EE641000 Quantum Information and Computation

Chung-Chin Lu

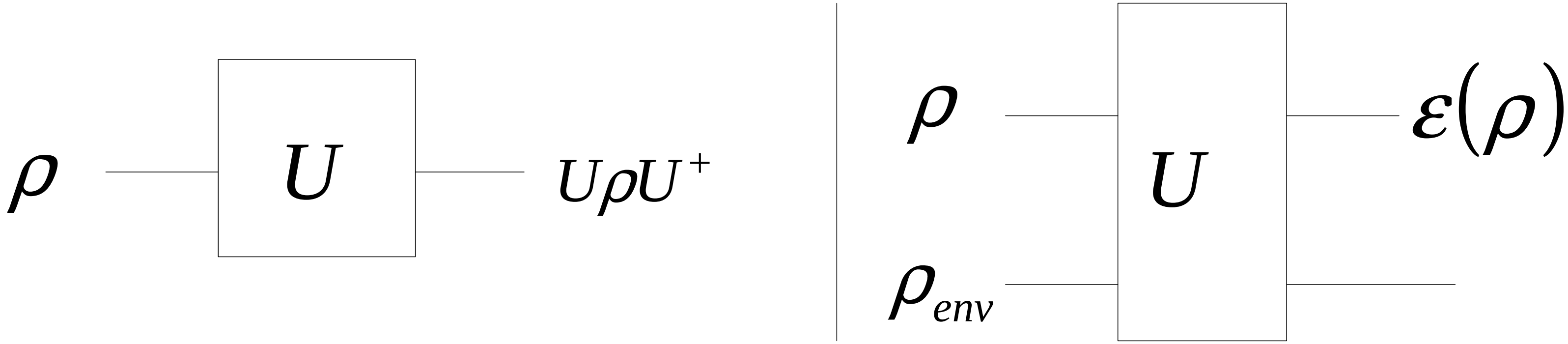
Department of Electrical Engineering

National Tsing Hua University

February 21, 2006

Unit Six – Quantum Operations

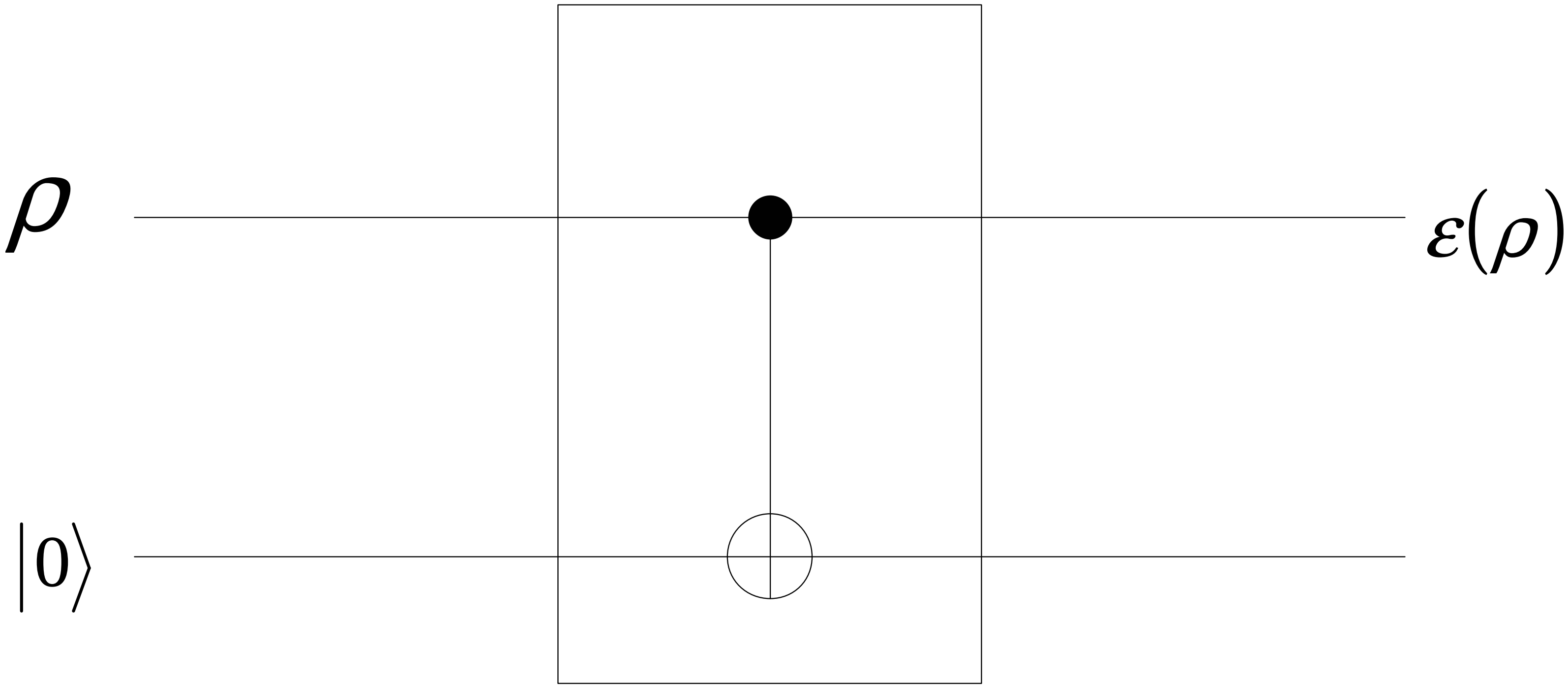
Open Quantum Systems



- Postulate 2 : the dynamics of a closed quantum system is described by a unitary transform
- An open quantum system together with its environment becomes a closed quantum system
- ρ : the density operator of the open quantum system, called the *principal* quantum system
- ρ_{env} : the density operator of the *environment*

- U : a unitary operator on the state space of the closed quantum system
- $\mathcal{E}(\rho)$: the density operator of the principal quantum system after the action of the unitary operator U
  - Closed system
  $$\mathcal{E}(\rho) = U\rho U^\dagger$$
  - Open system
  $$\mathcal{E}(\rho) = \mathrm{tr}_{env}\left(U\left(\rho \otimes \rho_{env}\right)U^\dagger\right)$$
    * Assume that the principal quantum system is prepared such that its correlation with the environment can be completely destroyed (Correlated initial state of the principal-environmental system will be discussed later)
    * If the state space of the principal system has dimension d, we will show that it is sufficient to model the environment to have state space of dimension no greater than d^2

An Example



- $\rho = \sum_{ij} \alpha_{ij} |i\rangle\langle j|$: the density operator of the principal system
- $\rho_{env} = |0\rangle\langle 0|$: the environment is in the pure state $|0\rangle$
- $\mathcal{E}(\rho) = P_0 \rho P_0 + P_1 \rho P_1$: $P_0 = |0\rangle\langle 0|$ and $P_1 = |1\rangle\langle 1|$ are projective operators

$$
\begin{aligned}
\mathcal{E}(\rho) &= \mathrm{tr}_2 \left(U \left(\sum_{ij} \alpha_{ij} |i\rangle\langle j| \otimes |0\rangle\langle 0| \right) U^\dagger \right) \\
&= \mathrm{tr}_2 \, (\alpha_{00}|0\rangle\langle 0| \otimes |0\rangle\langle 0| + \alpha_{10}|1\rangle\langle 0| \otimes |1\rangle\langle 0| + \\
&\quad \alpha_{01}|0\rangle\langle 1| \otimes |0\rangle\langle 1| + \alpha_{11}|1\rangle\langle 1| \otimes |1\rangle\langle 1|) \\
&= \alpha_{00}|0\rangle\langle 0| + \alpha_{11}|1\rangle\langle 1| \\
P_0 \rho P_0 &= \alpha_{00}|0\rangle\langle 0| \\
P_1 \rho P_1 &= \alpha_{11}|1\rangle\langle 1| \\
\mathcal{E}(\rho) &= P_0 \rho P_0 + P_1 \rho P_1
\end{aligned}
$$

Quantum Operations Formalism

The Input-Output Formalism

$$\boxed{\mathcal{E}(\rho^A) = \mathrm{tr}_B \left(U \left(\rho^A \otimes \rho^B\right) U^\dagger\right)}$$

or

$$\boxed{\mathcal{E}(\rho^A) = \mathrm{tr}_A \left(U \left(\rho^A \otimes \rho^B\right) U^\dagger\right)}$$

- ρ^A : the input density operator of quantum systems A
- ρ^B : the density operators of quantum system B
- $\mathcal{E}(\rho^A)$: the output density operator of quantum systems A or B

A General Definition of Triplet $\langle\psi|U|\varphi\rangle$

- H_1 and H_2 : complex inner product spaces
- U : a linear operator on the tensor product space $H_1 \otimes H_2$

$$U = \sum_i \alpha_i T_i^{H_1} \otimes T_i^{H_2},$$

∞

where $T_i^{H_1}$ and $T_i^{H_2}$ are linear operators on H_1 and H_2 respectively

- $|\psi\rangle$ and $|\varphi\rangle$: two vectors in H_2
- $\langle\psi|U|\varphi\rangle$: a linear operator on H_1

$$\langle\psi|U|\varphi\rangle \triangleq \sum_i \alpha_i T_i^{H_1} \langle\psi|T_i^{H_2}|\varphi\rangle$$

- $\langle\psi|\cdot|\varphi\rangle$: a linear map from $L(H_1 \otimes H_2, H_1 \otimes H_2)$ to $L(H_1, H_1)$

Well-defined

- $\{|j\rangle\}$: an orthonormal basis of H_2
- $\{|j\rangle\langle k|\}$: a basis of $L(H_2, H_2)$
- Unique representation : with $T_i^{H_2} = \sum_{jk} \beta_{ijk} |j\rangle\langle k|$, we have

$$U = \sum_{jk} \left(\sum_i \alpha_i \beta_{ijk} T_i^{H_1} \right) \otimes |j\rangle\langle k|$$

Thus we have

$$\begin{aligned} \langle\psi|U|\varphi\rangle &\triangleq \sum_i \alpha_i T_i^{H_1} \langle\psi|T_i^{H_2}|\varphi\rangle = \sum_i \alpha_i T_i^{H_1} \sum_{jk} \beta_{ijk} \langle\psi|j\rangle\langle k|\varphi\rangle \\ &= \sum_{jk} \left(\sum_i \alpha_i \beta_{ijk} T_i^{H_1} \right) \langle\psi|j\rangle\langle k|\varphi\rangle. \end{aligned}$$

A Theorem

$$\boxed{\mathrm{tr}_2(U) = \sum_i \langle i|U|i\rangle}$$

- H_1 and H_2 : complex inner product spaces
- U : a linear operator on the tensor product space $H_1 \otimes H_2$
- $\{|i\rangle\}$: an orthonormal basis of H_2

