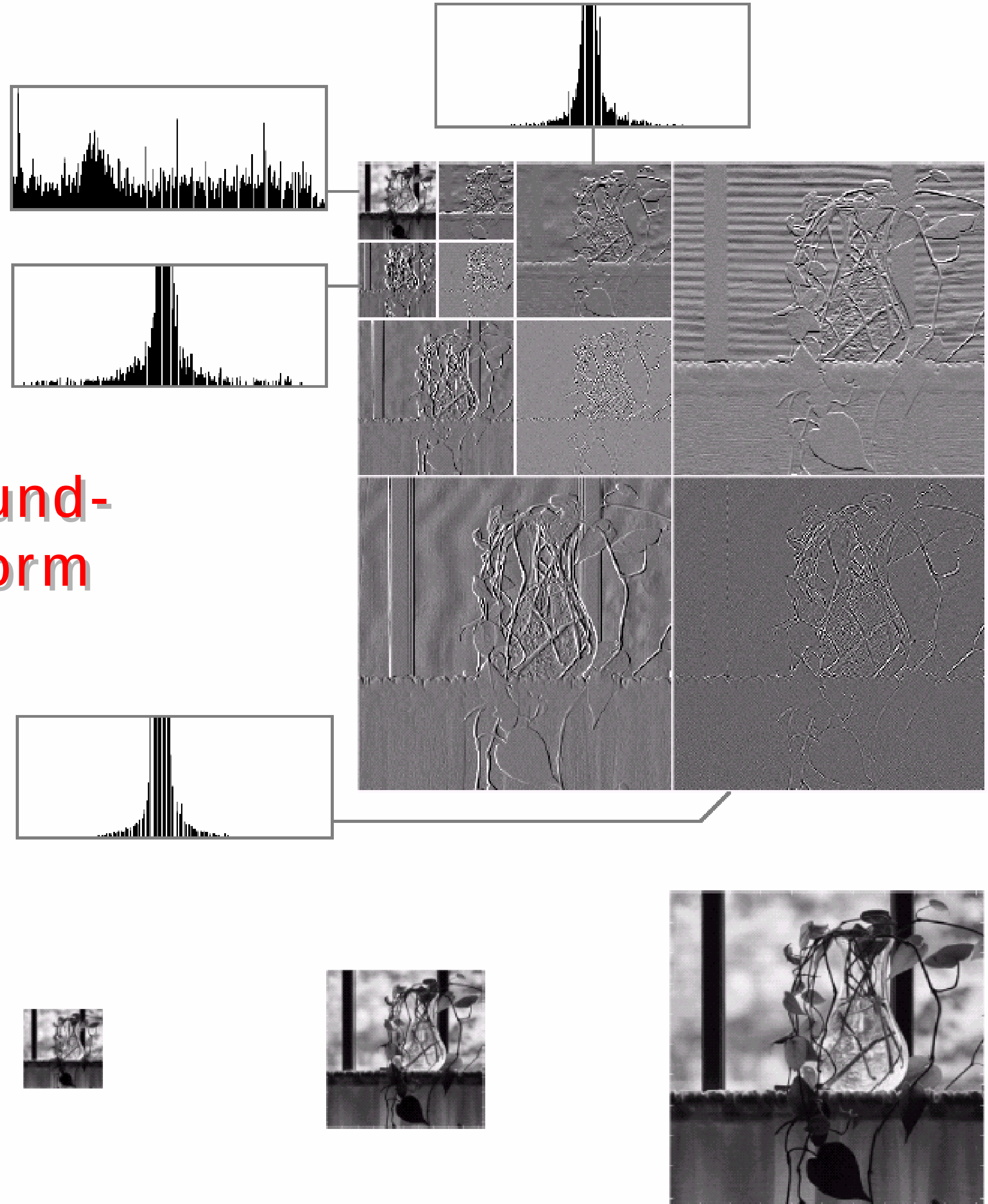

FIGURE 7.8 (a) A discrete wavelet transform using Haar basis functions. Its local histogram variations are also shown; (b)–(d) Several different approximations (64 × 64, 128 × 128, and 256 × 256) that can be obtained from (a).

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7.2 Multiresolution Expansions

- ***Multi-resolution analysis*** (MRA)
- A ***scaling function*** is used to create a series of approximations of a function or image, each different by a factor of 2 from its neighboring approximation
- Additional functions, called ***wavelet***, are used to encode the difference between two adjacent approximation.

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7.2 Multiresolution Expansions

- ***Series expansion***
- A signal *f(x)* can be represented as a linear combination of expansion functions as

$$f(x) = \sum_k \alpha_k \varphi_k(x)$$

- The expressible functions form a function space that is the closed span of the expansion set denoted as

$$V = \overline{\underset{k}{Span}\{\varphi_k(x)\}}$$

- $f(x) \in V$ means that *f(x)* is in the closed span of $\{\varphi_k(x)\}$

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7.2 Multiresolution Expansions

- For any function space V and the corresponding expansion set $\{\varphi_k(x)\}$ there is a set of dual functions denoted as $\{\tilde{\varphi}_k(x)\}$ that can be used to compute the α_k as

$$\alpha_k = \langle \tilde{\varphi}_k(x) \quad f(x) \rangle = \int \tilde{\varphi}_k^{\,*}(x) f(x) dx$$

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7.2 Multiresolution Expansions

- *Case 1*: The expansion functions form orthonormal basis for V that $\langle \varphi_j(x) \quad \varphi_k(x) \rangle = \delta_{jk}$
- The basis and its dual are equivalent $\varphi_k(x) = \tilde{\varphi}_k(x)$
- and then $\alpha_k = \langle \varphi_k(x) \quad f(x) \rangle$
- *Case 2*: If the expansion functions are not orthonormal, but orthogonal basis for V then

$$\langle \varphi_j(x) \quad \varphi_k(x) \rangle = 0 \quad if \;\; j \neq k$$

- and the basis functions and their duals are called bi-orthogonal.
- The biorthogonal basis and their duals are

$$\langle \varphi_j(x) \quad \tilde{\varphi}_k(x) \rangle = \delta_{jk}$$

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7.2 Multiresolution Expansions

- *Case 3*: If the expansion set is not a basis for V, but support the expansion, then it is a spanning set in which there is more than one set of α_k for any $f(x) \in V$
- The expansions and their duals are said to be overcomplete or redundant. They form a frame in which

$$A\|f(x)\|^2 \leq \sum_k \left| \langle \varphi_k(x) \quad f(x) \rangle \right|^2 \leq B\|f(x)\|^2$$

- A and B "frame" the normalized inner products of the expansion coefficients and the function.

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7.2 Multiresolution Expansions

- If A=B the expansion is called a *tight frame*, and it can shown that

$$f(x) = \frac{1}{A}\sum_k \langle \varphi_k(x) \quad f(x) \rangle \varphi_k(x)$$

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7.2 Multiresolution Expansions

- ***Scaling Functions***
- Consider a set of expansion functions composed of integer translations and binary scaling of the real square-integrable function $\varphi(x)$, that is the set $\{\varphi_{j,k}(x)\}$ where $\varphi_{j,k}(x)= 2^{j/2}\varphi(2^j x-k)$
- For all $j, k \in I$ *and* $\varphi(x) \in L^2(R)$
- j determines the position of $\varphi_{j,k}(x)$, and k determines the width of $\varphi_{j,k}(x)$
- $\varphi_{j,k}(x)$ *is called the scaling function*

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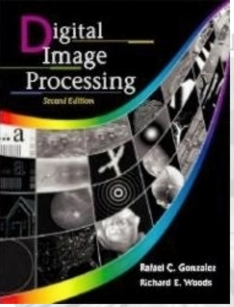

7.2 Multiresolution Expansions

- If we restrict j to a specific value $j=j_0$ then $\{\varphi_{j0,k}(x)\}$ *is a subset of* $\{\varphi_{j,k}(x)\}$

  and $$f(x)=\sum_k \alpha_k \varphi_{j,k}(x) \qquad V_j=\overline{\underset{k}{Span}\{\varphi_{j,k}(x)\}}$$

- *Figure 7.9*

  $$f(x)= 0.5\varphi_{1,0}(x)+\varphi_{1,1}(x)-0.25\varphi_{1,4}(x)$$

- *Expansion function can be decomposed as*

  $$\varphi_{0,k}(x)= 1/2^{0.5}\varphi_{1,2k}(x)+ 1/2^{0.5}\varphi_{1,2k}(x)$$

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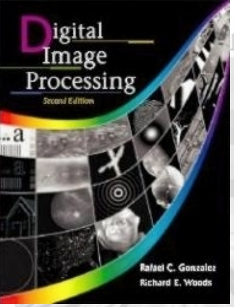
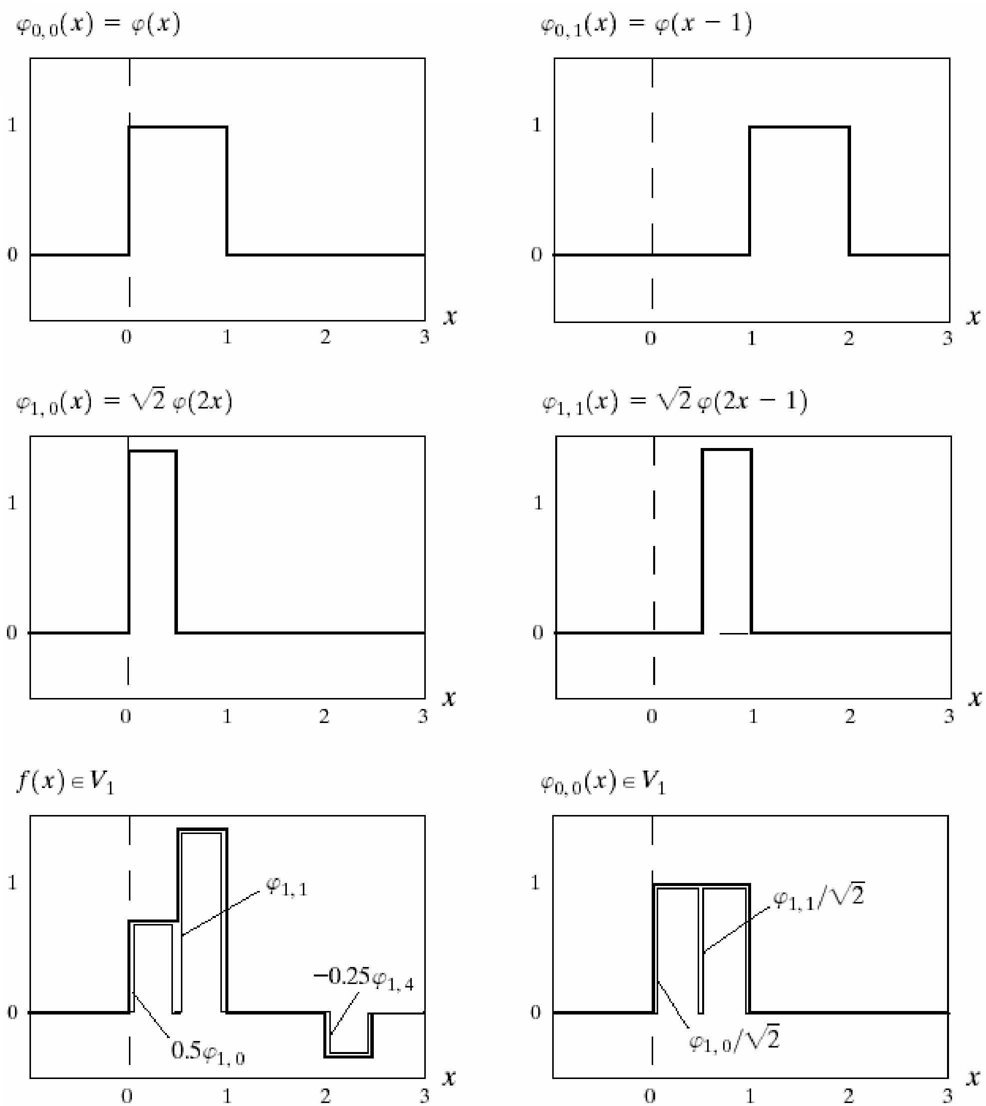


7.2 Multiresolution Expansions



a b
c d
e f

FIGURE 7.9 Haar scaling functions in V_0 in V_1.

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7.2 Multiresolution Expansions

- ***Four requirements for MRA***
- MRA requirement 1: *The scaling function is orthogonal to its integer translates*
- MRA requirement 2: *The subspace spanned by the scaling function at low scales are nested within those spanned at higher scales.*
- *Subspaces containing high resolution function must also contain all lower resolution functions*

$$V_{-\infty} \subset \ldots\ldots \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \ldots\ldots \subset V_{\infty}$$

- *If* $f(x) \in V_j$ *then* $f(2x) \in V_{j+1}$

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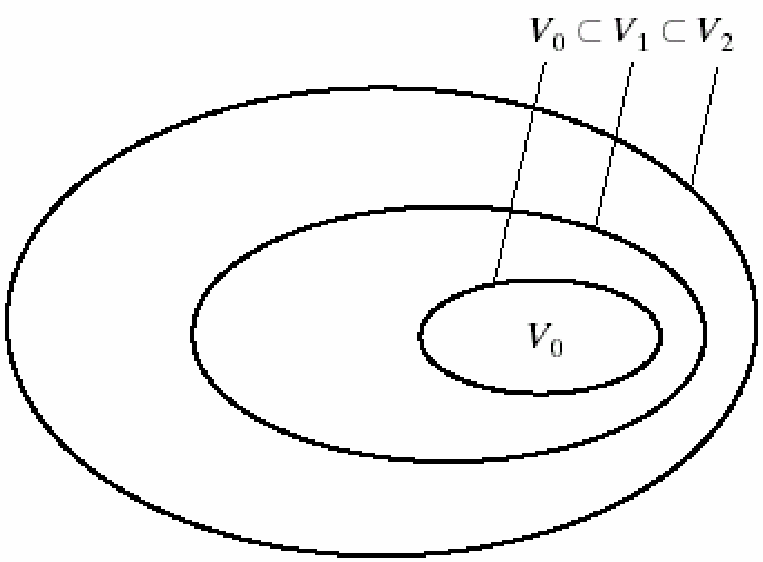
7.2 Multiresolution Expansions

FIGURE 7.10 The nested function spaces spanned by a scaling function.