Proof

$$
\begin{aligned}
\mathrm{tr}_2(U) &= \mathrm{tr}_2\left(\sum_j \alpha_j T_j^{H_1} \otimes T_j^{H_2}\right) \\
&= \sum_j \alpha_j T_j^{H_1} \mathrm{tr}(T_j^{H_2}) \\
&= \sum_j \alpha_j T_j^{H_1} \sum_i \langle i | T_j^{H_2} | i \rangle \\
&= \sum_i \sum_j \alpha_j T_j^{H_1} \langle i | T_j^{H_2} | i \rangle \\
&= \sum_i \langle i | \left(\sum_j \alpha_j T_j^{H_1} \otimes T_j^{H_2}\right) | i \rangle
\end{aligned}
$$

$$= \sum_i \langle i|U|i\rangle$$

A General Definition of Inner Product $\langle\psi|\varphi\rangle$

- H_1 and H_2 : complex inner product spaces
- $|\psi\rangle$: a vectors in H_1
- $|\varphi\rangle = \sum_i \alpha_i |v_i\rangle \otimes |w_i\rangle$: a vector in $H_1 \otimes H_2$
- $\langle\psi|\varphi\rangle$: a vector in H_2

$$\langle\psi|\varphi\rangle \stackrel{\triangle}{=} \sum_i \alpha_i \langle\psi|v_i\rangle |w_i\rangle$$

- $\langle\psi|\cdot\rangle$: a linear transformation from $H_1 \otimes H_2$ to H_2

Well-defined

- $\{|j\rangle\}$: an orthonormal basis of H_2
- Unique representation : with $|w_i\rangle = \sum_j \beta_{ij}|j\rangle$, we have

$$|\varphi\rangle = \sum_j (\sum_i \alpha_i \beta_{ij} |v_i\rangle) \otimes |j\rangle$$

Thus we have

$$\begin{aligned}
\langle\psi|\varphi\rangle &\triangleq \sum_i \alpha_i \langle\psi|v_i\rangle |w_i\rangle \\
&= \sum_i \alpha_i \langle\psi|v_i\rangle \sum_j \beta_{ij}|j\rangle \\
&= \sum_j \langle\psi| \left(\sum_i \alpha_i \beta_{ij} |v_i\rangle \right) |j\rangle
\end{aligned}$$

The Operator-Sum Representation of $\mathcal{E}$

$$\mathcal{E}(\rho) = \sum_k \langle e_k|U(\rho \otimes \rho_{env})U^\dagger|e_k\rangle = \sum_{km} E_{km}\rho E_{km}^\dagger$$

where $E_{km} = \langle e_k|U\sqrt{\lambda_m}|\psi_m\rangle$

- ρ : the density operator of the principal quantum system
- $\{|e_k\rangle\}$: an orthonormal basis of the (finite-dimensional) state space of the environment
- $\rho_{env} = \sum_m \lambda_m|\psi_m\rangle\langle\psi_m|$: the density operator of the environment with ensemble $\{\lambda_m, |\psi_m\rangle\}$
- $\{E_{km}\}$: operation elements for the quantum operation $\mathcal{E}$

Proof

- $U = \sum_i \alpha_i T_i^{pri} \otimes T_i^{env}$
- $\rho_{env} = \sum_m \lambda_m |\psi_m\rangle\langle\psi_m|$

$$
\begin{aligned}
& \mathcal{E}(\rho) \\
= \quad & \mathrm{tr}_{env}\left(U\left(\rho \otimes \rho_{env}\right)U^\dagger\right) \\
= \quad & \sum_k \langle e_k|(\sum_i \alpha_i T_i^{pri} \otimes T_i^{env})\left(\rho \otimes \rho_{env}\right)(\sum_j \alpha_j T_j^{pri} \otimes T_j^{env})^\dagger |e_k\rangle \\
= \quad & \sum_k \sum_{ij} \alpha_i \alpha_j^* \langle e_k| \left(T_i^{pri} \rho T_j^{pri\dagger} \otimes T_i^{env} \rho_{env} T_j^{env\dagger}\right)|e_k\rangle \\
= \quad & \sum_k \sum_{ij} \alpha_i \alpha_j^* T_i^{pri} \rho T_j^{pri\dagger} \langle e_k| T_i^{env} \rho_{env} T_j^{env\dagger} |e_k\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_k \sum_{ij} \alpha_i \alpha_j^* T_i^{pri} \rho T_j^{pri\dagger} \sum_m \lambda_m \langle e_k | T_i^{env} | \psi_m \rangle \langle \psi_m | T_j^{env\dagger} | e_k \rangle \\
&= \sum_k \sum_m \lambda_m (\sum_i \alpha_i T_i^{pri} \langle e_k | T_i^{env} | \psi_m \rangle) \rho (\sum_j \alpha_j T_j^{pri} \langle e_k | T_j^{env} | \psi_m \rangle)^\dagger \\
&= \sum_k \sum_m \lambda_m \langle e_k | \sum_i \alpha_i T_i^{pri} \otimes T_i^{env} | \psi_m \rangle \rho (\langle e_k | \sum_j \alpha_j T_j^{pri} \otimes T_j^{env} | \psi_m \rangle)^\dagger \\
&= \sum_k \sum_m \langle e_k | U \sqrt{\lambda_m} | \psi_m \rangle \rho (\langle e_k | U \sqrt{\lambda_m} | \psi_m \rangle)^\dagger \\
&= \sum_k \sum_m E_{km} \rho E_{km}^\dagger
\end{aligned}
$$

where $E_{km} = \langle e_k | U \sqrt{\lambda_m} | \psi_m \rangle$

Lemma

Let T be a linear operator on a Hilbert space H. If

$$\mathrm{tr}(T\rho) = 1$$

for any density operators ρ on H, then we have

$$T = I$$

Proof.

Note that $\mathrm{tr}(T\rho) = 1$ for any density operator ρ on H if and only if $\mathrm{tr}(T|\psi\rangle\langle\psi|) = 1$, i.e. $\langle\psi|T|\psi\rangle = 1$, for any unit vector $|\psi\rangle$ in H. Let $\{|e_k\rangle\}$ be an orthonormal basis of H. For any $i \neq j$ and any non-zero complex numbers a, b such that $|a|^2 + |b|^2 = 1$, we have

$$1 = (a|e_i\rangle + b|e_j\rangle)^\dagger T(a|e_i\rangle + b|e_j\rangle)$$

which implies that

$$a^* b\langle e_i|T|e_j\rangle + ab^*\langle e_j|T|e_i\rangle = 0.$$

By taking a, b both real, we have

$$\langle e_i|T|e_j\rangle + \langle e_j|T|e_i\rangle = 0.$$

But by taking a real and b pure imaginary, we have

$$\langle e_i|T|e_j\rangle - \langle e_j|T|e_i\rangle = 0.$$

Thus we conclude that $\langle e_i|T|e_j\rangle = \langle e_j|T|e_i\rangle = 0$ and the matrix representation of T relative to the orthonormal basis $\{|e_k\rangle\}$ is the identity matrix which implies that $T = I$.

Completeness Relation on Operation Elements

$$\boxed{\sum_{km} E_{km}^\dagger E_{km} = I}$$

- $\mathcal{E}$ is trace-preserving : for for any density operator ρ,

$$\begin{aligned}\mathrm{tr}(\mathcal{E}(\rho)) &= \mathrm{tr}(\mathrm{tr}_{env}(U(\rho \otimes \rho_{env})U^\dagger)) = \mathrm{tr}(U(\rho \otimes \rho_{env})U^\dagger) \\ &= \mathrm{tr}(U^\dagger U(\rho \otimes \rho_{env})) = \mathrm{tr}(\rho \otimes \rho_{env}) = \mathrm{tr}(\rho)\mathrm{tr}(\rho_{env}) \\ &= 1\end{aligned}$$

- $1 = \mathrm{tr}(\mathcal{E}(\rho)) = \sum_{km} \mathrm{tr}(E_{km}\rho E_{km}^\dagger) = \sum_{km} \mathrm{tr}(E_{km}^\dagger E_{km}\rho) = \mathrm{tr}((\sum_{km} E_{km}^\dagger E_{km})\rho)$ for any density operator ρ

Purification of the Environment

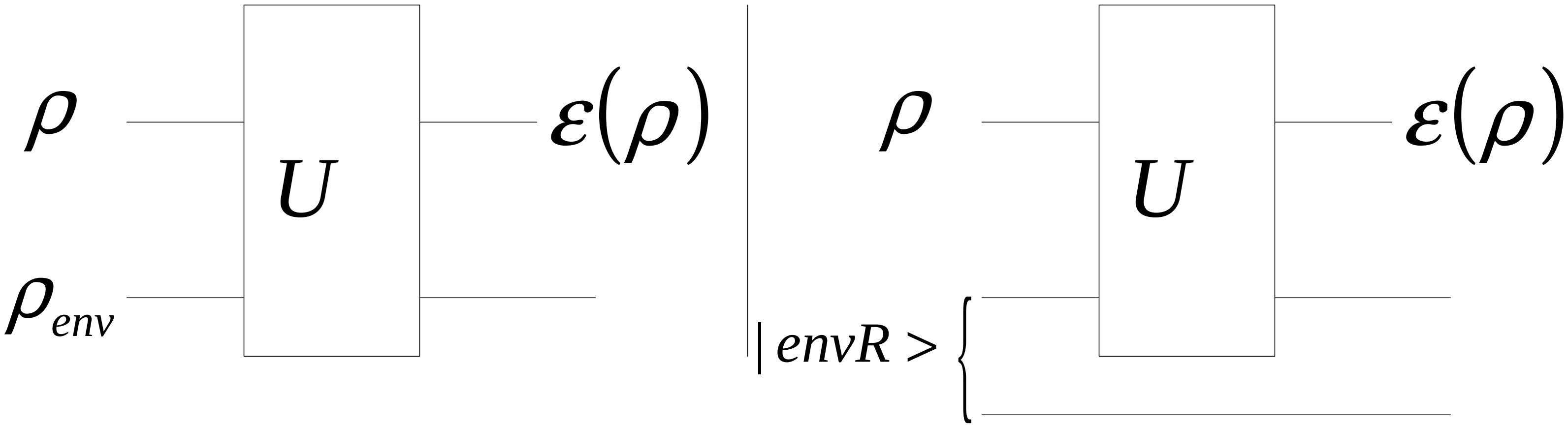
- ρ : the density operator of the principal quantum system
- $\rho_{env} = \sum_m \lambda_m |\psi_m\rangle\langle\psi_m|$: the density operator of the environment with ensemble $\{\lambda_m, |\psi_m\rangle\}$
- $\{|m_R\rangle\}$: an orthonormal basis of the state space of a reference system R, having the same cardinality as that of $\{|\psi_m\rangle\}$
- $|envR\rangle$: a pure state of the composite environment-R system

$$|envR\rangle = \sum_m \sqrt{\lambda_m} |\psi_m\rangle |m_R\rangle$$

such that

$$\rho_{env} = \mathrm{tr}_R(|envR\rangle\langle envR|)$$

Purification of the Environment (Cont')



$$
\begin{aligned}
\mathcal{E}(\rho) &= \mathrm{tr}_{env}\left(U\left(\rho\otimes\rho_{env}\right)U^{\dagger}\right)\\
&= \mathrm{tr}_{envR}\left((U\otimes I_R)(\rho\otimes|envR\rangle\langle envR|)(U\otimes I_R)^{\dagger}\right)
\end{aligned}
$$

Proof.