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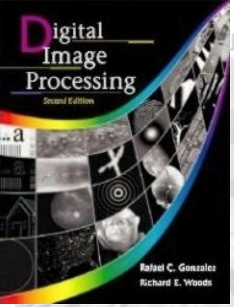

7.2 Multiresolution Expansions

- MRA requirement 3: *The only function that is common to all V_j is $f(x)=0$*
- MRA requirement 4: *Any function can be represented with arbitrary precision.*
- All measureable square-integrable functions can be represented in the limit as $j \to \infty$ as

$$V_{\infty} = \{L^2(r)\}$$

- The expansion function of subspace V_j can be expressed as a weighted sum of expansion functions in V_{j+1} space as

$$\varphi_{j,k}(x) = \Sigma_n \, \alpha_n \, \varphi_{j+1,n}(x)$$

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7.2 Multiresolution Expansions

- *Change* α_n *to* $h_\varphi(n)$

$$\varphi_{j,k}(x) = \Sigma_n h_\varphi(n) 2^{(j+1)/2} \varphi(2^{(j+1)/2} x - n)$$

- Set $j=k=0$ then $\varphi_{0,0}(x) = \varphi(x)$

$$\varphi(x) = \Sigma_n h_\varphi(n) 2^{1/2} \varphi(2x - n)$$

- It is called *the refinement equation, MRA equation, or the dilation equation.*
- $h_\varphi(n)$ *coefficients are called scaling function coefficients.*
- h_φ *is referred as scaling vector*
- *The scaling function for Harr function are* $h_\varphi(0) = h_\varphi(1) = 1/2^{1/2}$ *then* $\varphi(x) = 1/2^{1/2}[2^{1/2}\varphi(2x) + 2^{1/2}\varphi(2x-1)]$

$$= \varphi(2x) + \varphi(2x-1)$$

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7.2 Multiresolution Expansions

- ***Wavelet functions***
- *We define the set* $\{\psi_{j,k}(x)\}$ *as*

$$\psi_{j,k}(x) = 2^{j/2}\,\psi(2^j x - k)$$

- *The* W_j *space is*

$$W_j = \overline{\operatorname*{Span}_k \{\psi_{j,k}(x)\}}$$

- *If* $f(x) \in W_j$ *then* $f(x) = \Sigma_k\, \alpha_k\, \psi_{j,k}(x)$

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7.2 Multiresolution Expansions

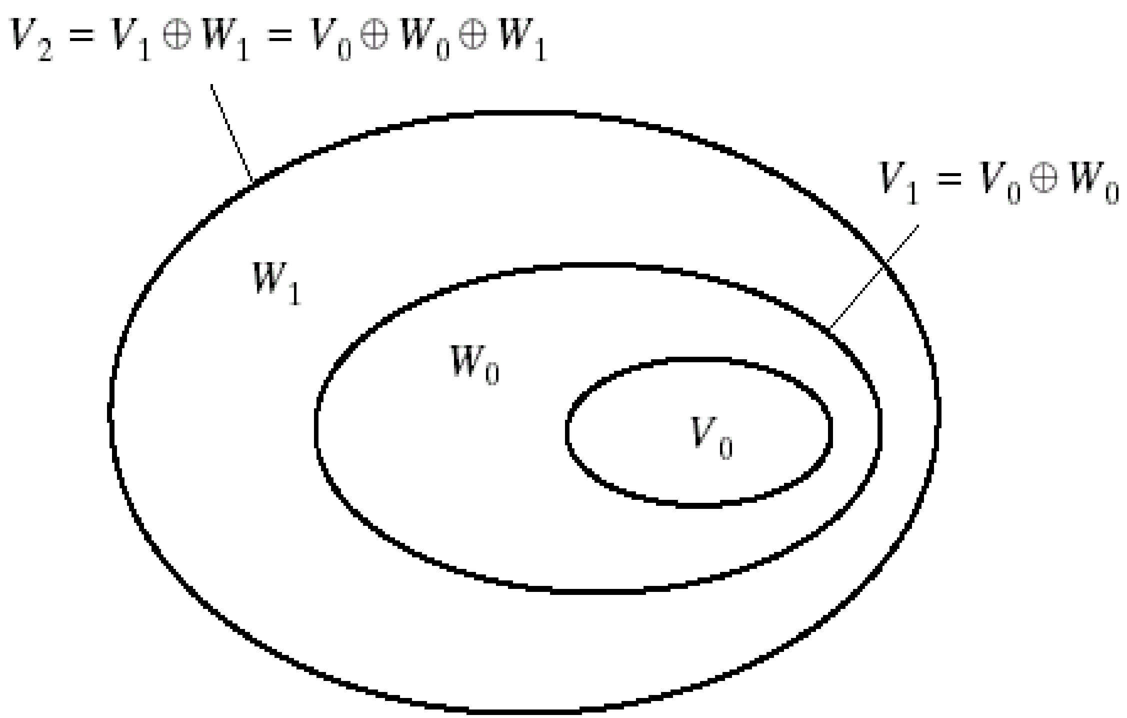

FIGURE 7.11 The relationship between scaling and wavelet function spaces.

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7.2 Multiresolution Expansions

- The scaling and wavelet subspaces in Figure 7.11 are related as $V_{j+1} = V_j \oplus W_j$

  where $\oplus$ denotes the unions of spaces.
- The orthogonal complement of V_j *in* V_{j+1} *is* W_j *and all members of* V_j *are orthogonal to the member of* W_j *, Thus*
- $<\varphi_{j,k}(x)\ \psi_{j,k}(x)> = 0$ *for all appropriate j, k,* $l \in Z$
- *We can express the space of all measurable square-integrable functions as*
- $L^2(\mathbf{R}) = V_0 \oplus W_0 \oplus W_1 \oplus \ldots$ *or* $L^2(\mathbf{R}) = V_1 \oplus W_1 \oplus W_2 \oplus \ldots..$
- *Or* $L^2(\mathbf{R}) = \ldots\ldots \oplus W_{-1} \oplus W_0 \oplus W_1 \oplus \ldots..$

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7.2 Multiresolution Expansions

- If *f(x)* is an element of V_1, but not V_0, an expansion of *f(x)* using V_0 scaling function; wavelet from W_0 would encode the difference between the approximation and the actual function. It can be generalized as

$$L^2(\mathbf{R}) = V_{j0} \oplus W_{j0} \oplus W_{j0+1} \oplus \ldots$$

where j0 is an arbitrary starting scale

- Any wavelet function can be expressed as a weighted sum of shifted double-resolution scaling functions that is $\psi(x) = \Sigma_n h_\psi(n) 2^{1/2} \varphi(2x-n)$
- *Where* $h_\psi(n)$ *is wavelet function coefficients and* h_ψ *is the wavelet vector.*

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7.2 Multiresolution Expansions

- $h_\psi(n)$ *is related to* $h_\varphi(n)$ *as*

$$h_\psi(n)=(-1)^n h_\varphi(1-n)$$

- *The Harr scaling factor illustrated as* $h_\varphi(0)=h_\varphi(1)=1/2^{1/2}$

$$h_\psi(0)=(-1)^0 h_\varphi(1-0)=1/2^{1/2}$$
$$h_\psi(1)=(-1)^1 h_\varphi(1-1)=-1/2^{1/2}$$

- *We get* $\psi(x)=\varphi(2x)-\varphi(2x-1)$

The Harr wavelet function is $$\psi(x)=\begin{cases} 1 & 0\le x<5 \\ -1 & 0.5\le x<1 \\ 0 & elsewhere \end{cases}$$

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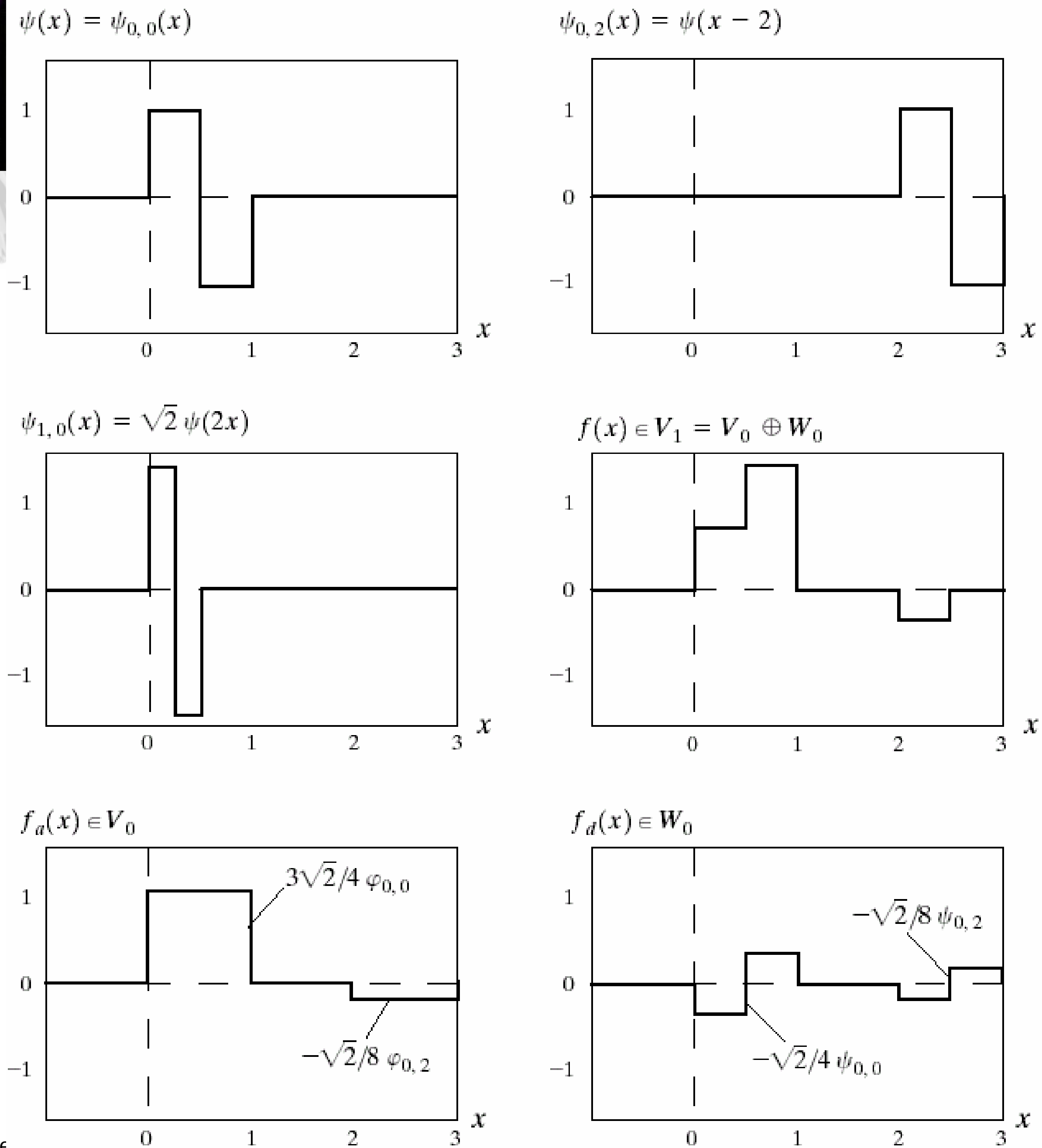

a b
c d
e f

FIGURE 7.12 Haar wavelet functions in W_0 and W_1.

7.2 Multiresolution Expansions

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7.2 Multiresolution Expansions

- A function in V_1 that is not subspace in V_0 can be expanded using V_0 and W_0 as

$$f(x)=f_a(x)+f_d(x)$$

where

$$f_a(x)=\frac{3\sqrt{2}}{4}\varphi_{0,0}(x)-\frac{\sqrt{2}}{8}\varphi_{0,2}(x)$$

$$f_d(x)=\frac{-\sqrt{2}}{4}\psi_{0,0}(x)-\frac{\sqrt{2}}{8}\psi_{0,2}(x)$$

$f_a(x)$ is an approximation of $f(x)$ using V_0 scaling function, whereas $f_d(x)$ is the difference $f(x)-f_a(x)$ as a sum of W_0 wavelets

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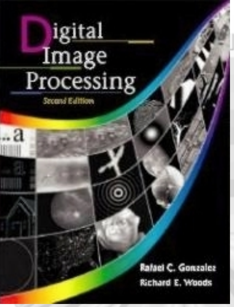

7.3 Wavelets transform in one dimension-the wavelet series expansion

- Expand *f(x)* related to wavelet $\psi(x)$ and scaling function $\varphi(x)$ *as*

$$f(x)=\sum_k c_0(k)\varphi_{0,k}(x)+\sum_{j=j0}^{\infty}\sum_k d_j(k)\psi_{j,k}(x)$$

- *Where j0 is an arbitrary starting scale and* $c_{j0}(k)$*'s and d(k) are relabeled* α*. The* $c_{j0}(k)$*'s are normally called the approximation or scaling coefficients and the* $d_j(k)$*'s are referred as the detail or wavelet coefficients*

- $$c_0(k)=\langle f(x) \quad \varphi_{0,k}(x)\rangle=\int f(x)\varphi_{0,k}(x)dx$$
$$d_j(k)=\langle f(x) \quad \psi_{j,k}(x)\rangle=\int f(x)\psi_{j,k}(x)dx$$

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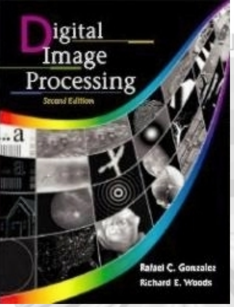
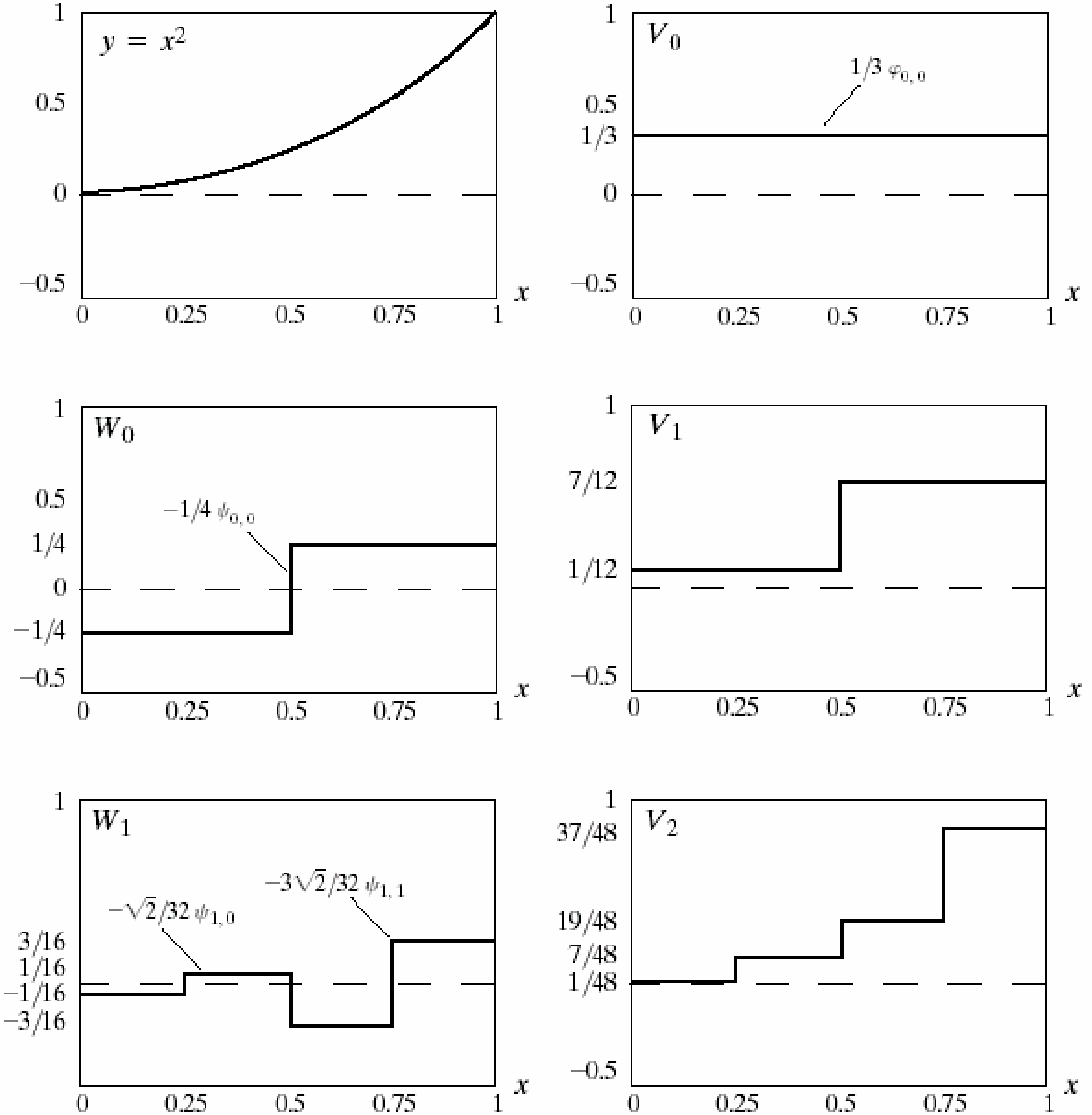


7.3 Wavelets transform in one dimension



a b
c d
e f

FIGURE 7.13 A wavelet series expansion of $y = x^2$ using Haar wavelets.

7.3 Wavelets transform in one dimension- the wavelet series expansion

- Wavelet series expansion

$$y = \frac{1}{3}\varphi_{00}(x) + \left[-\frac{1}{4}\psi_{00}(x)\right] + \left[-\frac{\sqrt{2}}{32}\psi_{10}(x) - \frac{3\sqrt{2}}{32}\psi_{11}(x)\right] + \ldots..$$

V_0 W_0 W_1

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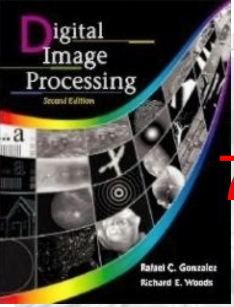

7.3 Wavelets transform in one dimension- the discrete wavelet transform

- For discrete case, the series expansion becomes

$$W_\varphi(j0,k)=\frac{1}{\sqrt{M}}\sum_x f(x)\varphi_{j0,k}(x)$$

for j≥j0

$$W_\psi(j,k)=\frac{1}{\sqrt{M}}\sum_x f(x)\psi_{j,k}(x)$$

$$f(x)=\frac{1}{\sqrt{M}}\sum_x W_\varphi(j0,k)\varphi_{j0,k}(x)+\frac{1}{\sqrt{M}}\sum_{j=j0}^{\infty}\sum_x W_\psi(j,k)\psi_{j0,k}(x)$$

- Here *f(x)*, $\varphi_{j0,k}(x)$, $\psi_{j,k}(x)$ *are functions of the discrete variable x=0,1,2,…M-1*
- For example *f(x)=f(*x_0*+x∆x) for some* x_0, *∆x, and x=0,1,2,…M-1.*
- *Normally, we select* $M=2^J$, *j=0,1,2,…J-1, k=0,1,…* 2^j*-1*

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7.3 Wavelets transform in one dimension- the discrete wavelet transform

- ***Example 7.8***
- *f(0)=1, f(1)=4, f(2)=-3, f(3)=0, M=4, J=2*, and with *j0=0*, the summation is performed over *x=0,1,2,3*, *j=0,1*, and *k=0 for j=0 or k=0, 1 for j=1.*
- Using Harr scaling and wavelet functions (the rows of $\mathbf{H}_4$)and assume that four samples of *f(x)* are distributed over the support of basis functions.

$$W_\varphi(0,0) = \frac{1}{2}\sum_x f(x)\varphi_{0,0}(x) = \frac{1}{2}[1\times1 + 4\times1 - 3\times1 + 0\times1] = 1$$

$$W_\psi(0,0) = \frac{1}{2}\sum_x f(x)\psi_{0,0}(x) = \frac{1}{2}[1\times1 + 4\times1 - 3\times(-1) + 0\times(-1)] = 4$$

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7.3 Wavelets transform in one dimension- the discrete wavelet transform

$$W_\psi(1,0) = \frac{1}{2}\sum_x f(x)\psi_{1,0}(x) = -1.5\sqrt{2}$$

$$W_\psi(1,1) = \frac{1}{2}\sum_x f(x)\psi_{1,1}(x) = -1.5\sqrt{2}$$

$$f(x) = \frac{1}{2}W_\varphi(0,0)\varphi_{0,0}(x) + W_\psi(0,1)\psi_{0,1}(x) + W_\psi(1,0)\psi_{1,0}(x) + W_\psi(1,1)\psi_{1,1}(x)$$

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7.3 Wavelets transform in one dimension- the continuous wavelet transform

- CWT: transform a continuous function into a highly redundant function of two continuous variables - translation and scale

$$W_\psi(s,\tau) = \int_{-\infty}^{\infty} f(x)\psi_{s,\tau}(x)dx$$

where

$$\psi_{s,\tau}(x) = \frac{1}{\sqrt{s}}\psi(\frac{x-\tau}{s})$$

s and τ are called the scale and translation parameters.

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7.3 Wavelets transform in one dimension- the continuous wavelet transform

- ICWT

$$f(x)=\frac{1}{C_\psi}\int_0^\infty\int_{-\infty}^{\infty}W_\psi(s,\tau)\frac{\psi_{s,\tau}(x)}{s^2}\,d\tau ds$$

where

$$C_\psi=\int_{-\infty}^{\infty}\frac{|\Psi(u)|^2}{u}du$$

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7.3 Wavelets transform in one dimension- the continuous wavelet transform

- The Mexican hat wavelet

$$\psi(x)=((2/3^{1/2})\pi^{-1/4})(1-x^2)e^{-x^2/2}$$

Figure 7.14(a) $f(x)=\psi_{1,10}(x)+\psi_{6,80}(x)$

Figure 7.14(c) shows a portion ($1\leq s\leq 10$ and $\tau\leq 100$) of the CWT of Figure 7.14(a)

- *Continuous translation* τ
- *Continuous scaling s*
- *The set of transformation coefficients* $\{W_\psi(s,\tau)\}$ *and basis functions* $\{\psi_{s,\tau}(x)\}$ *are infinite.*

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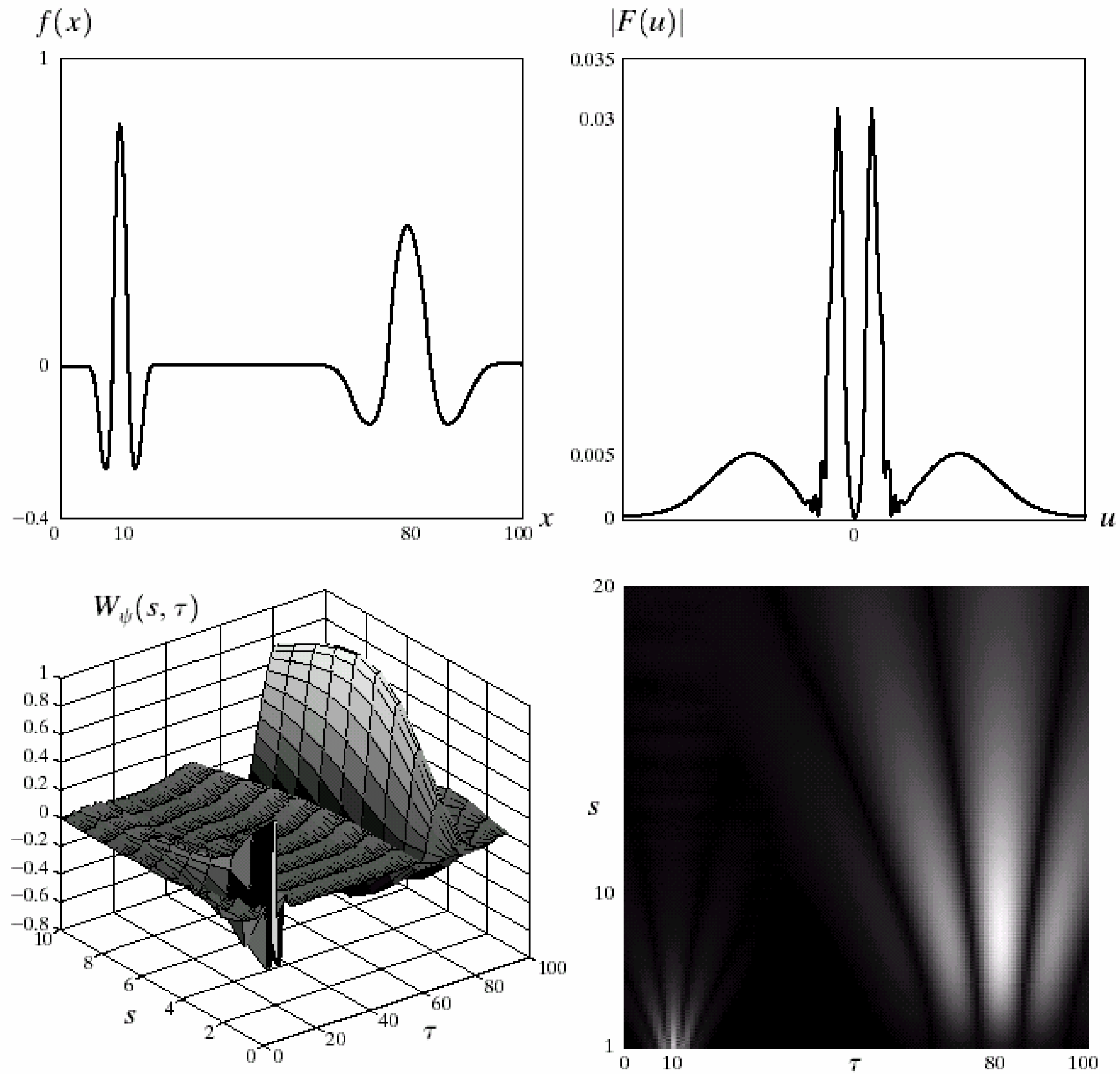
7.3 Wavelets transform in one dimension



a b
c d

FIGURE 7.14 The continuous wavelet transform (c and d) and Fourier spectrum (b) of a continuous one-dimensional function (a).

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7.4 Fast Wavelet Transform (FWT)

- *FWT* exploits a surprising but fortunate relationship between the coefficients of the *DWT* at adjacent scales *(known as Herringbone algorithm)*
- *Consider the multi-resolution refinement equation:*
  $\varphi(x)= \Sigma_n h_\varphi(n)2^{1/2}\ \varphi(2x-n)$
- *Scaling x by 2, translating by k, and letting m=2k+n then*

$$\varphi(2^j x-k)= \Sigma_n h_\varphi(n)2^{1/2}\ \varphi(2(2^j x-k)-n)$$
$$=\Sigma_n h_\varphi(m-2k)2^{1/2}\ \varphi(2^{j+1}x-m)$$

- h_φ *can be thought of as the "weights" used to expand* $\varphi(2^j x-k)$ *as sum of the scale j+1 scaling function.*

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7.4 Fast Wavelet Transform (FWT)

Similarly, for $\psi(2^j x-k)$ *we have*

$$\psi(2^j x-k) = \Sigma_n h_\psi(m-2k)2^{1/2}\, \psi(2^{j+1}x-m)$$

For DWT,

$$W_\psi(j,k) = \frac{1}{\sqrt{M}}\sum_x f(x)2^{j/2}\psi(2^j x-k)$$

Replacing $\psi(2^j x-k)$*, we have*

$$W_\psi(j,k) = \frac{1}{\sqrt{M}}\sum_x f(x)2^{j/2}\left[\sum_m h_\psi(m-2k)\sqrt{2}\varphi(2^{j+1}x-m)\right]$$

or

$$W_\psi(j,k) = \sum_m h_\psi(m-2k)\left[\frac{1}{\sqrt{M}}\sum_x f(x)2^{(j+1)/2}\sqrt{2}\varphi(2^{j+1}x-m)\right]$$

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7.4 Fast Wavelet Transform (FWT)

- where the bracketed quantity is identical to the DWT transform pair with $j0=j+1$, so

$$W_\psi(j,k)=\sum_m h_\psi(m-2k)W_\varphi(j+1,m)$$

- The DWT detail coefficients at scale j are a function of the DWT approximation coefficients at scale $j+1$.
- Similarly, we have

$$W_\varphi(j,k)=\sum_m h_\varphi(m-2k)W_\psi(j+1,m)$$

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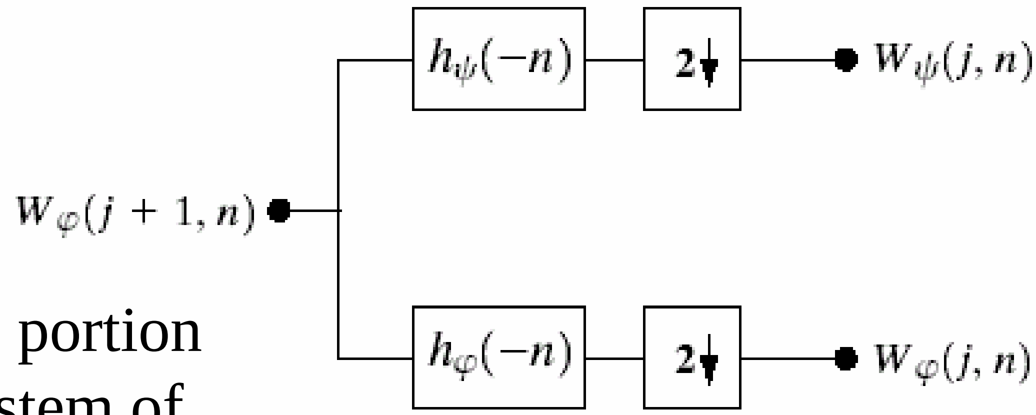
7.4 Fast Wavelet Transform

FIGURE 7.15 An FWT analysis bank.



It is identical to the analysis portion of the two-band subband system of Figure 7.4 with $h_0(n) = h_\varphi(-n)$ and $h_1(n) = h_\psi(-n)$

$$W_\psi(j,k) = h_\psi(-n) \otimes W_\psi(j+1,n)\Big|_{n=2k,k\geq 0}$$

$$W_\varphi(j,k) = h_\varphi(-n) \otimes W_\psi(j+1,n)\Big|_{n=2k,k\geq 0}$$

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7.4 Fast Wavelet Transform

- Figure 7.16 shows a two-stage filterbank for generating the coefficients at two highest scales of the transform.
- The highest level is $W_{\varphi}(J, n)=f(n)$, where J is the highest scale.
- *$W_{\varphi}(J\text{-}1, n)$ is the low-pass approximation component, and $W_{\psi}(J\text{-}1, n)$ is the high-pass detail component*
- *The second filter bank split the spectrum and the subspace V_{J-1}, the lower half-band, into quarter-band subspaces W_{J-2}, and V_{J-2}, with corresponding DWT coefficients: $W_{\varphi}(J\text{-}2, n)$ and $W_{\psi}(J\text{-}2, n)$.*

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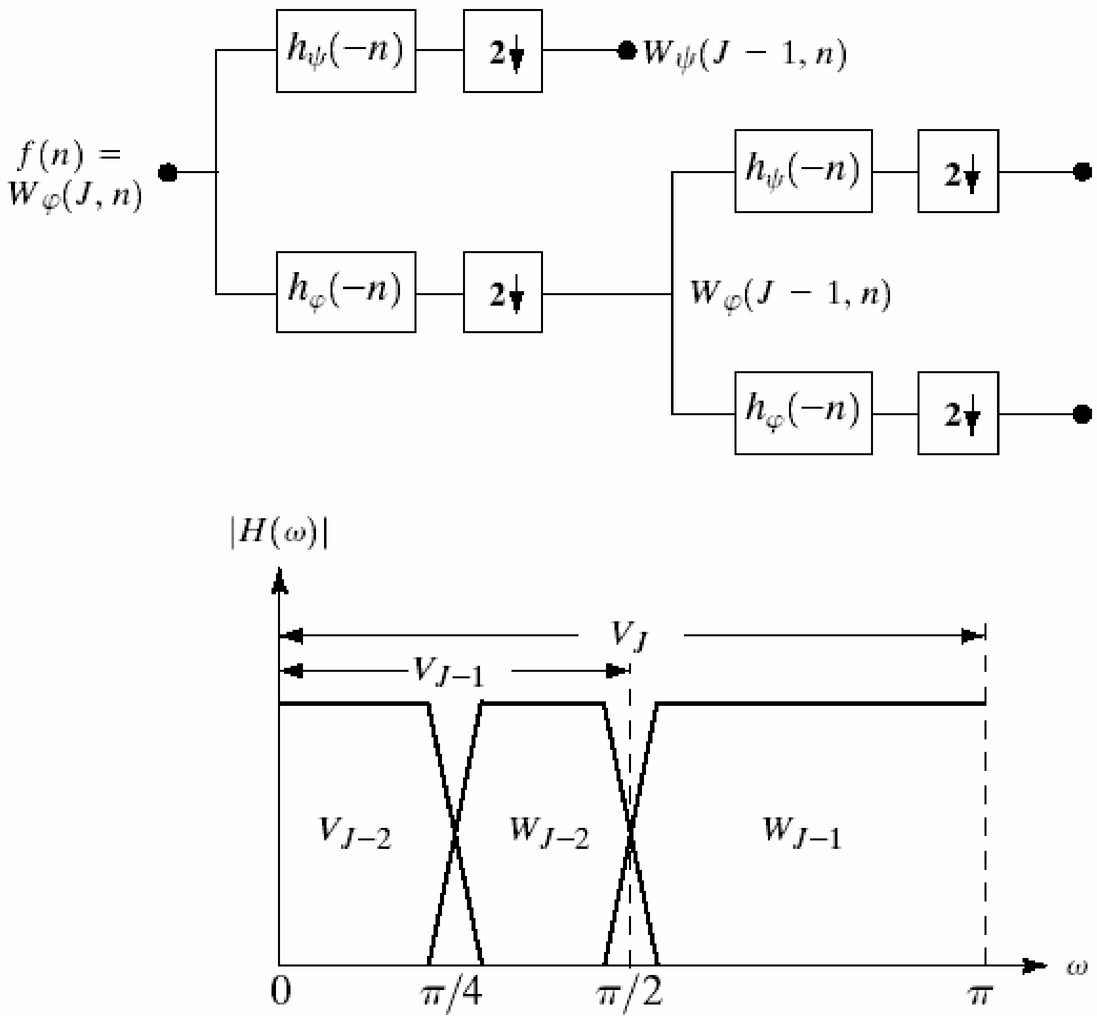
7.4 Fast Wavelet Transform



a
b

FIGURE 7.16 (a) A two-stage or two-scale FWT analysis bank and (b) its frequency splitting characteristics.