Note that for an orthonormal basis $\{|e_k\rangle\}$ of the state space of the environment, $\{|e_k\rangle|m_R\rangle\}$ is an orthonormal basis of the state space of the composite environment-R system. We have

$$\mathrm{tr}_{envR}\left((U\otimes I_R)(\rho\otimes|envR\rangle\langle envR|)(U\otimes I_R)^\dagger\right) = \sum_{km} F_{km}\rho F_{km}^\dagger$$

where

$$\begin{aligned}
F_{km} &= \langle e_k|\langle m_R|(U\otimes I_R)|envR\rangle \\
&= \langle e_k|\langle m_R|(U\otimes I_R)\sum_j \sqrt{\lambda_j}|\psi_j\rangle|j_R\rangle \\
&= \sum_j \sqrt{\lambda_j}\langle e_k|\langle m_R|(U\otimes I_R)|\psi_j\rangle|j_R\rangle \\
&= \sum_j \sqrt{\lambda_j}\langle e_k|U|\psi_j\rangle\langle m_R|j_R\rangle \\
&= \langle e_k|U\sqrt{\lambda_m}|\psi_m\rangle \\
&= E_{km}
\end{aligned}$$

Three Features of the Operator-Sum Representation

Physical Interpretation

- ρ : the density operator of the principal quantum system
- $\rho_{env} = \sum_m \lambda_m |\psi_m\rangle\langle\psi_m|$: the density operator of the environment
- $\{|e_k\rangle\}$: an orthonormal basis of the state space of the environment
- Principle of implicit measurement : the state of the principal system will not be affected if measurement is performed on the environment
- $\{|e_k\rangle\langle e_k|\}$: a projective measurement on the environment
- ρ_k : the state of the principal system given that outcome k occurs

$$
\begin{aligned}
\rho_k &= \mathrm{tr}_{env}\left(\frac{(I\otimes|e_k\rangle\langle e_k|)U(\rho\otimes\rho_{env})U^\dagger(I\otimes|e_k\rangle\langle e_k|)}{\mathrm{tr}((I\otimes|e_k\rangle\langle e_k|)U(\rho\otimes\rho_{env})U^\dagger(I\otimes|e_k\rangle\langle e_k|))}\right)\\
&= \mathrm{tr}_{env}\left(\frac{\langle e_k|U(\rho\otimes\rho_{env})U^\dagger|e_k\rangle\otimes|e_k\rangle\langle e_k|}{\mathrm{tr}(\langle e_k|U(\rho\otimes\rho_{env})U^\dagger|e_k\rangle\otimes|e_k\rangle\langle e_k|)}\right)\\
&= \frac{\langle e_k|U(\rho\otimes\rho_{env})U^\dagger|e_k\rangle\mathrm{tr}(|e_k\rangle\langle e_k|)}{\mathrm{tr}(\langle e_k|U(\rho\otimes\rho_{env})U^\dagger|e_k\rangle)\mathrm{tr}(|e_k\rangle\langle e_k|)}\\
&= \frac{\sum_m E_{km}\rho E_{km}^\dagger}{\mathrm{tr}(\sum_m E_{km}\rho E_{km}^\dagger)}
\end{aligned}
$$

where $E_{km} = \langle e_k|U\sqrt{\lambda_m}|\psi_m\rangle$

- $\mathcal{P}(k)$: the probability that outcome k occurs

$$
\begin{aligned}
\mathcal{P}(k) &= \mathrm{tr}((I\otimes|e_k\rangle\langle e_k|)U(\rho\otimes\rho_{env})U^\dagger(I\otimes|e_k\rangle\langle e_k|))\\
&= \mathrm{tr}(\sum_m E_{km}\rho E_{km}^\dagger)
\end{aligned}
$$

$$\boxed{\mathcal{E}(\rho) = \sum_k \mathcal{P}(k)\rho_k = \sum_{km} E_{km}\rho E_{km}^\dagger}$$

- The action of the quantum operation $\mathcal{E}$ is equivalent to taking the state ρ as input and randomly replacing it by

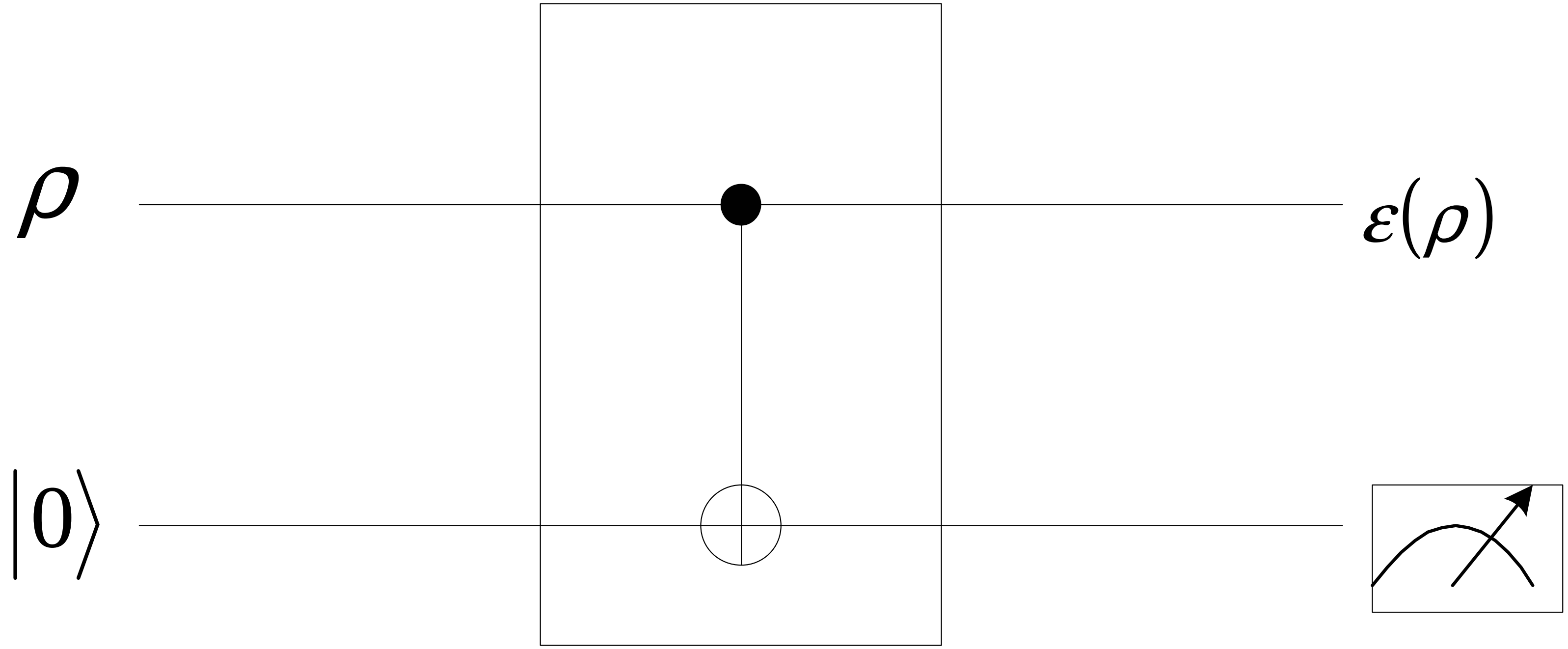
$$\frac{\sum_m E_{km}\rho E_{km}^\dagger}{\mathrm{tr}(\sum_m E_{km}\rho E_{km}^\dagger)}$$

  with probability

$$\mathrm{tr}(\sum_m E_{km}\rho E_{km}^\dagger)$$

- A quantum operation which describes a quantum noise process will be referred to as a noisy quantum channel

An Example (Revisited)



- ρ : the density operator of the principal system
- $\rho_{env} = |0_E\rangle\langle 0_E|$: the environment is in the pure state $|0_E\rangle$
- $U =$
  $|0_P 0_E\rangle\langle 0_P 0_E| + |0_P 1_E\rangle\langle 0_P 1_E| + |1_P 1_E\rangle\langle 1_P 0_E| + |1_P 0_E\rangle\langle 1_P 1_E|$

- $\{|0_E\rangle, |1_E\rangle\}$: an orthonormal basis of the state space of the environment
- $E_0 = \langle 0_E|U|0_E\rangle = |0_P\rangle\langle 0_P| = P_0$
- $E_1 = \langle 1_E|U|0_E\rangle = |1_P\rangle\langle 1_P| = P_1$
- $\mathcal{E}(\rho) = P_0\rho P_0 + P_1\rho P_1$

Effect of Global Measurement

- ρ : the density operator of the principal quantum system
- $\rho_{env} = \sum_j \lambda_j |\psi_j\rangle\langle\psi_j|$: the density operator of the environment
- $\{|e_k\rangle\}$: an orthonormal basis of the state space of the environment
- $\{P_m\}$: projective measurement after the unitary operation U
- The state of the principal system given that outcome m occurs is

$$\mathrm{tr}_{env}\left(\frac{P_m U(\rho \otimes \rho_{env})U^\dagger P_m}{\mathrm{tr}(P_m U(\rho \otimes \rho_{env})U^\dagger P_m)}\right)$$

with probability

$$\mathrm{tr}(P_m U(\rho \otimes \rho_{env})U^\dagger P_m)$$

Define a map

$$\mathcal{E}_m(\rho) \stackrel{\triangle}{=} \mathrm{tr}_{env}(P_m U(\rho \otimes \rho_{env}) U^\dagger P_m) = \sum_{kj} E_{kj}^{(m)} \rho E_{kj}^{(m)\dagger}$$

where

$$E_{kj}^{(m)} = \langle e_k | P_m U \sqrt{\lambda_j} | \psi_j \rangle$$

- The state of the principal system given that outcome m occurs is
$$\frac{\mathcal{E}_m(\rho)}{\mathrm{tr}(\mathcal{E}_m(\rho))}$$
with probability $\mathrm{tr}(\mathcal{E}_m(\rho))$
- $\{\mathcal{E}_m(\rho)\}$: a kind of measurement process which generalizes the measurement described in Unit Three where $\mathcal{E}_m(\rho) = E_m \rho E_m^\dagger$ for a quantum measurement $\{E_m\}$

The Converse Problem

- $\{E_k\}$: a given collection of operator elements acting on a *principal* quantum system and satisfying the completeness relation

$$\sum_k E_k^\dagger E_k = I$$

- The problem : find a system-environment model, i.e., an environment, a unitary operator U on the composite system-environment model such that $\{E_k\}$ is the operator elements in the operator-sum representation of the quantum operation $\mathcal{E}$

A System-Environment Model

- $\rho = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$: the density operator of the principal system
- $\{|e_k\rangle\}$: an orthonormal basis of the state space of a chosen environment, having the same cardinality as that of $\{E_k\}$
- $|0\rangle$: an arbitrarily chosen state of the environment
- U : a linear operator acting on the states of the form $|\psi\rangle|0\rangle$ where $|\psi\rangle$ is any state of the principal system

$$U|\psi\rangle|0\rangle = \sum_k E_k|\psi\rangle|e_k\rangle$$

and for any states $|\psi\rangle$ and $|\varphi\rangle$ of the principal system,

$$\langle\psi|\langle 0|U^\dagger U|\varphi\rangle|0\rangle = \sum_{kj}(E_k|\psi\rangle|e_k\rangle)^\dagger(E_j|\varphi\rangle|e_j\rangle)$$

$$= \sum_{kj} \langle \psi | E_k^\dagger E_j | \varphi \rangle \langle e_k | e_j \rangle = \langle \psi | \sum_k E_k^\dagger E_k | \varphi \rangle = \langle \psi | \varphi \rangle$$

Thus U can be extended to a unitary operator $U = \sum_i \alpha_i T_i^{pri} \otimes T_i^{env}$ on the composite system and we have for any state $|\psi\rangle$ of the principal system

$$\begin{aligned}
(\langle e_k | U | 0 \rangle) | \psi \rangle &= \sum_i \alpha_i T_i^{pri} \langle e_k | T_i^{env} | 0 \rangle | \psi \rangle \\
&= \sum_i \alpha_i T_i^{pri} | \psi \rangle \langle e_k | T_i^{env} | 0 \rangle \\
&= \langle e_k | (\sum_i \alpha_i T_i^{pri} \otimes T_i^{env}) | \psi \rangle | 0 \rangle \\
&= \langle e_k | U | \psi \rangle | 0 \rangle = \langle e_k | \sum_j E_j | \psi \rangle | e_j \rangle = E_k | \psi \rangle
\end{aligned}$$

which says that $E_k = \langle e_k | U | 0 \rangle$

Axiomatic Approach to Quantum Operations

Formal Definition of Quantum Operations

A quantum operation $\mathcal{E}$ is a map from the set of density operators of the input space Q_1 to the set of positive operators of the output space Q_2, satisfying the three axiomatic properties as follows.

Three Axioms

- Axiom I : $\mathrm{tr}(\mathcal{E}(\rho))$ is the probability that the process represented by $\mathcal{E}$ occurs, when ρ is the input density operator,

$$0 \leq \mathrm{tr}(\mathcal{E}(\rho)) \leq 1$$

- Axiom II : $\mathcal{E}$ is a convex-linear map on the set of density operators, i.e., for non-negative numbers λ_i with $\sum_i \lambda_i = 1$ and density operators ρ_i, $\mathcal{E}(\sum_i \lambda_i \rho_i) = \sum_i \lambda_i \mathcal{E}(\rho_i)$
- Axiom III : $\mathcal{E}$ is a completely positive map, i.e., for an arbitrarily introduced system R of arbitrary dimension and the identity map $\mathcal{I}$ on the set of all linear operators on R, $I \otimes \mathcal{E}$ is a well-defined map from the set of positive operators of the composite system RQ_1 to the set of positive operators of the composite system RQ_2

Axiom I

$\mathrm{tr}(\mathcal{E}(\rho))$ is the probability that the process represented by $\mathcal{E}$ occurs, when ρ is the input density operator,

$$0 \le \mathrm{tr}(\mathcal{E}(\rho)) \le 1$$

- This is a mathematical convenience to include the case of measurement as quantum operation, where the trace may not be preserved and $\mathrm{tr}(\mathcal{E}(\rho))$ is exactly the probability of the occurrence of a particular measurement outcome when the state before measurement is ρ. The state after measurement becomes $\mathcal{E}(\rho)/\mathrm{tr}(\mathcal{E}(\rho))$

Axiom II

$\mathcal{E}$ is a convex-linear map on the set of density operators, i.e., for non-negative numbers p_i with $\sum_i p_i = 1$ and density operators ρ_i,

$$\mathcal{E}(\sum_i p_i \rho_i) = \sum_i p_i \mathcal{E}(\rho_i)$$

- $\rho = \sum_i p_i \rho_i$: the input quantum state as a random selection from the ensemble $\{p_i, \rho_i\}$ of quantum (mixed) states
- $\mathcal{E}(\rho)/\mathrm{tr}(\mathcal{E}(\rho)) = \mathcal{E}(\rho)/p(\mathcal{E})$: the resulting state as a random selection from the ensemble $\{p(i|\mathcal{E}), \mathcal{E}(\rho_i)/\mathrm{tr}(\mathcal{E}(\rho_i))\}$ of quantum (mixed) states, where $p(i|\mathcal{E})$ is the probability that the state prepared is ρ_i, given that the process described by $\mathcal{E}$ occurs, i.e., we demand

$$\frac{\mathcal{E}(\rho)}{p(\mathcal{E})} = \sum_i p(i|\mathcal{E}) \frac{\mathcal{E}(\rho_i)}{\mathrm{tr}(\mathcal{E}(\rho_i))}$$

- A Bayesian rule :

$$p(i|\mathcal{E}) = \frac{p(\mathcal{E}|i)p_i}{p(\mathcal{E})} = \frac{\mathrm{tr}(\mathcal{E}(\rho_i))p_i}{p(\mathcal{E})}$$

- Justification :

$$\mathcal{E}(\rho) = p(\mathcal{E}) \sum_i p(i|\mathcal{E}) \frac{\mathcal{E}(\rho_i)}{\mathrm{tr}(\mathcal{E}(\rho_i))} = \sum_i p_i \mathcal{E}(\rho_i)$$

Axiom III

$\mathcal{E}$ is a completely positive map, i.e., for an arbitrarily introduced system R of arbitrary dimension and the identity map $\mathcal{I}$ on the set of all linear operators on R, $\mathcal{I} \otimes \mathcal{E}$ is a well-defined map from the set of positive operators of the composite system RQ_1 to the set of positive operators of the composite system RQ_2

- It is required for a physical system that if ρ^{RQ_1} is a (mixed) state of a composite system RQ_1 and the quantum operation $\mathcal{E}$ acts solely on the system Q_1, then the result $\mathcal{I} \otimes \mathcal{E}(\rho^{RQ_1})$ must also be a state (up to a normalization factor) of the composite system RQ_2

Theorem

A map $\mathcal{E}$ from the set of density operators of the input space Q_1 to the set of positive operators of the output space Q_2 satisfies the above three axiomatic properties if and only if

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger \tag{1}$$

for a collection of linear transformations E_k from the input space Q_1 to the output space Q_2 and

$$\sum_k E_k^\dagger E_k \leq I.$$

Furthermore, $\mathcal{E}$ is trace-preserving, i.e., $\mathcal{E}(\rho)$ is a density operator for any density operator ρ if and only if $\sum_k E_k^\dagger E_k = I$.

Proof $\Longleftarrow$

From (1), it is clear that $\mathcal{E}(\rho)$ is a positive operator on Q_2 for any density operator ρ on Q_1 and $\mathcal{E}$ is a linear map.

- Axiom I : for a density operator $\rho = \sum_j \lambda_j |j\rangle\langle j|$ on Q_1, we have

$$\begin{aligned} 0 \leq \mathrm{tr}(\mathcal{E}(\rho)) &= \sum_k \mathrm{tr}(E_k \rho E_k^\dagger) = \sum_k \sum_j \lambda_j \mathrm{tr}(E_k |j\rangle\langle j| E_k^\dagger) \\ &= \sum_k \sum_j \lambda_j \langle j| E_k^\dagger E_k |j\rangle = \sum_j \lambda_j \langle j| (\sum_k E_k^\dagger E_k) |j\rangle \\ &\leq \sum_j \lambda_j \langle j|j\rangle = 1 \end{aligned}$$

  since $\sum_k E_k^\dagger E_k \leq I$

- Axiom II : it is clear from the linearity of $\mathcal{E}$

- Axiom III : for any positive operator $B = \sum_i \alpha_i T_i^R \otimes T_i^{Q_1}$ on the composite system RQ_1, we define

$$\begin{aligned}
& (\mathcal{I} \otimes \mathcal{E})(B) \\
\stackrel{\triangle}{=} \; & \sum_i \alpha_i \mathcal{I}(T_i^R) \otimes (\sum_k E_k T_i^{Q_1} E_k^\dagger) \\
= \; & \sum_k \sum_i \alpha_i (I_R T_i^R I_R) \otimes (E_k T_i^{Q_1} E_k^\dagger) \\
= \; & \sum_k \sum_i \alpha_i (I_R \otimes E_k)(T_i^R \otimes T_i^{Q_1})(I_R \otimes E_k^\dagger) \\
= \; & \sum_k (I_R \otimes E_k) B (I_R \otimes E_k^\dagger)
\end{aligned}$$

which is clearly well-defined .

Let $|\psi\rangle$ be a state of the composite system RQ_2 and let $|\varphi_k\rangle = (I_R \otimes E_k^\dagger)|\psi\rangle$ for all k, we have

$$\begin{aligned} \langle\psi|(\mathcal{I} \otimes \mathcal{E})(B)|\psi\rangle &= \sum_k \langle\psi|(I_R \otimes E_k)B(I_R \otimes E_k^\dagger)|\psi\rangle \\ &= \sum_k \langle\varphi_k|B|\varphi_k\rangle \geq 0 \end{aligned}$$

which implies that $(\mathcal{I} \otimes \mathcal{E})(B)$ is a positive operator on RQ_2

Proof $\Longrightarrow$

- R : an arbitrarily introduced space, with the same dimension as Q_1
- $\{|i_R\rangle\}$ and $\{|i_{Q_1}\rangle\}$: orthonormal bases for R and Q_1 respectively
- $|u\rangle = \sum_i |i_R\rangle|i_{Q_1}\rangle$: a (maximally entangled) vector in the composite system RQ_1
- $\mathcal{I} \otimes \mathcal{E}$: a map from the set of positive operators on RQ_1 to the set of positive operators on RQ_2 by the complete positivity of $\mathcal{E}$ from Axiom III
- $\sigma = (\mathcal{I} \otimes \mathcal{E})(|u\rangle\langle u|)$: a positive operator on RQ_2

$$\sigma = \sum_{ij} |i_R\rangle\langle j_R| \otimes \mathcal{E}(|i_{Q_1}\rangle\langle j_{Q_1}|)$$

– It will be seen that the positive operator σ completely specifies the quantum operation $\mathcal{E}$