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7.4 Fast Wavelet Transform

- Consider discrete function *f(n)={1, 4, -3, 0}*
- The corresponding scaling and wavelet vectors

$$h_\varphi(n) = \begin{cases} 1/\sqrt{2} & n = 0,1 \\ 0 & otherwise \end{cases} \qquad h_\psi(n) = \begin{cases} 1/\sqrt{2} & n = 0 \\ -1/\sqrt{2} & n = 1 \\ 0 & otherwise \end{cases}$$

- $\{1, 4, -3, 0\} * \{-(½)^{0.5}, (½)^{0.5}\} = \{-(½)^{0.5}, -3(½)^{0.5}, 7(½)^{0.5}, -3(½)^{0.5}, 0\}$
- $W_\psi(1,k) = \{-3(½)^{0.5}, -3(½)^{0.5}\}$
- Or $W_\psi(1,k) = h_\psi(-n) W_\varphi(2,n)|_{n=2k,\, k\geq 0}$

$$= h_\psi(n) f(n)\,|_{n=2k,\, k\geq 0} = \Sigma_l h_\psi(2k-l) x(l)\,|_{k=0,1}$$

$$= (½)^{0.5} x(2k) - (½)^{0.5} x(2k+1)$$

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7.4 Fast Wavelet Transform

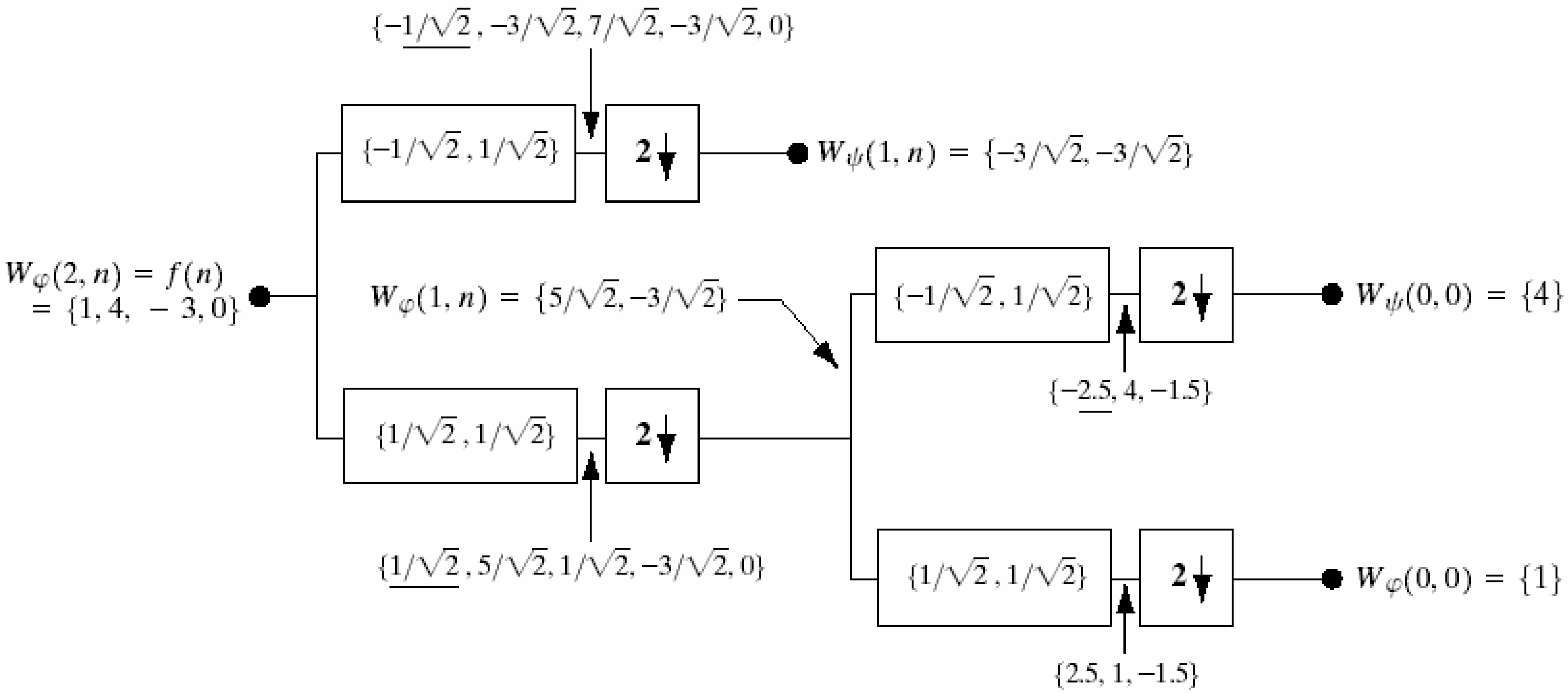

FIGURE 7.17 Computing a two-scale fast wavelet transform of sequence {1, 4, −3, 0} using Haar scaling and wavelet vectors.

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7.4 Fast Wavelet Transform- Inverse FWT

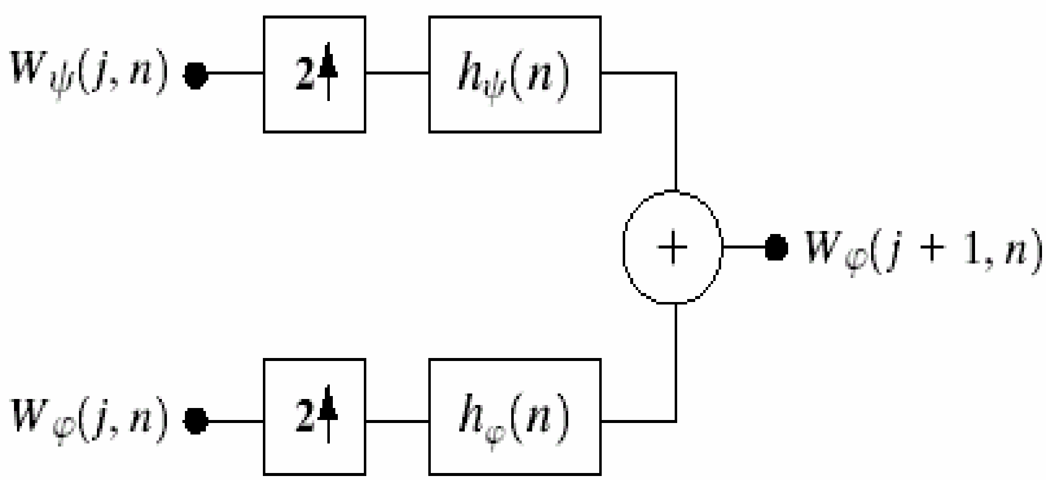

FIGURE 7.18 The FWT^{-1} synthesis filter bank.

$$W_\varphi(j+1,k)=h_\varphi(k)\otimes W_\varphi^{up}(j,k)|_{k\geq 0}+h_\psi(k)\otimes W_\psi^{up}(j,k)|_{k\geq 0}$$

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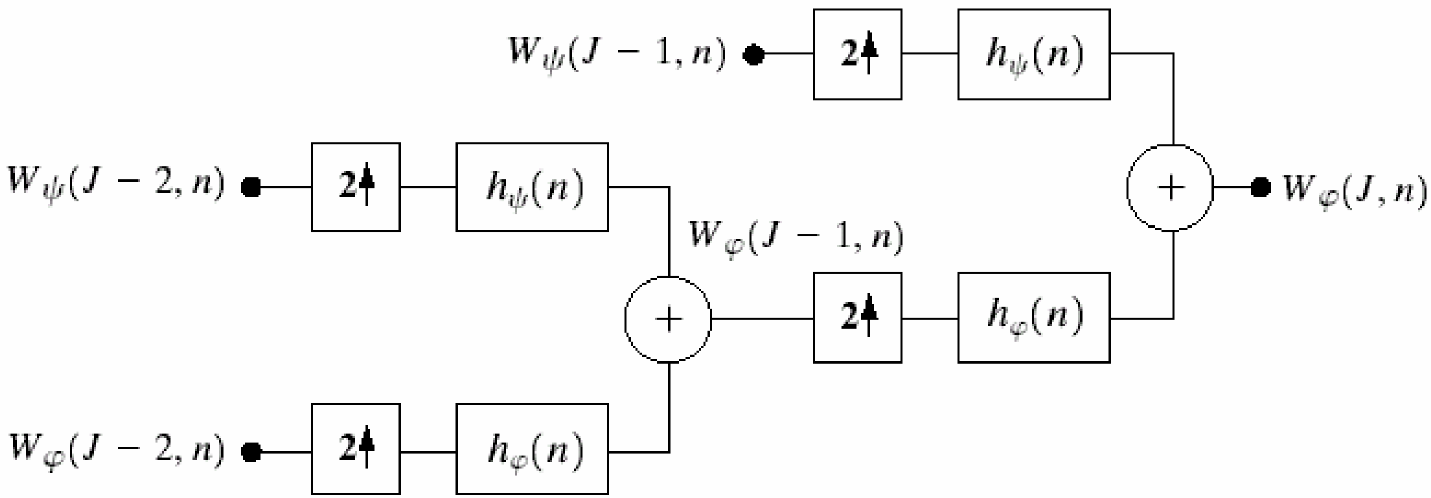
7.4 Fast Wavelet Transform- Inverse FWT

FIGURE 7.19 A two-stage or two-scale FWT^{-1} synthesis bank.



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7.4 Fast Wavelet Transform- Inverse FWT

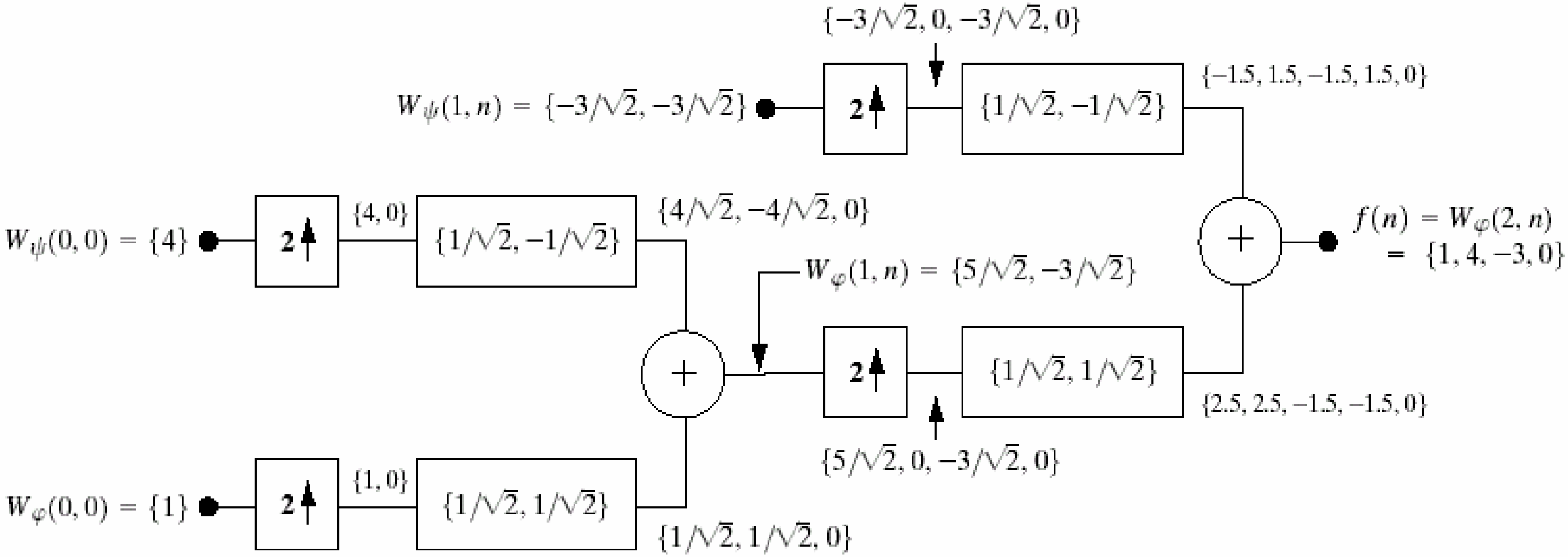

FIGURE 7.20 Computing a two-scale inverse fast wavelet transform of sequence $\{1, 4, -1.5\sqrt{2}, -1.5\sqrt{2}\}$ with Haar scaling and wavelet vectors.

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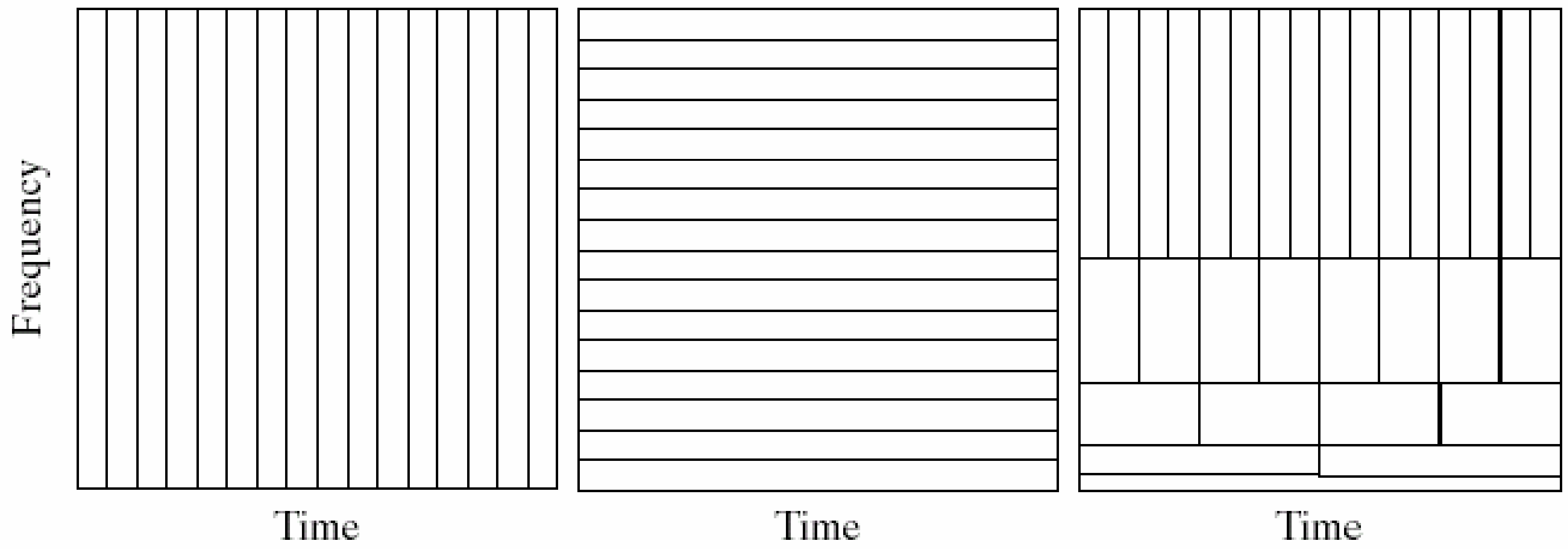
7.4 Fast Wavelet Transform- Inverse FWT



a b c

FIGURE 7.21 Time-frequency tilings for (a) sampled data, (b) FFT, and (c) FWT basis functions.