- $|v\rangle = \sum_i \alpha_i |i_{Q_1}\rangle$: a vector in Q_1
- $|\tilde{v}\rangle = \sum_i \alpha_i^* |i_R\rangle$: a vector in R corresponding to the vector $|v\rangle$ in Q_1
- An identity : a strong connection between σ and $\mathcal{E}$

$$\begin{aligned}
\langle \tilde{v}|\sigma|\tilde{v}\rangle &= \langle \tilde{v}| \left(\sum_{ij} |i_R\rangle\langle j_R| \otimes \mathcal{E}(|i_{Q_1}\rangle\langle j_{Q_1}|) \right) |\tilde{v}\rangle \\
&= \sum_{ij} \langle \tilde{v}|i_R\rangle\langle j_R|\tilde{v}\rangle \mathcal{E}(|i_{Q_1}\rangle\langle j_{Q_1}|) \\
&= \sum_{ij} \alpha_i \alpha_j^* \mathcal{E}(|i_{Q_1}\rangle\langle j_{Q_1}|) \\
&= \mathcal{E}(|v\rangle\langle v|)
\end{aligned}$$

- $\sigma = \sum_k \lambda_k |s_k\rangle\langle s_k|$: spectral decomposition of σ

$$\langle\tilde{v}|\sigma|\tilde{v}\rangle = \sum_k \lambda_k \langle\tilde{v}|s_k\rangle\langle s_k|\tilde{v}\rangle$$

- E_k : a linear transformation from Q_1 to Q_2 defined as

$$E_k(|v\rangle) \triangleq \sqrt{\lambda_k}\langle\tilde{v}|s_k\rangle$$

  for any vector $|v\rangle$ in Q_1

Thus for any state $|\psi\rangle$ in Q_1, we have

$$\boxed{\mathcal{E}(|\psi\rangle\langle\psi|) = \langle\tilde{\psi}|\sigma|\tilde{\psi}\rangle = \sum_k E_k|\psi\rangle\langle\psi|E_k^\dagger}$$

and then for any density operator $\rho = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$ with ensemble $\{\lambda_i, |\psi_i\rangle\}$, we have

$$
\begin{aligned}
\mathcal{E}(\rho) &= \mathcal{E}(\sum_i \lambda_i |\psi_i\rangle\langle\psi_i|) \\
&= \sum_i \lambda_i \mathcal{E}(|\psi_i\rangle\langle\psi_i|) \text{ by Axiom II} \\
&= \sum_i \lambda_i \sum_k E_k |\psi_i\rangle\langle\psi_i| E_k^\dagger \\
&= \sum_k E_k (\sum_i \lambda_i |\psi_i\rangle\langle\psi_i|) E_k^\dagger \\
&= \sum_k E_k \rho E_k^\dagger
\end{aligned}
$$

To show that

$$\sum_k E_k^\dagger E_k \leq I,$$

we need to show that

$$\langle\psi|(I - \sum_k E_k^\dagger E_k)|\psi\rangle \geq 0$$

for any state ψ in Q_1. But

$$\begin{aligned} \langle\psi|(I - \sum_k E_k^\dagger E_k)|\psi\rangle &= \langle\psi|\psi\rangle - \langle\psi|(\sum_k E_k^\dagger E_k)|\psi\rangle \\ = \quad & 1 - \sum_k \langle\psi|E_k^\dagger E_k|\psi\rangle = 1 - \sum_k \mathrm{tr}(E_k|\psi\rangle\langle\psi|E_k^\dagger) \\ = \quad & 1 - \mathrm{tr}(\sum_k E_k|\psi\rangle\langle\psi|)E_k^\dagger) = 1 - \mathrm{tr}(\mathcal{E}(|\psi\rangle\langle\psi|)) \geq 0 \end{aligned}$$

by Axiom I

Freedom in the Operator-Sum Representation

Unitary Freedom in the Operator-Sum Representation

- $\{E_1, E_2, \ldots, E_m\}$: operation elements of a quantum operation $\mathcal{E}$
- $\{F_1, F_2, \ldots, F_n\}$: operation elements of a quantum operation $\mathcal{F}$
- $m = n$: by appending zero operators in the shorter list of operation elements

Then $\mathcal{E} = \mathcal{F}$ if and only if

$$E_i = \sum_j u_{ij} F_j$$

where $[u_{ij}]$ is an $m \times m$ unitary matrix

Proof $\Longrightarrow$

Suppose that for any density operator ρ, we have

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger = \sum_k F_k \rho F_k^\dagger = \mathcal{F}(\rho)$$

- R : an introduced space, with the same dimension as Q_1
- $\{|i_R\rangle\}$ and $\{|i_{Q_1}\rangle\}$: orthonormal bases for R and Q_1 respectively
- $|u\rangle = \sum_i |i_R\rangle|i_{Q_1}\rangle$: a (maximally entangled) vector in the composite system RQ_1
- $\mathcal{I} \otimes \mathcal{E} = \mathcal{I} \otimes \mathcal{F}$: a map from the set of positive operators on RQ_1 to the set of positive operators on RQ_2 by the complete positivity of $\mathcal{E}$ from the 3rd axiomatic property
- $\sigma = (\mathcal{I} \otimes \mathcal{E})(|u\rangle\langle u|) = (\mathcal{I} \otimes \mathcal{F})(|u\rangle\langle u|)$

Now,

$$
\begin{aligned}
(\mathcal{I}\otimes\mathcal{E})(|u\rangle\langle u|) &= \sum_{ij}|i_R\rangle\langle j_R|\otimes\mathcal{E}(|i_{Q_1}\rangle\langle j_{Q_1}|) \\
&= \sum_{ij}|i_R\rangle\langle j_R|\otimes(\sum_k E_k|i_{Q_1}\rangle\langle j_{Q_1}|E_k^\dagger) \\
&= \sum_k(\sum_i|i_R\rangle(E_k|i_{Q_1}\rangle))(\sum_j|j_R\rangle(E_k|j_{Q_1}\rangle))^\dagger \\
&= \sum_k|e_k\rangle\langle e_k| \\
(\mathcal{I}\otimes\mathcal{F})(|u\rangle\langle u|) &= \sum_k|f_k\rangle\langle f_k|
\end{aligned}
$$

where we define

$$
|e_k\rangle \stackrel{\triangle}{=} \sum_i|i_R\rangle(E_k|i_{Q_1}\rangle),\quad |f_k\rangle \stackrel{\triangle}{=} \sum_i|i_R\rangle(F_k|i_{Q_1}\rangle).
$$

Now we have

$$\boxed{\sigma = \sum_k |e_k\rangle\langle e_k| = \sum_k |f_k\rangle\langle f_k|}$$

and for any vector $|v\rangle = \sum_i \alpha_i |i_{Q_1}\rangle$ in Q_1, we have

$$E_k|v\rangle = \langle\tilde{v}|e_k\rangle, \quad F_k|v\rangle = \langle\tilde{v}|f_k\rangle,$$

where $|\tilde{v}\rangle = \sum_i \alpha_i^* |i_R\rangle$ is the vector in R corresponding to the vector $|v\rangle$

- Unitary freedom in the ensemble for density operators : there exists an $m \times m$ unitary matrix $[u_{kl}]$ such that

$$|e_k\rangle = \sum_l u_{kl} |f_l\rangle$$

Thus we have $E_k|v\rangle = \langle\tilde{v}|e_k\rangle = \sum_l u_{kl}\langle\tilde{v}|f_l\rangle = \sum_l u_{kl} F_l |v\rangle$, i.e.,

$$E_k = \sum_l u_{kl} F_l$$

Two Examples of Quantum Operations

Trace as a Quantum Operation

- Q: a quantum systems with state space H_Q
- $\{|i\rangle\}$: an orthonormal basis of the state space H_Q of Q
- Q': a quantum systems with one-diemnsional state space $H_{Q'}$
- $\{|0\rangle\}$: an orthonormal basis of the state space $H_{Q'}$ of Q'
- $E_i = |0\rangle\langle i|$: a linear transformation from H_Q to $H_{Q'}$
- A completeness relation : $\sum_i E_i^\dagger E_i = I$
- $\mathcal{E}$: a quantum operation defined as

$$\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger = \sum_i |0\rangle\langle i|\rho|i\rangle\langle 0|$$

It is clear that

$$\boxed{\mathcal{E}(\rho) = \mathrm{tr}(\rho)|0\rangle\langle 0|}$$

58

Partial Trace as a Quantum Operation

- Q and R : two quantum systems with state spaces H_Q and H_R
- $\{|j\rangle\}$: an orthonormal basis of the state space H_R of R
- $\sum_j |v_j\rangle|j\rangle$: a vector (in a unique representation format) in $H_Q \otimes H_R$
- E_i : a linear transformation from $H_Q \otimes H_R$ to H_Q defined as

$$E_i\left(\sum_j |v_j\rangle|j\rangle\right) = |v_i\rangle$$

- $E_i^\dagger$: the adjoint of E_i, which is a linear transformation from H_Q to $H_Q \otimes H_R$ and can be shown to be

$$E_i^\dagger(|v\rangle) = |v\rangle|i\rangle$$

- A completeness relation : $\sum_i E_i^\dagger E_i = I$
- $\mathcal{E}$: a quantum operation can be defined as

$$\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger$$

for all density operators ρ on the composite quantum QR. In fact, $\mathcal{E}$ is a linear map from $L(H_Q \otimes H_R, H_Q \otimes H_R)$ to $L(H_Q, H_Q)$

- T^Q : a linear operator on H_Q, we have

$$\mathcal{E}(T^Q \otimes |j\rangle\langle j'|) = T^Q \delta_{jj'} = \mathrm{tr}_R(T^Q \otimes |j\rangle\langle j'|)$$

Proof. Let $|v\rangle \in H_Q$. Then

$$\mathcal{E}(T^Q \otimes |j\rangle\langle j'|)(|v\rangle) = \sum_i E_i (T^Q \otimes |j\rangle\langle j'|) E_i^\dagger (|v\rangle)$$

$$= \sum_i E_i (T^Q \otimes |j\rangle\langle j'|)(|v\rangle|i\rangle) = \sum_i E_i (T^Q|v\rangle \otimes \delta_{ij'}|j\rangle)$$

$$= \sum_i \delta_{ij}\delta_{ij'} T^Q|v\rangle = \delta_{jj'} T^Q|v\rangle = \mathrm{tr}_R(T^Q \otimes |j\rangle\langle j'|)(|v\rangle)$$

For each linear operator $T^{QR} = \sum_{jj'} T^Q_{jj'} \otimes |j\rangle\langle j'|$ on the composite system QR, we have

$$\boxed{\mathcal{E}(T^{QR}) = \sum_i E_i T^{QR} E_i^\dagger = \mathrm{tr}_R(T^{QR})}$$

by linearity of $\mathcal{E}$ and tr_R.