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7.5 Wavelet transform in Two Dimension

- *2-D scaling function:*

$$\varphi(x, y) = \varphi(x)\varphi(y)$$

- *2-D wavelets:*

$$\psi^H(x, y) = \psi(x)\ \varphi(y)$$

$$\psi^V(x, y) = \varphi(x)\ \psi(y)$$

$$\psi^D(x, y) = \psi(x)\ \psi(y)$$

- Translated and scaled basis functions:

$$\varphi_{j,m,n}(x, y) = 2^{j/2}\ \varphi(2^j x - m,\ 2^j y - n)$$

$$\psi^i_{j,m,n}(x, y) = 2^{j/2}\ \psi\,(2^j x - m,\ 2^j y - n),\quad i = \{H, V, D\}$$

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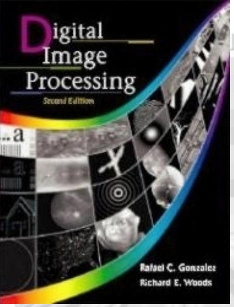

7.5 Wavelet transform in Two Dimension

- The DWT of function *f(x,y)* of size M×N is

$$W_\varphi(j_0, m, n) = \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \varphi_{j0,m,n}(x, y)$$

$$W_\psi(j_0, m, n) = \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \psi^i_{j0,m,n}(x, y)$$

where *i={H, V, D}*

Normally we let j0=0 and M=N=2^J*, j=0,1,2,…J-1*

Inverse DWT

$$f(x, y) = \frac{1}{\sqrt{MN}} \sum_m \sum_n W_\varphi(j0, m, n) \varphi_{j0,m,n} + \frac{1}{\sqrt{MN}} \sum_{i=H,V,D} \sum_m \sum_n W_\varphi(j0, m, n) \psi^i_{j0,m,n}$$

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7.5 Wavelet transform in Two Dimension

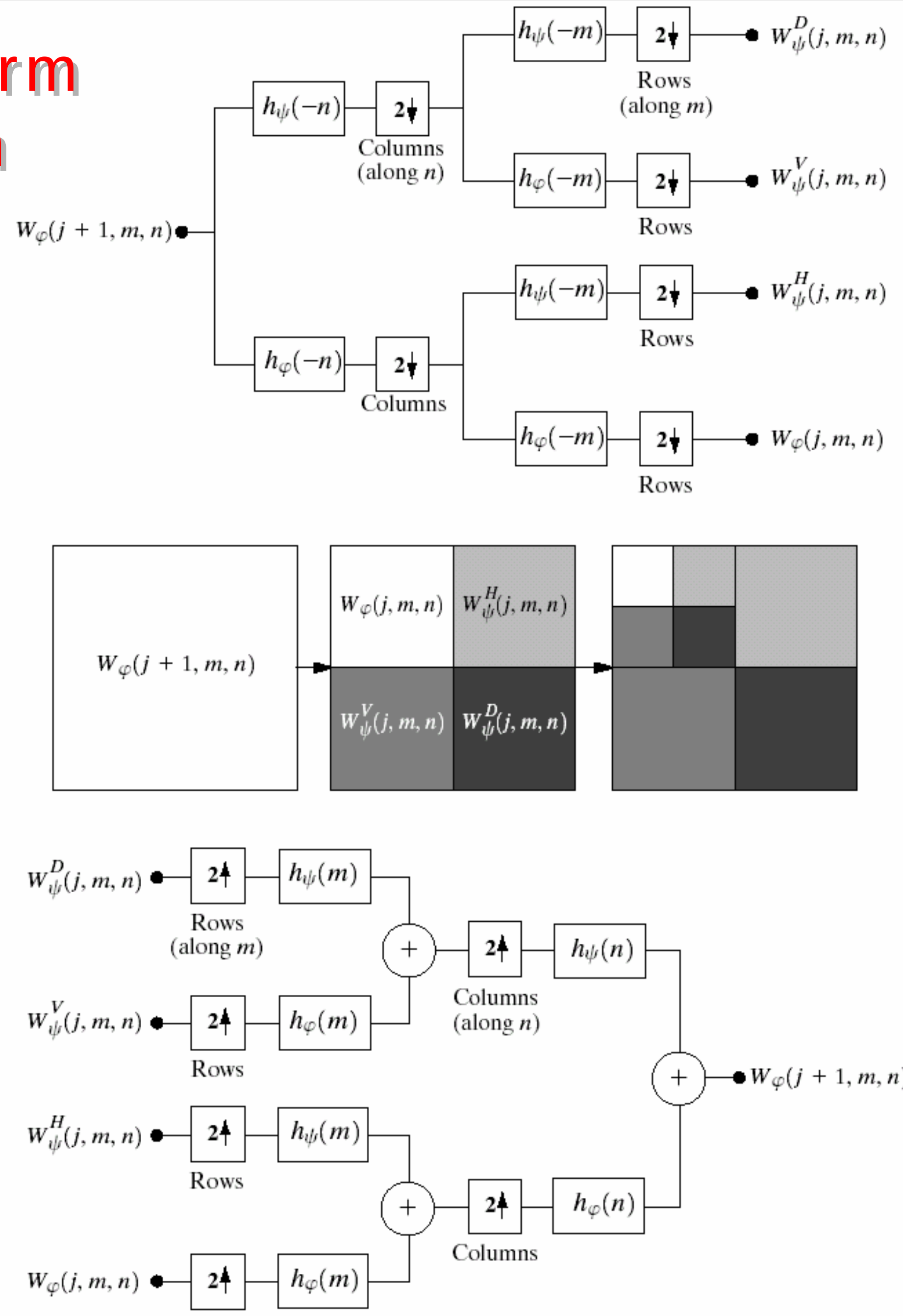

Figure 7.22 the 2-D fast wavelet transform (a) the analysis filter bank; (b) the results of decomposition; (c) the synthesis filter bank

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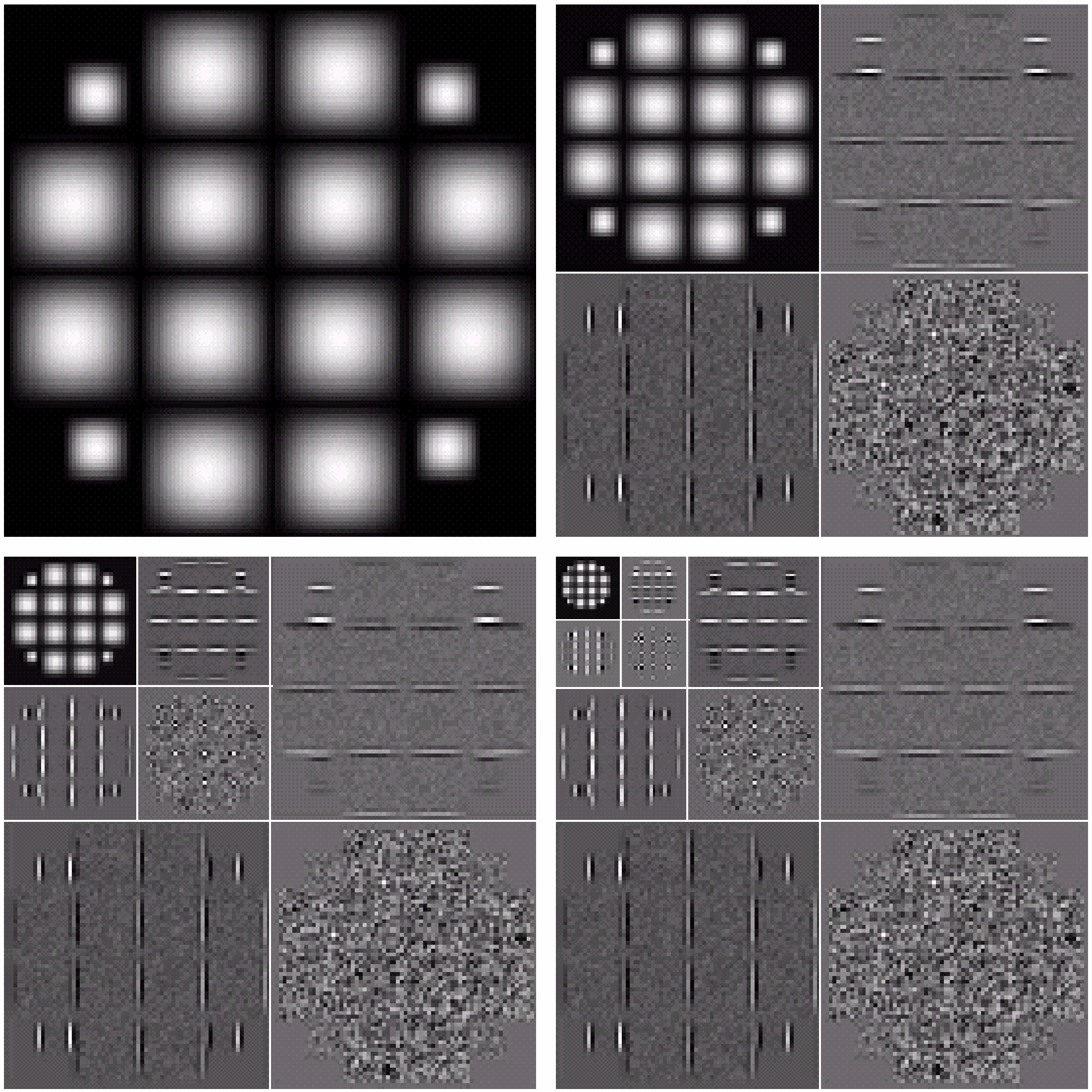
7.5 Wavelet transform in Two Dimension



a b
c d

FIGURE 7.23 A three-scale FWT.

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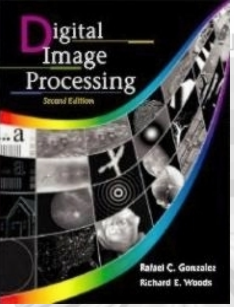

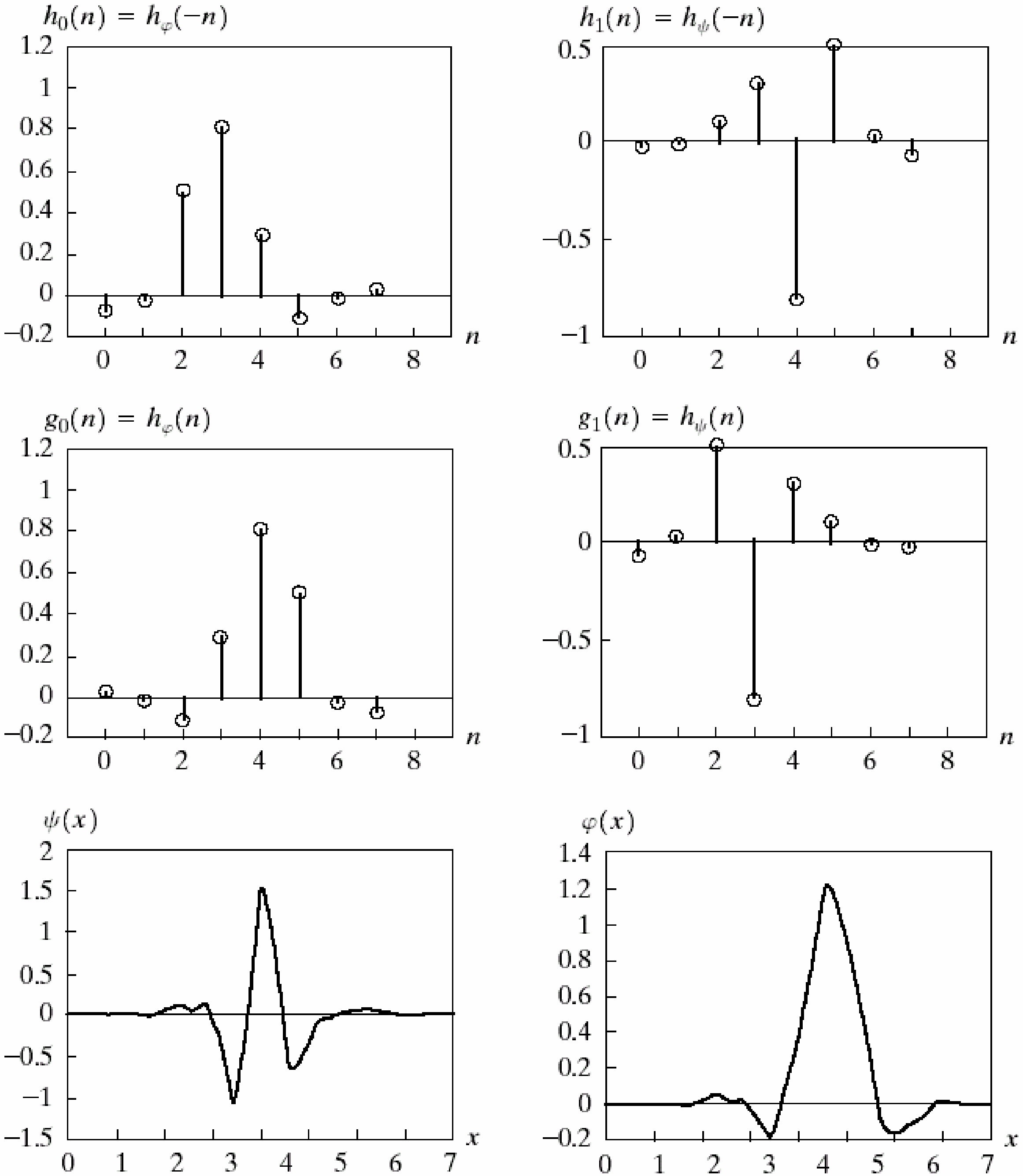

a b
c d
e f
g

FIGURE 7.24 Fourth-order symlets: (a)–(b) decomposition filters; (c)–(d) reconstruction filters; (e) the one-dimensional wavelet; (f) the one-dimensional scaling function; and (g) one of three two-dimensional wavelets, $\psi^H(x, y)$.

7.5 Wavelet transform in Two Dimension

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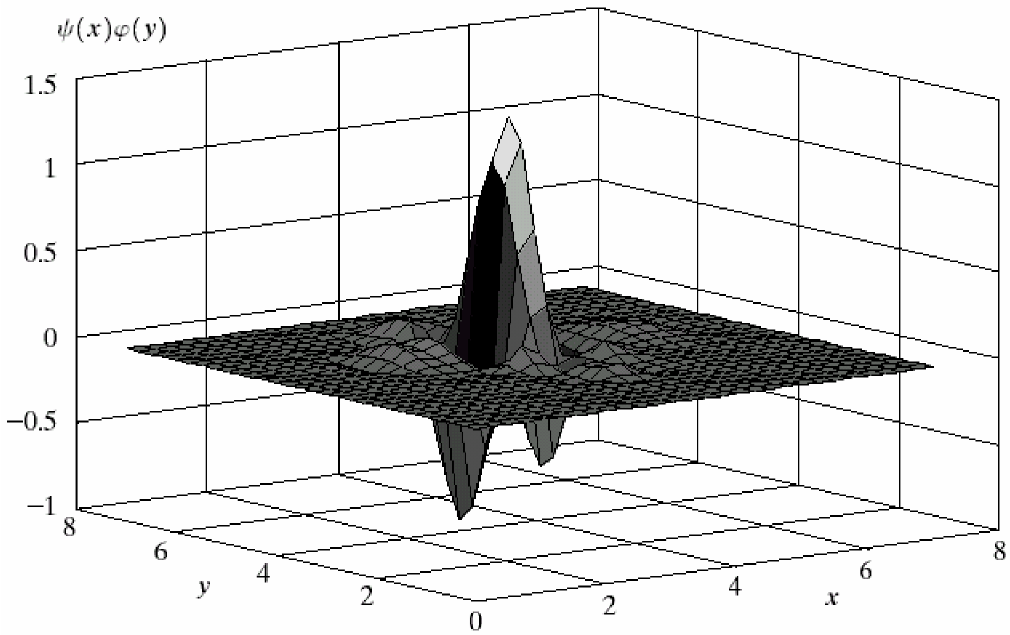
7.5 Wavelet Transform in Two Dimension

Fig. 7.24 (Con't)



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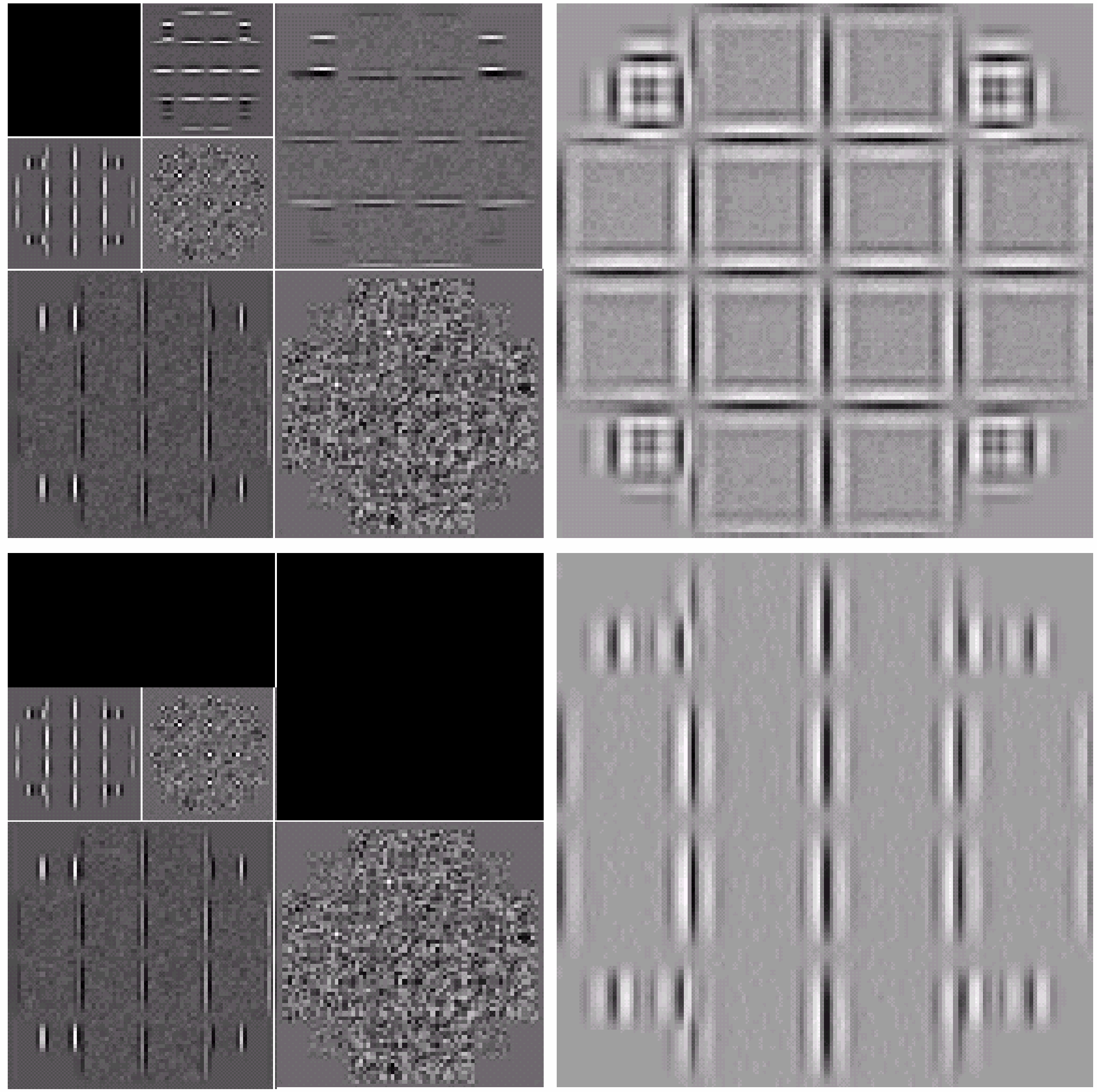
7.5 Wavelet Transform in Two Dimension



a b
c d

FIGURE 7.25 Modifying a DWT for edge detection: (a) and (c) two-scale decompositions with selected coefficients deleted; (b) and (d) the corresponding reconstructions.

DWT for edge detection

(a) Emphasize the reconstructed image edge

(b) Isolate the vertical edge

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7.5 Wavelet Transform in Two Dimension

- The general wavelet-based for de-noising the image.
- Step 1: choose a wavelet and number of levels or scales, p, for decomposition, then compute FWT.
- Step 2: Thresholding the detail coefficients. Select and apply a threshold to the detail coefficients from scales J-1 to J-P. (a) Hard thresholding; (b) Soft thresholding, first hard-thresholding and then followed by scaling the nonzero coefficients toward zero to eliminate the discontinuity at the threshold.
- Step 3: Perform a wavelet reconstruction based on the original approximation coefficients at level J-P and the modified detail coefficients for level from J-1 to J-P.

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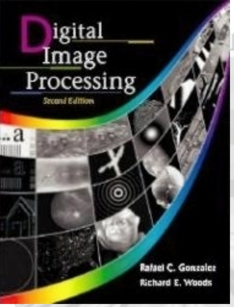

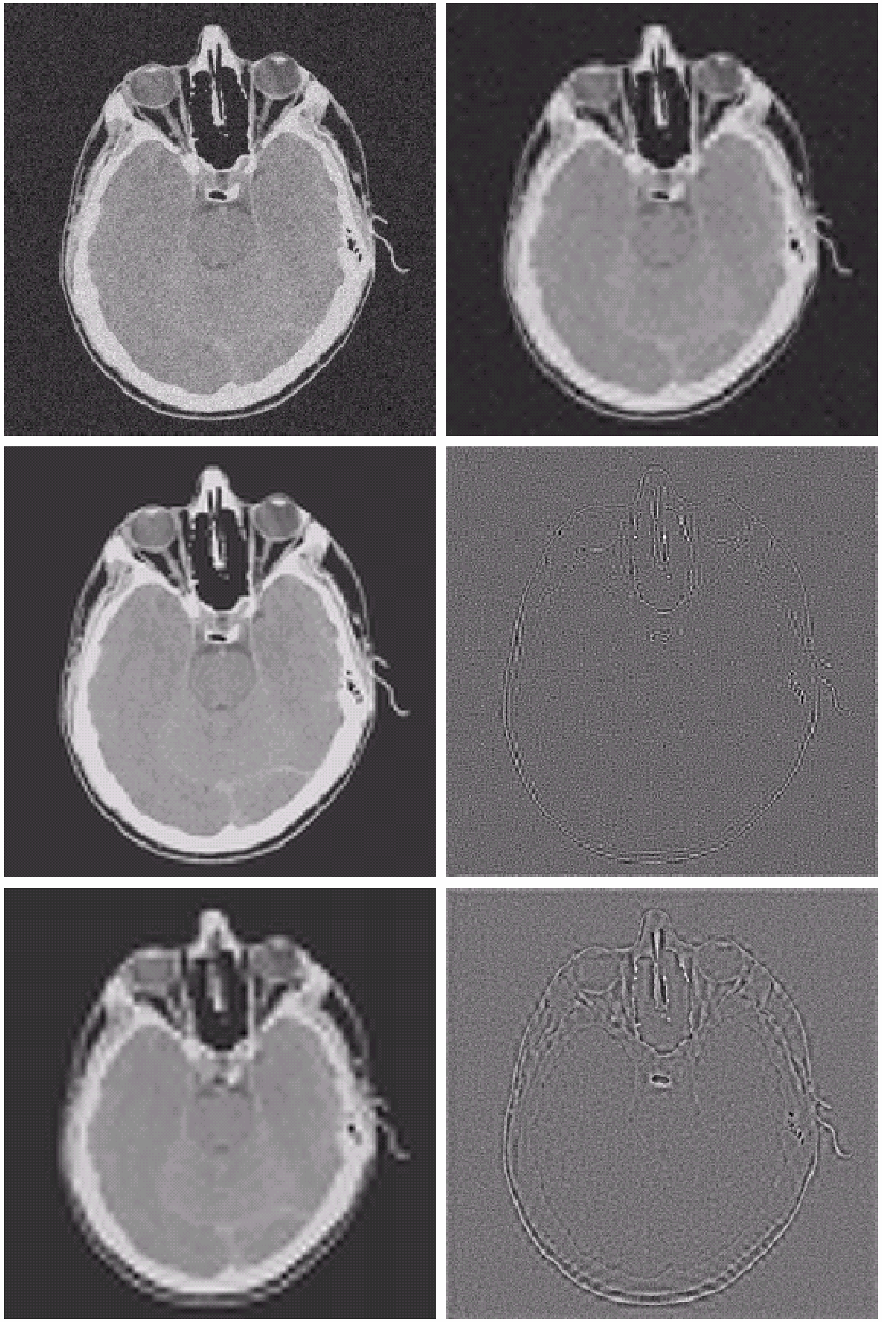

a b
c d
e f

FIGURE 7.26 Modifying a DWT for noise removal: (a) a noisy MRI of a human head; (b), (c) and (e) various reconstructions after thresholding the detail coefficients; (d) and (f) the information removed during the reconstruction of (c) and (e). (Original image courtesy Vanderbilt University Medical Center.)

7.5 Wavelet Transform in Two Dimension

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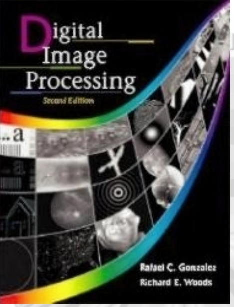

7.6 Wavelet packet

- DWT: the low frequencies are group into narrow bands, while the high frequencies are grouped into wider bands - constant-Q filters.
- ***Wavelet packet*** – a Generalized DWT, in which the details are teratively filtered and separated.
- Increased computation complexity raised from *O(M)* to *O(MlogM)*

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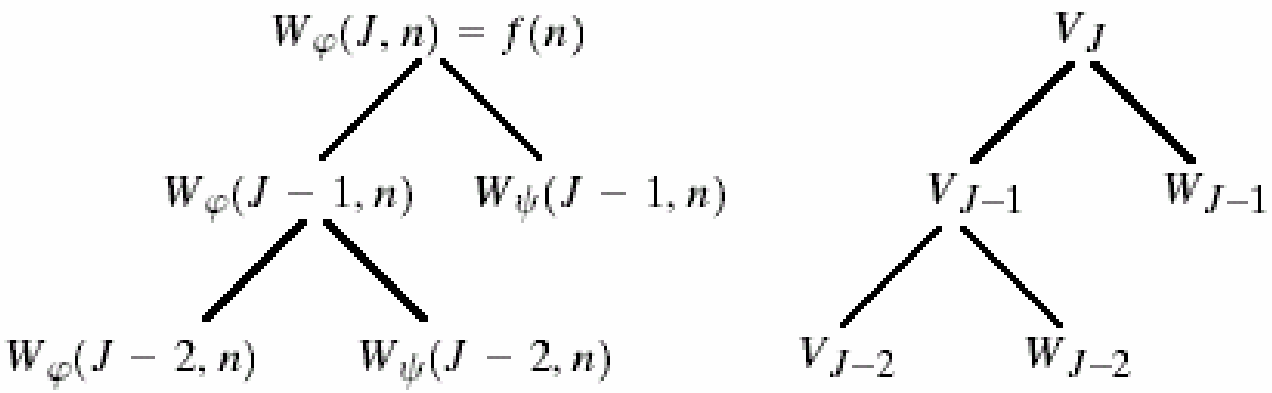
7.6 Wavelet packet



a b

FIGURE 7.27 A coefficient (a) and analysis (b) tree for the two-scale FWT analysis bank of Fig. 7.16.

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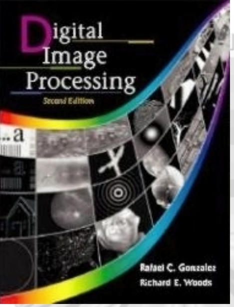
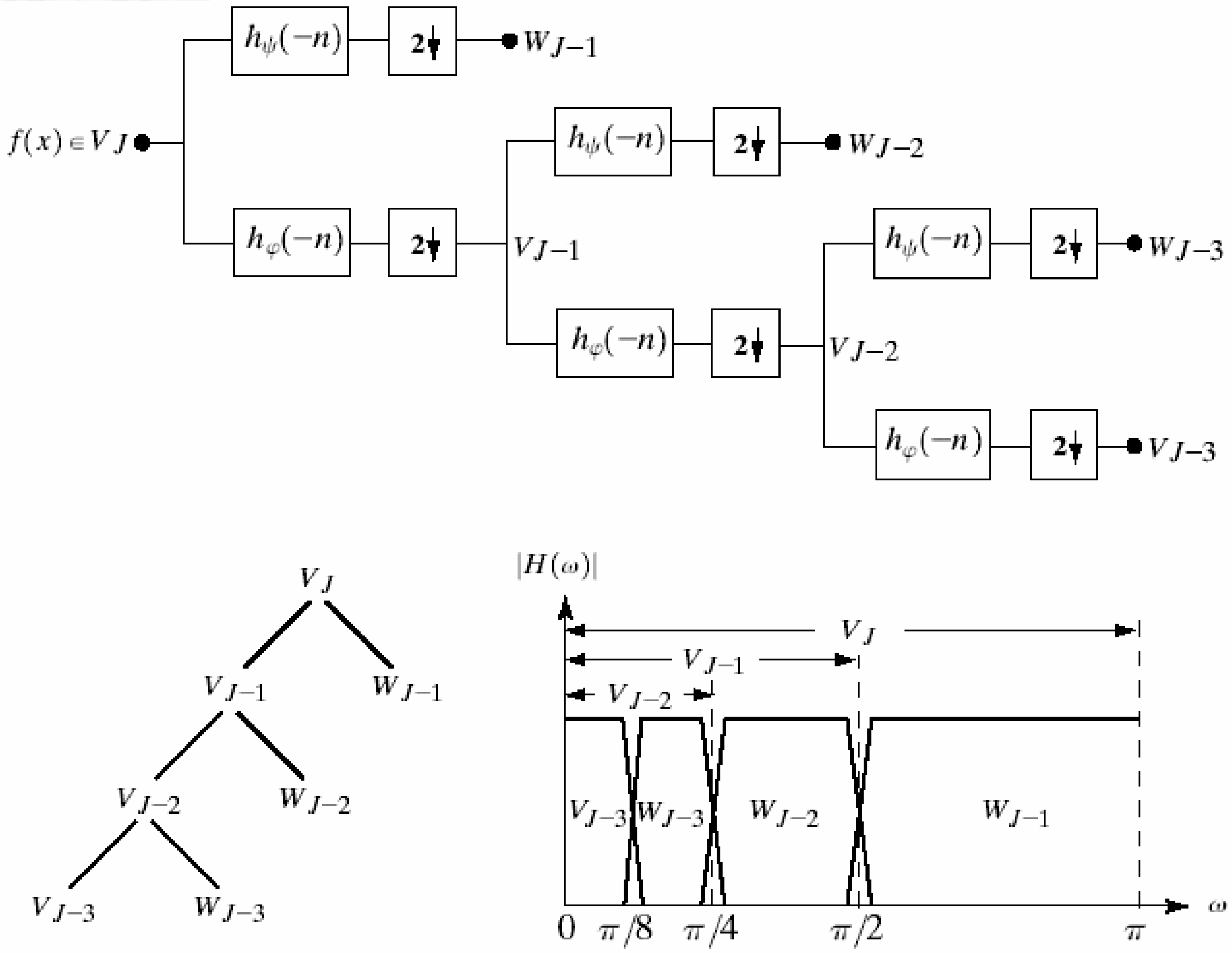


7.6 Wavelet packet



a
b c

FIGURE 7.28 A three-scale FWT filter bank: (a) block diagram; (b) decomposition space tree; and (c) spectrum splitting characteristics.

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7.6 Wavelet packet

- Double subscripts are introduced for Wavelet packet
- The first identifies the scale of the FWT parent node, and the second is a variable length string of A's and D's.

  A indicates the approximation filter.