Geometric Visualization of Quantum Operations on a Qubit

Bloch Vector Representation of Density Operators of a Qubit

- $\{|0\rangle, |1\rangle\}$: a computational basis of the state space H of a qubit
- ρ : a density operator of the qubit with matrix representation relative to the computational basis

$$[\rho] = \begin{bmatrix} \frac{1+r_z}{2} & \frac{r_x - i r_y}{2} \\ \frac{r_x + i r_y}{2} & \frac{1-r_z}{2} \end{bmatrix}$$

- $\sigma_x, \sigma_y, \sigma_z$: Pauli operators
- Bloch vector representation :

$$\rho = \frac{I + r_x\sigma_x + r_y\sigma_y + r_z\sigma_z}{2} = \frac{I + \vec{r}\cdot\vec{\sigma}}{2}$$

  – ρ is a density operator $\Longrightarrow ||\vec{r}||^2 = r_x^2 + r_y^2 + r_z^2 \leq 1$

- $\rho = I/2 \Longleftrightarrow \vec{r} = \vec{0}$

- $\mathrm{tr}(\rho^2) = (1 + ||\vec{r}||^2)/2$
- $\rho = |\psi\rangle\langle\psi|$ is a pure state if and only if $||\vec{r}|| = 1$
  - $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$ with the visualizing representation

$$(\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta)$$

on the Bloch sphere in Unit Four, which is equal to the Bloch vector $\vec{r}$ in above, i.e.,

$$[\rho] = \begin{bmatrix} \frac{1+\cos\theta}{2} & \frac{\cos\varphi\sin\theta - i\sin\varphi\sin\theta}{2} \\ \frac{\cos\varphi\sin\theta + i\sin\varphi\sin\theta}{2} & \frac{1-\cos\theta}{2} \end{bmatrix}$$

The Vector Space $L(H)$ with Trace Inner Product

- $L(H)$: the vector space of all linear operators on H
- $(T,S) \stackrel{\triangle}{=} \operatorname{tr}(T^\dagger S)$: the Hilbert-Schmidt or trace inner product of T and S in $L(H)$
  - $(T,T) = \operatorname{tr}(T^\dagger T) \geq 0$: $T^\dagger T$ is a positive operator
    - $(T,T) = \operatorname{tr}(T^\dagger T) = 0 \Longleftrightarrow$ all singular values of T are zeros $\Longleftrightarrow$ rank(T)=0 $\Longleftrightarrow T = 0$
  - $(T,S) = \operatorname{tr}(T^\dagger S) = \operatorname{tr}((S^\dagger T)^\dagger) = \overline{\operatorname{tr}(S^\dagger T)} = \overline{(S,T)}$: Hermitian symmetry
  - $(T, \alpha_1 S_1 + \alpha_2 S_2) = \operatorname{tr}(T^\dagger(\alpha_1 S_1 + \alpha_2 S_2)) =$ $\alpha_1 \operatorname{tr}(T,S_1) + \alpha_2 \operatorname{tr}(T,S_2) = \alpha(T,S_1) + \alpha_2(T,S_2)$: linearity
- $L(H)$: a 4-dimensional complex inner product space with trace inner product

- $\{I/\sqrt{2}, \sigma_x/\sqrt{2}, \sigma_y/\sqrt{2}, \sigma_z/\sqrt{2}\}$: an orthonormal basis of $L(H)$

Trace-Preserving Quantum Operations as Affine Maps from Bloch Sphere to Itself

- $\sigma_1\sigma_2 = i\sigma_3$, $\sigma_2\sigma_3 = i\sigma_1$, $\sigma_3\sigma_1 = i\sigma_2$
- $\mathcal{E} = \sum_k E_k \rho E_k^\dagger$: a trace-preserving quantum operation on a qubit (where input system = output system) with

$$E_k = \alpha_{k0} I + \sum_{i=1}^{3} \alpha_{ki}\sigma_i$$

where $\sum_k E_k^\dagger E_k = I$ which implies that

$$\sum_k \sum_{i=0}^{3} |\alpha_{ki}|^2 = 1, \sum_k \Re\{\alpha_{k0}\alpha_{k1}^*\} + \Im\{\alpha_{k2}\alpha_{k3}^*\} = 0,$$

$$\sum_k \Re\{\alpha_{k0}\alpha_{k2}^*\} + \Im\{\alpha_{k3}\alpha_{k1}^*\} = 0, \sum_k \Re\{\alpha_{k0}\alpha_{k3}^*\} + \Im\{\alpha_{k1}\alpha_{k2}^*\} = 0$$

- $\rho = \frac{I + \vec{r} \cdot \vec{\sigma}}{2}$ and $\mathcal{E}(\rho) = \frac{I + \vec{r}' \cdot \vec{\sigma}}{2}$

$$\vec{r} \xrightarrow{\mathcal{E}} \vec{r}' = M\vec{r} + \vec{c}$$

where

$$M_{ij} = \sum_k \left(2\Re\{\alpha_{ki}\alpha^*_{kj}\} + 2\Im\{\alpha^*_{k0} \sum_{p=1}^{3} \epsilon_{ijp}\alpha_{kp}\} + \left(|\alpha_{k0}|^2 - \sum_{p=1}^{3} |\alpha_{kp}|^2 \right) \delta_{ij}$$

and

$$c_j = 2i \sum_k \sum_{mn} \epsilon_{mnj}\alpha_{kn}\alpha^*_{km}$$

Examples of Single Qubit Noisy Quantum Channels

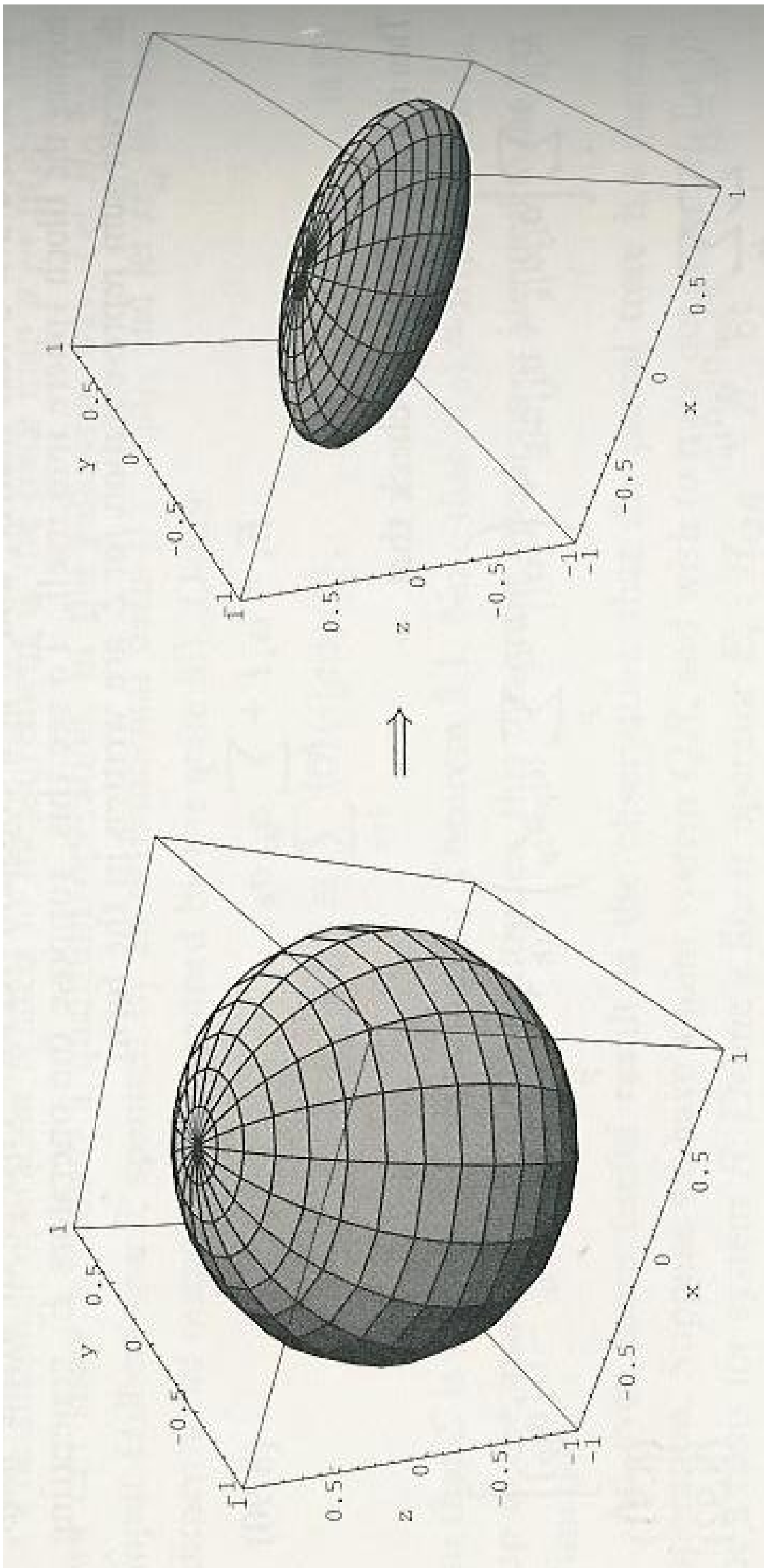
Bit Flip Channel

- The bit flip channel flips the state of a qubit from $|0\rangle$ to $|1\rangle$ (and vice versa) with probability p
- $\mathcal{E}(\rho) = E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger$
  - ρ : the density operator of single qubit
  - $E_0 = \sqrt{1-p} I = \sqrt{1-p} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
  - $E_1 = \sqrt{p} \sigma_x = \sqrt{p} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
- Trace-preserving : $E_0^\dagger E_0 + E_1^\dagger E_1 = I$

Visualization of the Bit Flip Channel

- $\rho = (I + \vec{r} \cdot \vec{\sigma})/2$ and $\mathcal{E}(\rho) = (I + \vec{r'} \cdot \vec{\sigma})/2$

$$\vec{r'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-2p & 0 \\ 0 & 0 & 1-2p \end{bmatrix} \vec{r}$$



x
y
z
-1
-0.5
0
0.5
1

Phase Flip Channel

- The phase flip channel flips the phase of the state $|1\rangle$ of a qubit with probability p
- $\mathcal{E}(\rho) = E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger$
  - ρ : the density operator of single qubit
  - $E_0 = \sqrt{1-p} I = \sqrt{1-p} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
  - $E_1 = \sqrt{p} \sigma_z = \sqrt{p} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
- Trace-preserving : $E_0^\dagger E_0 + E_1^\dagger E_1 = I$

Visualization of the Phase Flip Channel

- $\rho = (I + \vec{r} \cdot \vec{\sigma})/2$ and $\mathcal{E}(\rho) = (I + \vec{r'} \cdot \vec{\sigma})/2$

$$\vec{r'} = \begin{bmatrix} 1-2p & 0 & 0 \\ 0 & 1-2p & 0 \\ 0 & 0 & 1 \end{bmatrix} \vec{r}$$