  D indicates the detail filter.
- This creates a fixed logarithmic relationship between the frequency bands.
- In general, P scale, 1-D wavelet packet transform support $D(P+1)=[D(P)]^2+1$ unique decompositions, where $D(1)=1$.

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7.6 Wavelet packet

$$V_J = V_{J-3} \oplus W_{J-3} \oplus W_{J-2,A} \oplus W_{J-2,D} \oplus W_{J-1,AA} \oplus W_{J-1,AD} \oplus W_{J-1,DA} \oplus W_{J-1,DD}$$

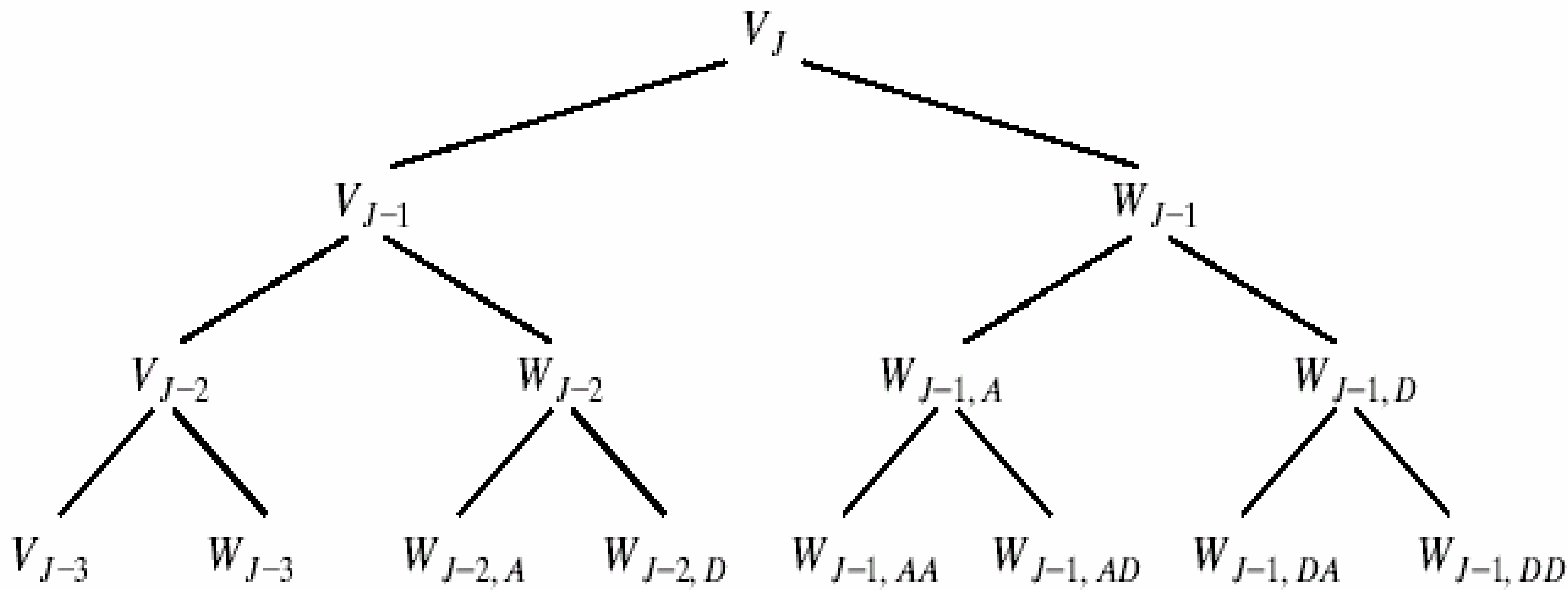

FIGURE 7.29 A three-scale wavelet packet analysis tree.

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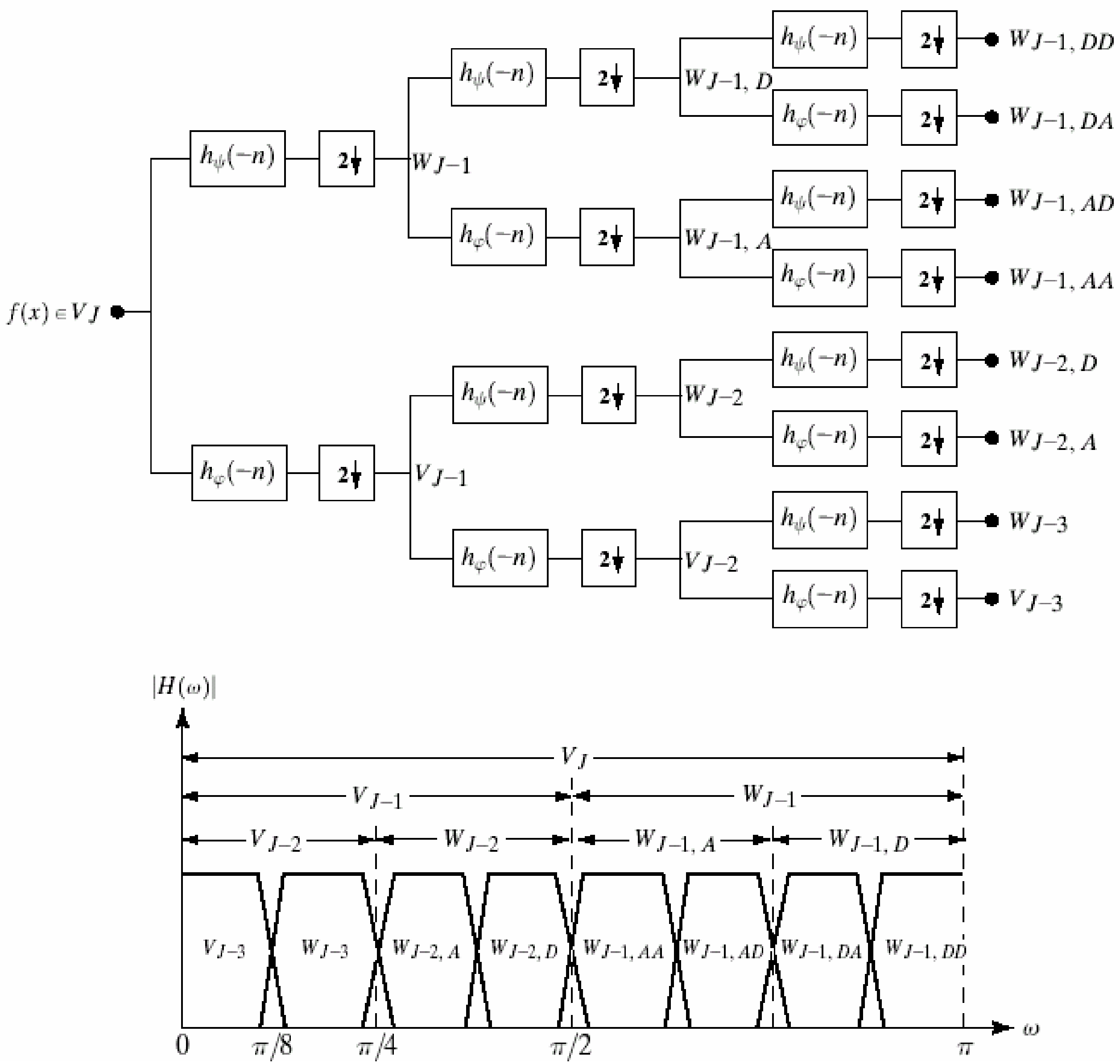

a
b

FIGURE 7.30 The (a) filter bank and (b) spectrum splitting characteristics of a three-scale full wavelet packet analysis tree.

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7.6 Wavelet packet

$$V_J = V_{J-1} \oplus W_{J-1,D} \oplus W_{J-1,AA} \oplus W_{J-1,AD}$$

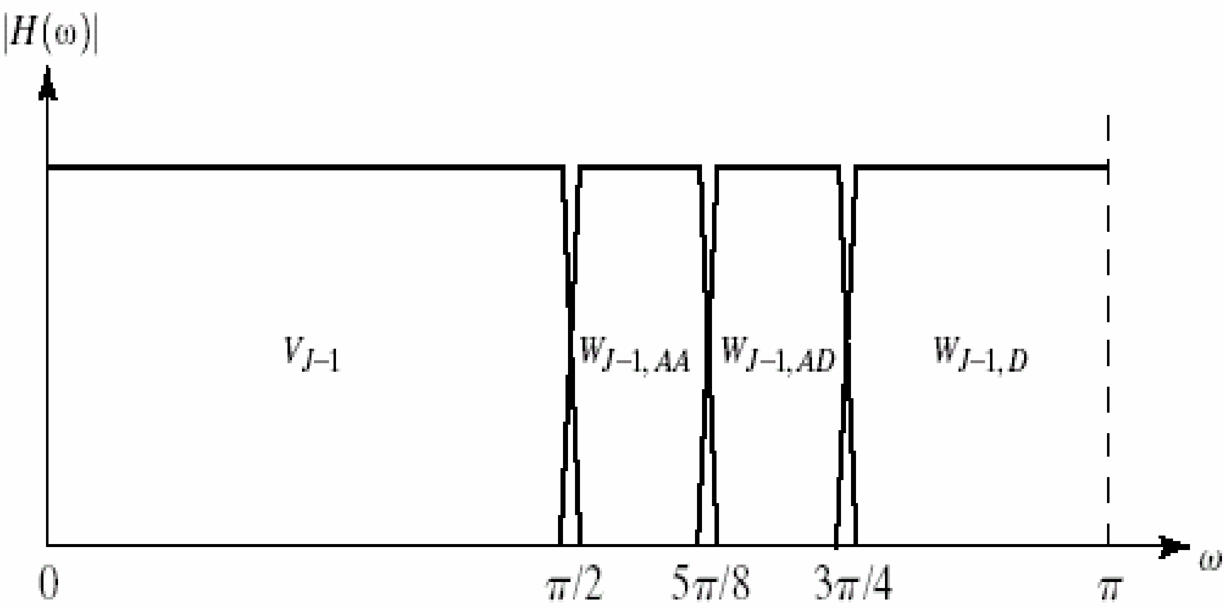

FIGURE 7.31 The spectrum of the decomposition in Eq. (7.6-5).

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7.6 Wavelet packet

- The spectrum resulting from th first iteration (j+1=J) is shown in Figure 7.32
- It divides the frequency plane into four equal areas.
- Figure 7.33, a three-scale full wavelet packet decomposition.
- In general, A P-scale, 2-D wavelet packet transform support $D(P+1)=[D(P)]^4+1$ unique decompositions, where $D(1)=1$
- *Three-scale tree offer 83522 possible decompositions*

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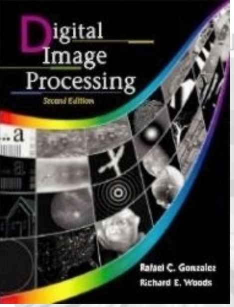

7.6 Wavelet packet

a b

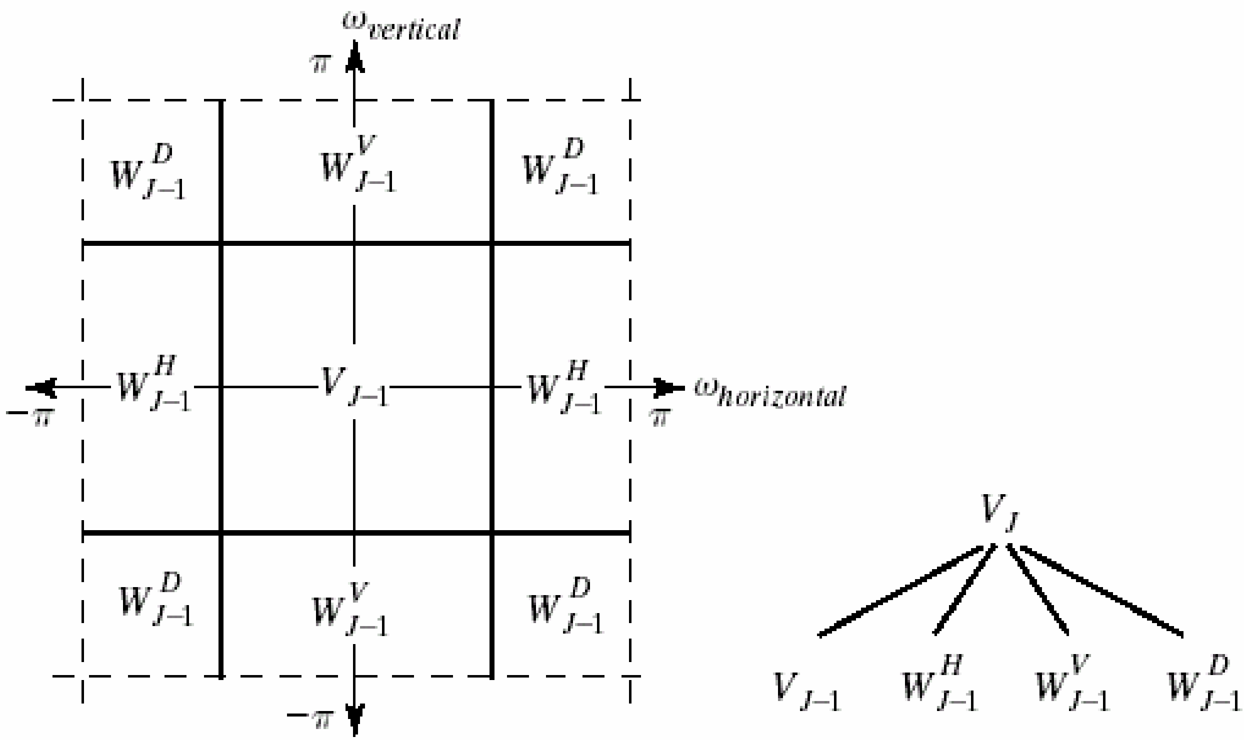

FIGURE 7.32 The first decomposition of a two-dimensional FWT: (a) the spectrum and (b) the subspace analysis tree.

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7.6 Wavelet packet

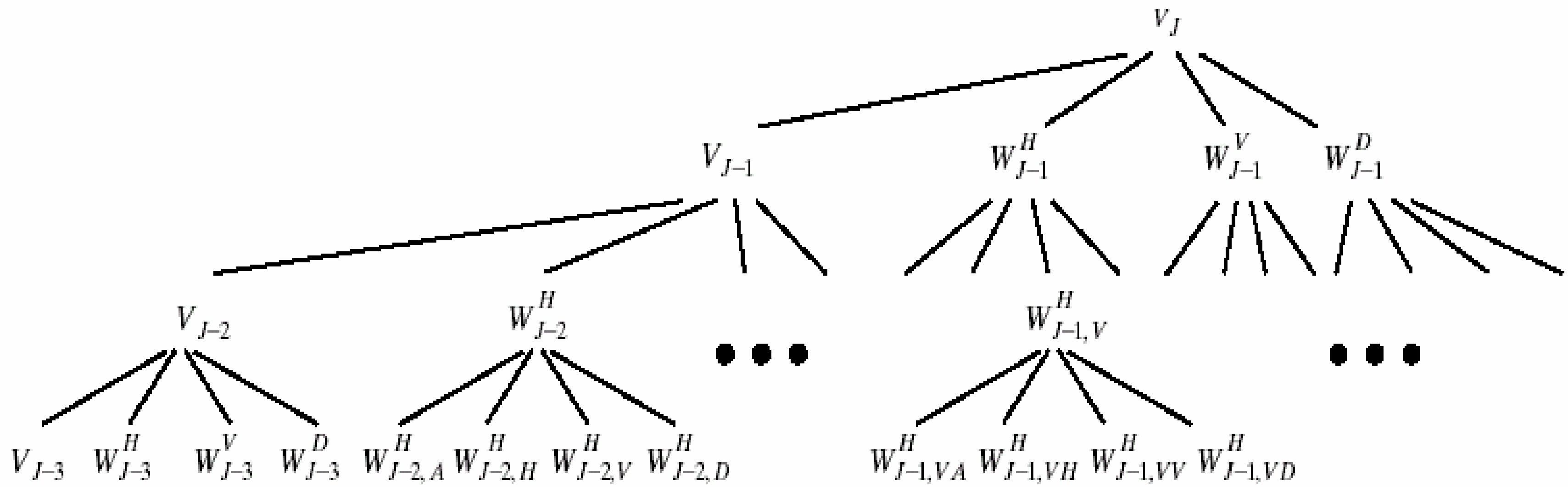

FIGURE 7.33 A three-scale, full wavelet packet decomposition tree. Only a portion of the tree is provided.