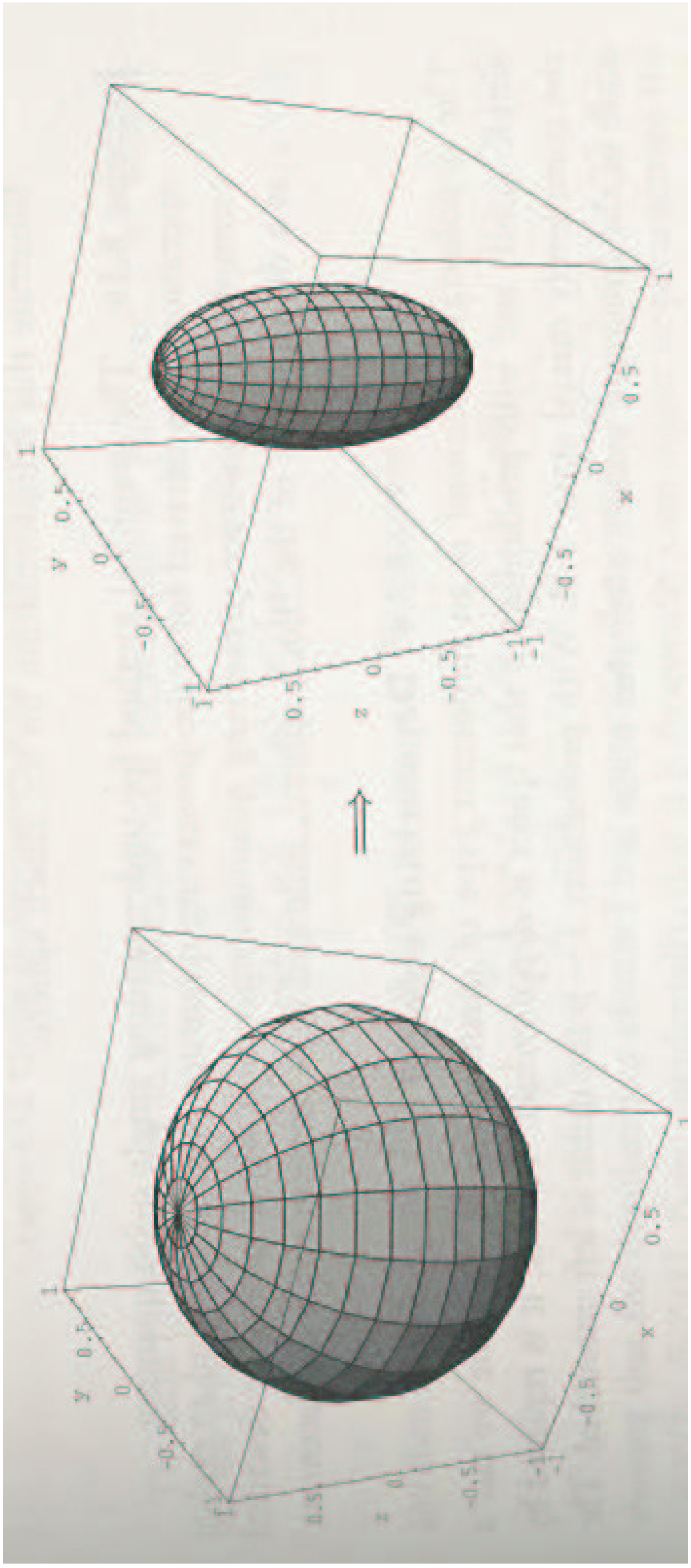
- $p = 1/2 :\ \vec{r'} = (0, 0, r_z)$ and the collection of operator elements

$$E_0 = \frac{1}{\sqrt{2}} I, \quad E_1 = \frac{1}{\sqrt{2}} \sigma_z,$$

is unitarily equivalent to the collection of operator elements

$$P_0 = |0\rangle\langle 0| = \frac{1}{\sqrt{2}} E_0 + \frac{1}{\sqrt{2}} E_1, \quad P_1 = |1\rangle\langle 1| = \frac{1}{\sqrt{2}} E_0 - \frac{1}{\sqrt{2}} E_1$$

which is the projective measurement on the basis $\{|0\rangle, |1\rangle\}$


x
y
z
-1
-0.5
0
0.5
1

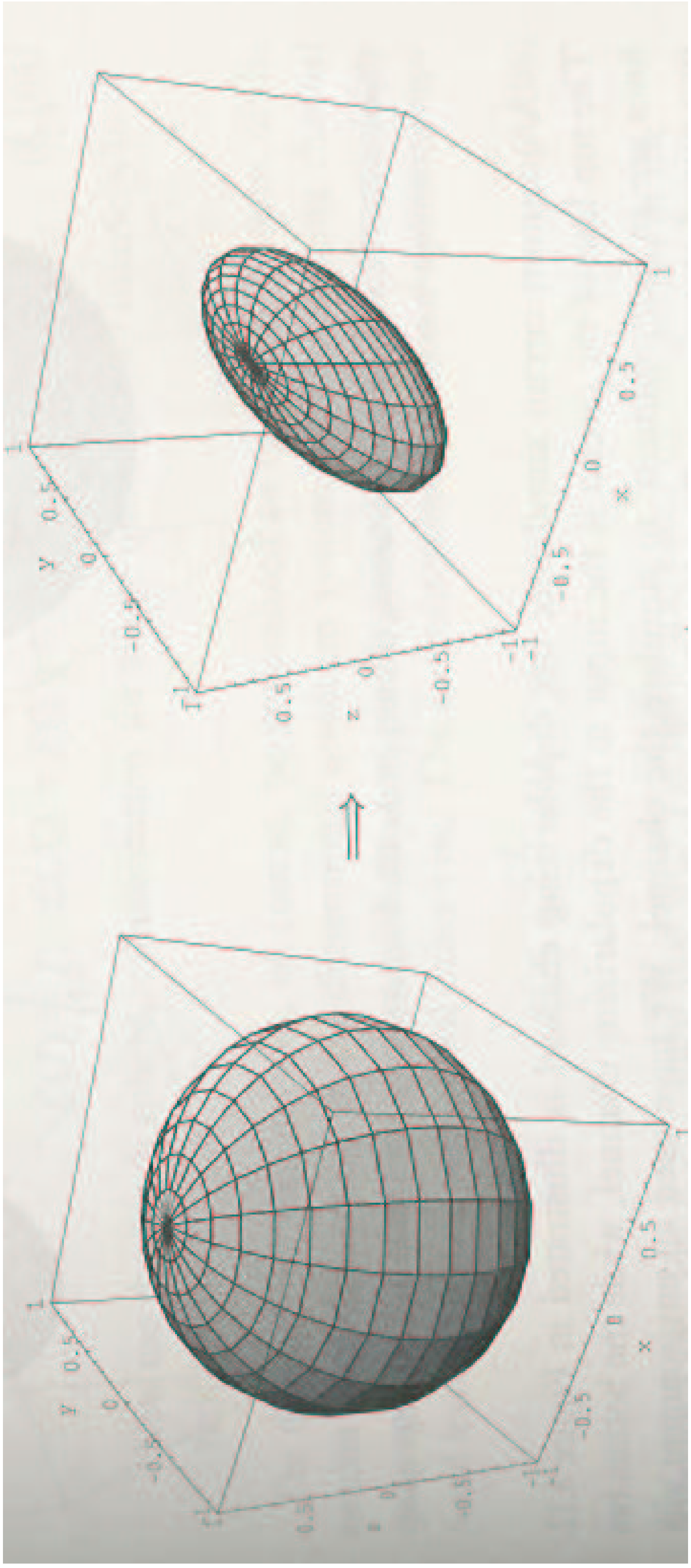
Bit-Phase Flip Channel

- The bit-phase flip channel flips the state of the qubit from $|0\rangle$ to $|1\rangle$ (and vice versa) and then flips the phase of the state $|1\rangle$ of the qubit with probability p
- $\sigma_y = -i\sigma_z\sigma_x$
- $\mathcal{E}(\rho) = E_0\rho E_0^\dagger + E_1\rho E_1^\dagger$
  - ρ : the density operator of single qubit
  - $E_0 = \sqrt{1-p}I = \sqrt{1-p}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
  - $E_1 = \sqrt{p}\sigma_y = \sqrt{p}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$
- Trace-preserving : $E_0^\dagger E_0 + E_1^\dagger E_1 = I$

Visualization of the Bit-Phase Flip Channel

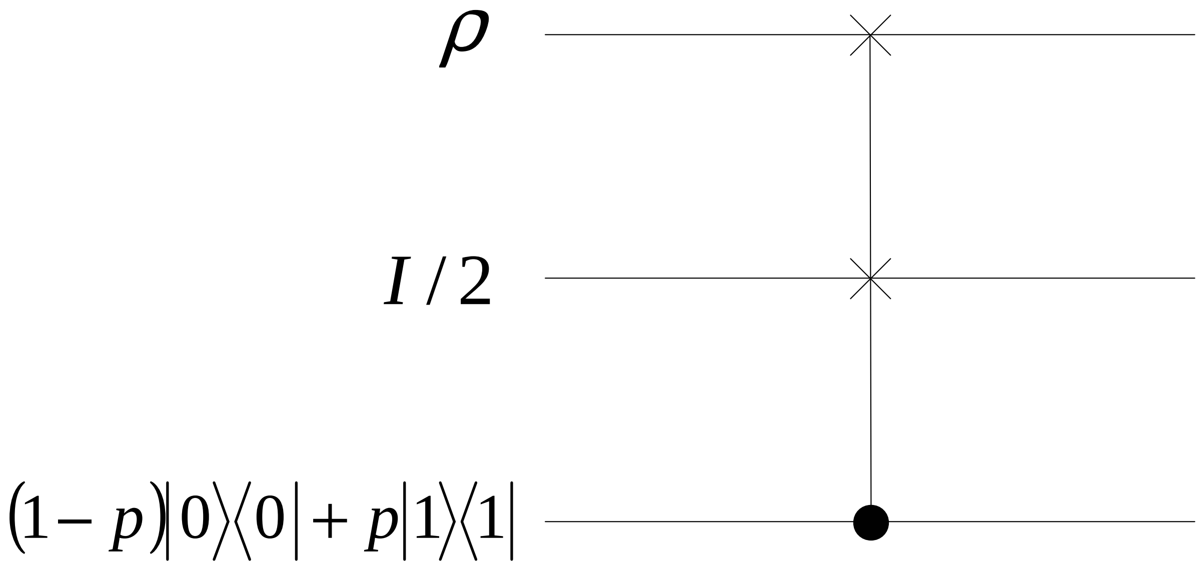
- $\rho = (I + \vec{r} \cdot \vec{\sigma})/2$ and $\mathcal{E}(\rho) = (I + \vec{r'} \cdot \vec{\sigma})/2$

$$\vec{r'} = \begin{bmatrix} 1-2p & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1-2p \end{bmatrix} \vec{r}$$