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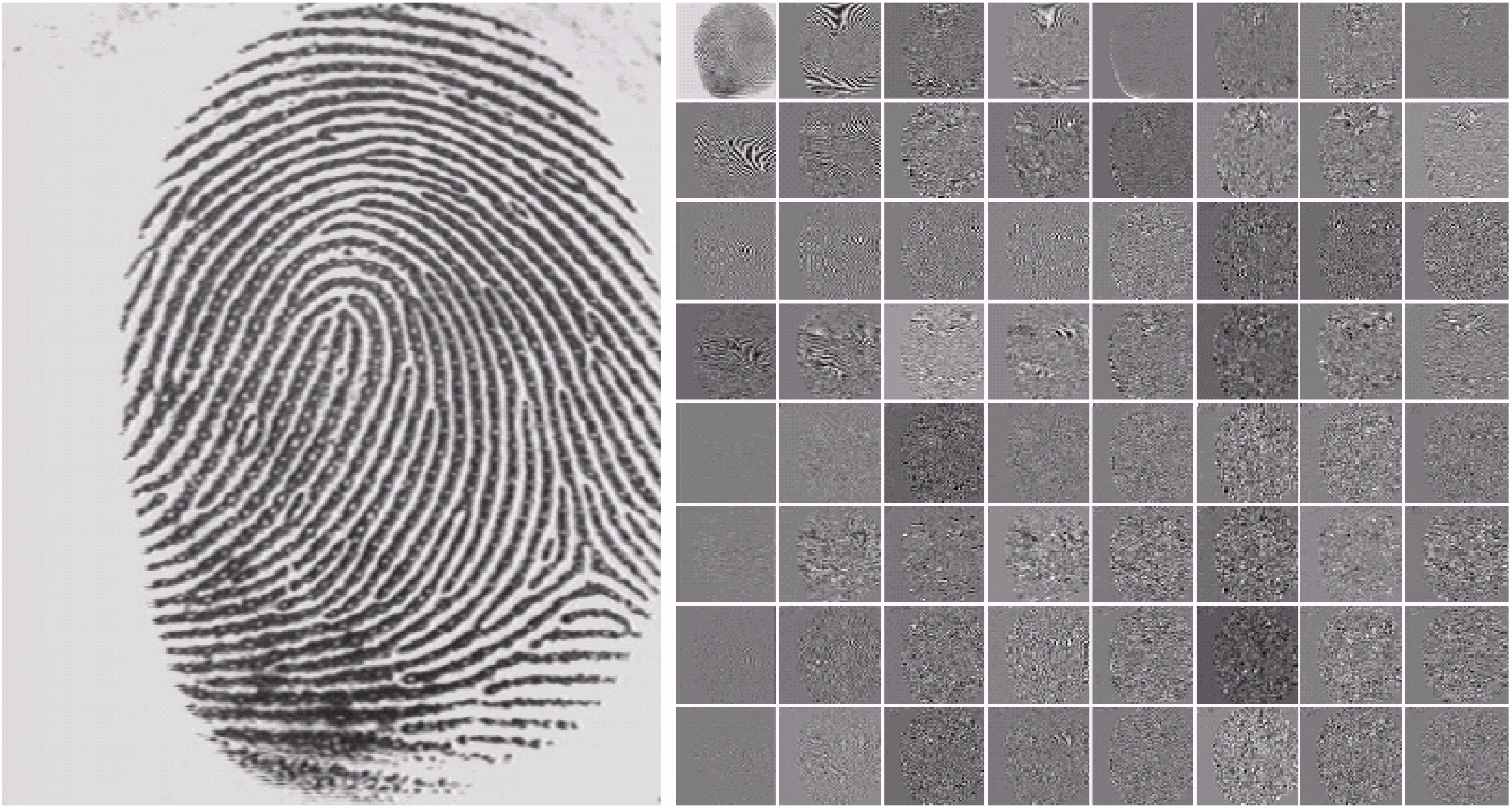

a b

FIGURE 7.34 (a) A scanned fingerprint and (b) its three-scale, full wavelet packet decomposition. (Original image courtesy of the National Institute of Standards and Technology.)

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7.6 Wavelet packet

- To select a optimal decompositions for the compression of the image is considering the cost function $E(f)=\Sigma\Sigma|f(x,y)|$, which measures the entropy or information content of 2-D function f.
- Minimal entropy leaf nodes should be favored because they have more near-zero values that lead to greater compression.
- The cost function E(f) can be used as a local measurement for the node under consideration.

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7.6 Wavelet packet

- For each node of the analysis tree

  1) compute both the entropy of the node E_P (parent entropy) and the entropy of the four offsprings, E_A , E_H , E_V , E_D .

  2) Compare E_p with $E_A + E_H + E_V + E_D$, If the combined entropy is greater than the entrop of the parent than prune the offspring, keep only the parent.

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7.6 Wavelet packet

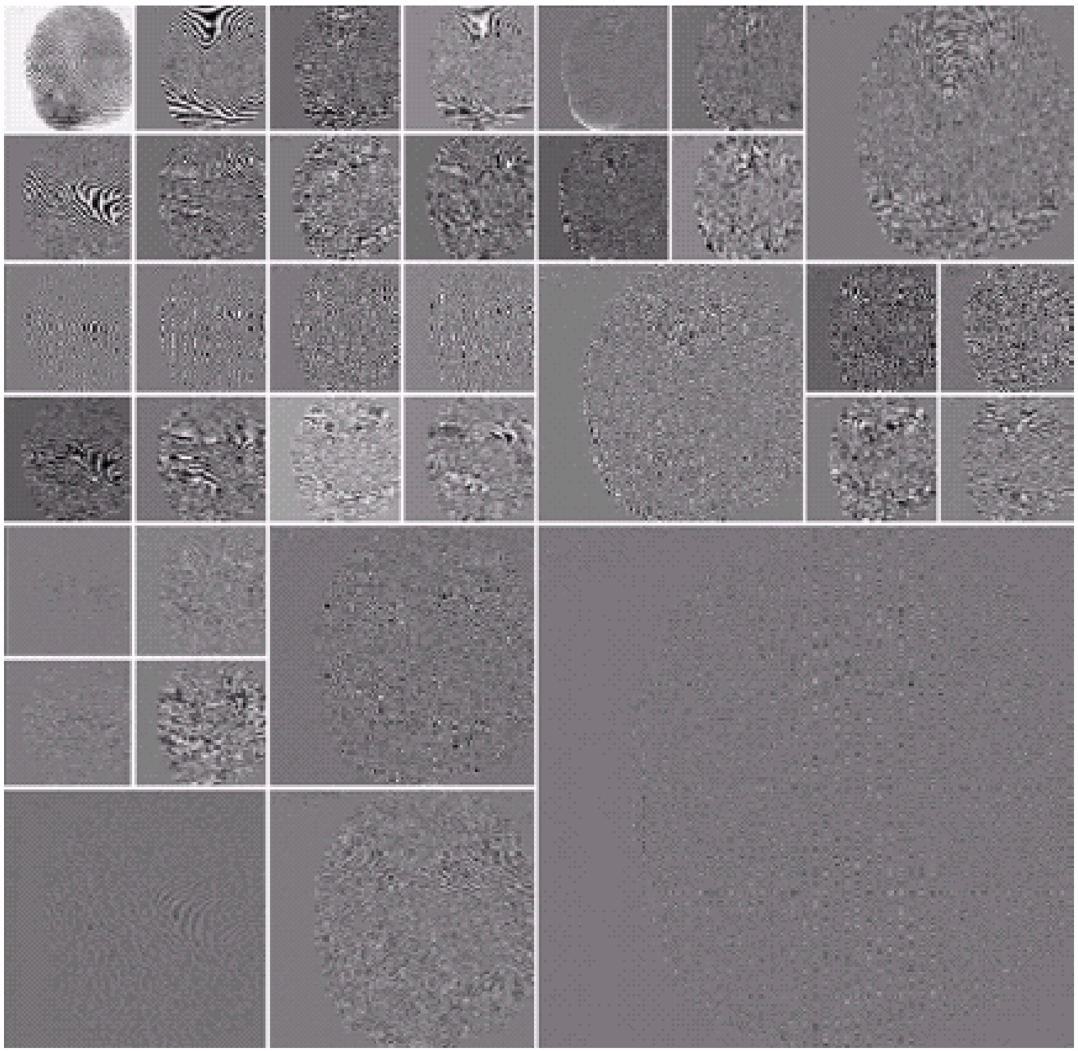

FIGURE 7.35 An optimal wavelet packet decomposition for the fingerprint of Fig. 7.34(a).

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7.6 Wavelet packet

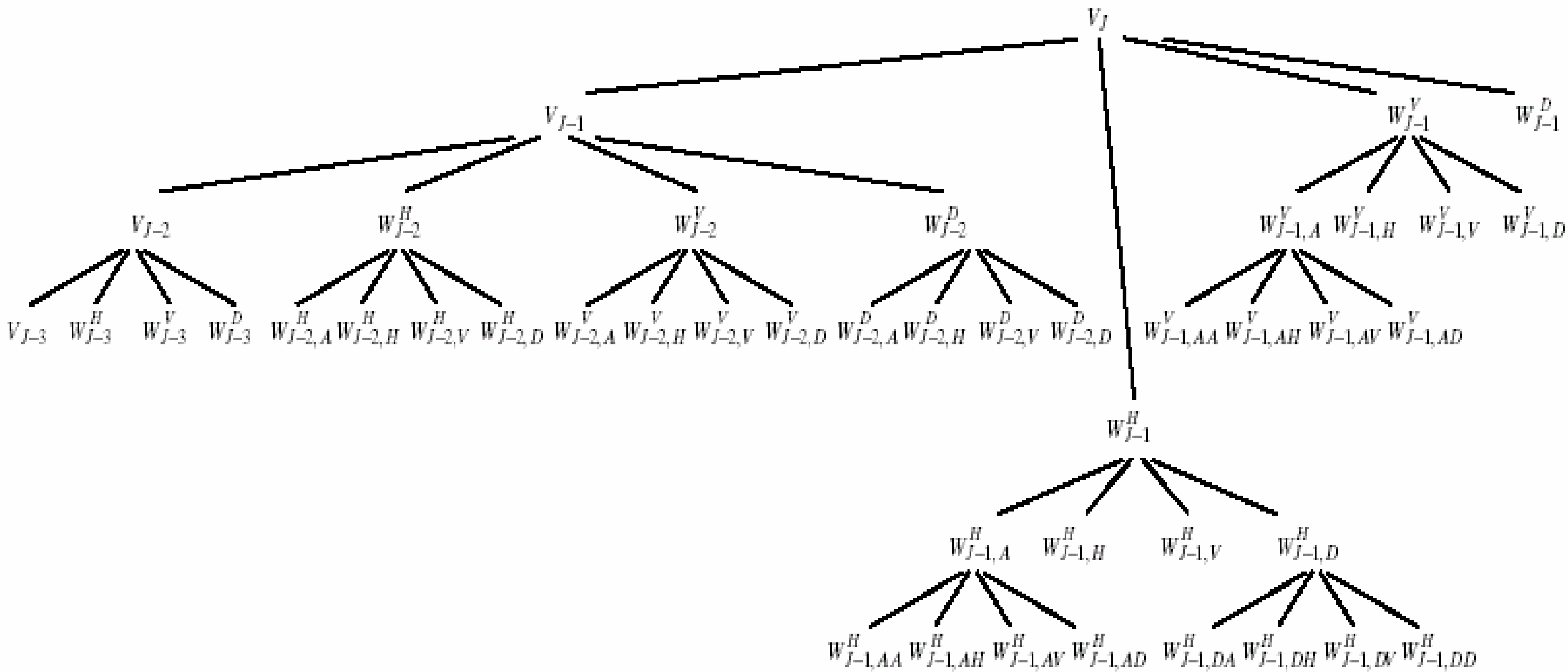

FIGURE 7.36 The optimal wavelet packet analysis tree for the decomposition in Fig. 7.35.

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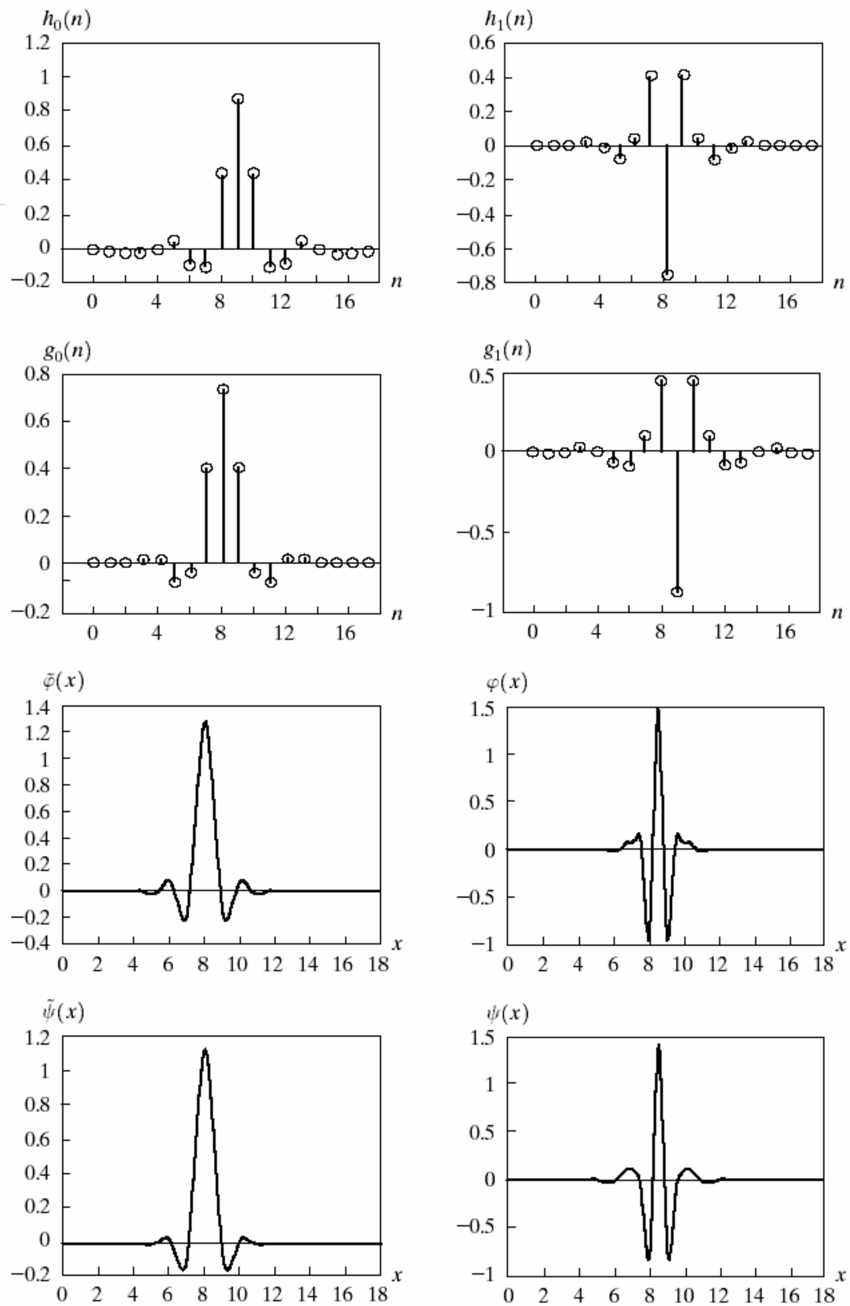

a b
c d
e f
g h

FIGURE 7.37 A member of the Cohen-Daubechies-Feauveau biorthogonal wavelet family: (a) and (b) decomposition filter coefficients; (c) and (d) reconstruction filter coefficients; (e)–(h) dual wavelet and scaling functions.

7.6 Wavelet packet

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