x
y
z
-0.5
0
0.5
1
-1

The Depolarizing Channel

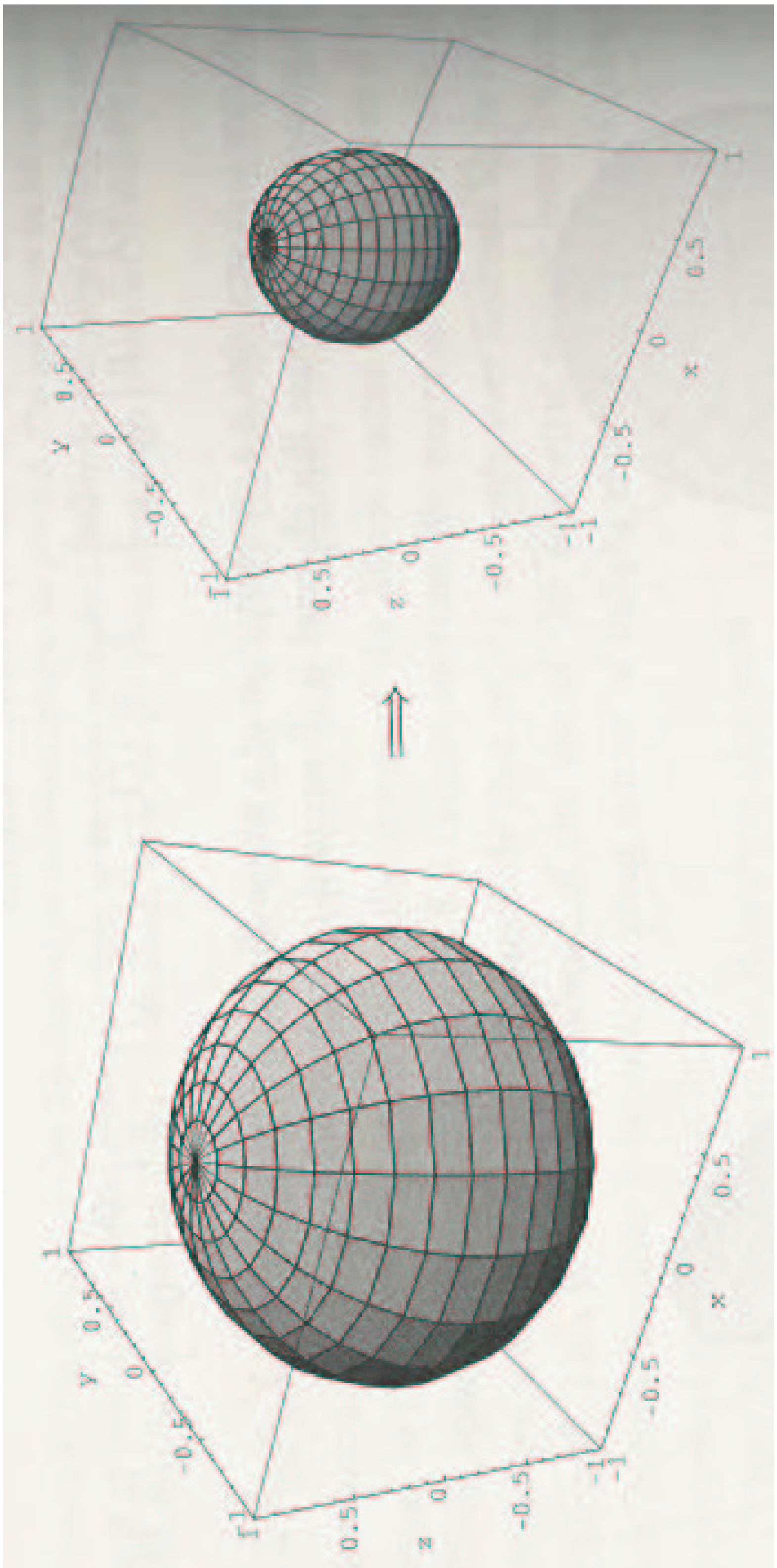
- With probability p, the single qubit is depolarized, i.e., its state is replaced by the completely mixed state $I/2$
- $\mathcal{E}(\rho) = p(I/2) + (1-p)\rho$
- A circuit implementation of the depolarizing channel



Visualization of the Depolarizing Channel

- $\rho = (I + \vec{r} \cdot \vec{\sigma})/2$ and $\mathcal{E}(\rho) = (I + \vec{r'} \cdot \vec{\sigma})/2$

$$\vec{r'} = \begin{bmatrix} 1-p & 0 & 0 \\ 0 & 1-p & 0 \\ 0 & 0 & 1-p \end{bmatrix} \vec{r}$$



x
y
z

The Operator-Sum Representation of Depolarizing Channels

- An identity :

$$\frac{I}{2} = \frac{\rho + \sigma_x \rho \sigma_x + \sigma_y \rho \sigma_y + \sigma_z \rho \sigma_z}{4}$$

- $\mathcal{E}(\rho) = (1 - (3/4)p)\rho + (p/4)(\sigma_x \rho \sigma_x + \sigma_y \rho \sigma_y + \sigma_z \rho \sigma_z)$
  - $E_0 = \sqrt{1 - 3p/4} I$
  - $E_1 = \sqrt{p}\sigma_x/2$
  - $E_2 = \sqrt{p}\sigma_y/2$
  - $E_3 = \sqrt{p}\sigma_z/2$
- Trace-preserving : $\sum_{k=0}^{3} E_k^{\dagger} E_k = I$
- Another expression :

$$\mathcal{E}(\rho) = (1 - p')\rho + (p'/3)(\sigma_x \rho \sigma_x + \sigma_y \rho \sigma_y + \sigma_z \rho \sigma_z)$$

The Depolarizing Channel for n-Qubit Systems

- With probability p, the n-qubit system is depolarized, i.e., its state is replaced by the completely mixed state $I/(2^n)$
- $\mathcal{E}(\rho) = pI/(2^n) + (1-p)\rho$

Amplitude Damping Channel

- $|0\rangle$: the ground state without a quantum of energy
- $|1\rangle$: the state with a quantum of energy
- $\mathcal{E}(\rho) = E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger$
  - ρ : the density operator of a single qubit
  - $E_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix}$: a quantum of energy was not lost to the environment such that the qubit must be more probably in the state $|0\rangle$ than in the state $|1\rangle$
  - $E_1 = \sqrt{p} \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix}$: a quantum of energy was lost to the environment with probability γ such that the the qubit must be in the state $|1\rangle$

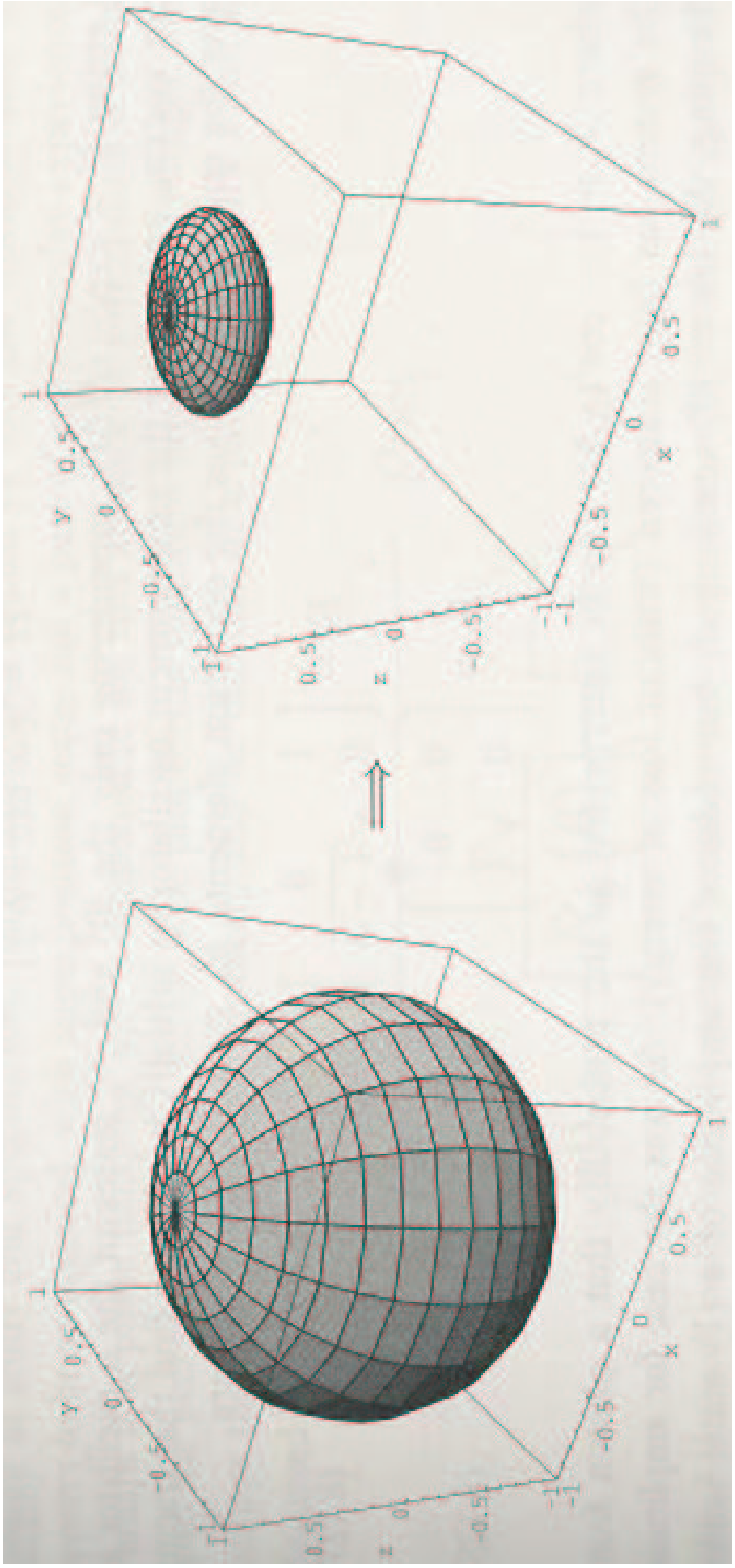
- Trace-preserving : $E_0^\dagger E_0 + E_1^\dagger E_1 = I$

Visualization of the Amplitude Damping Channel

- $\rho = (I + \vec{r} \cdot \vec{\sigma})/2$ and $\mathcal{E}(\rho) = (I + \vec{r'} \cdot \vec{\sigma})/2$

$$\vec{r'} = \begin{bmatrix} \sqrt{1-\gamma} & 0 & 0 \\ 0 & \sqrt{1-\gamma} & 0 \\ 0 & 0 & 1-\gamma \end{bmatrix} \vec{r} + \begin{bmatrix} 0 \\ 0 \\ \gamma \end{bmatrix}$$

- When describing $\gamma = 1 - e^{-t/T_1}$ as a time-varying function, the effect of amplitude damping is as a flow on the Bloch sphere, which moves every points in the unit sphere to the north pole of the unit sphere on which the state $|0\rangle$ resides


y
x
z
-0.5
0
0.5
1
-1

Phase Damping Channel

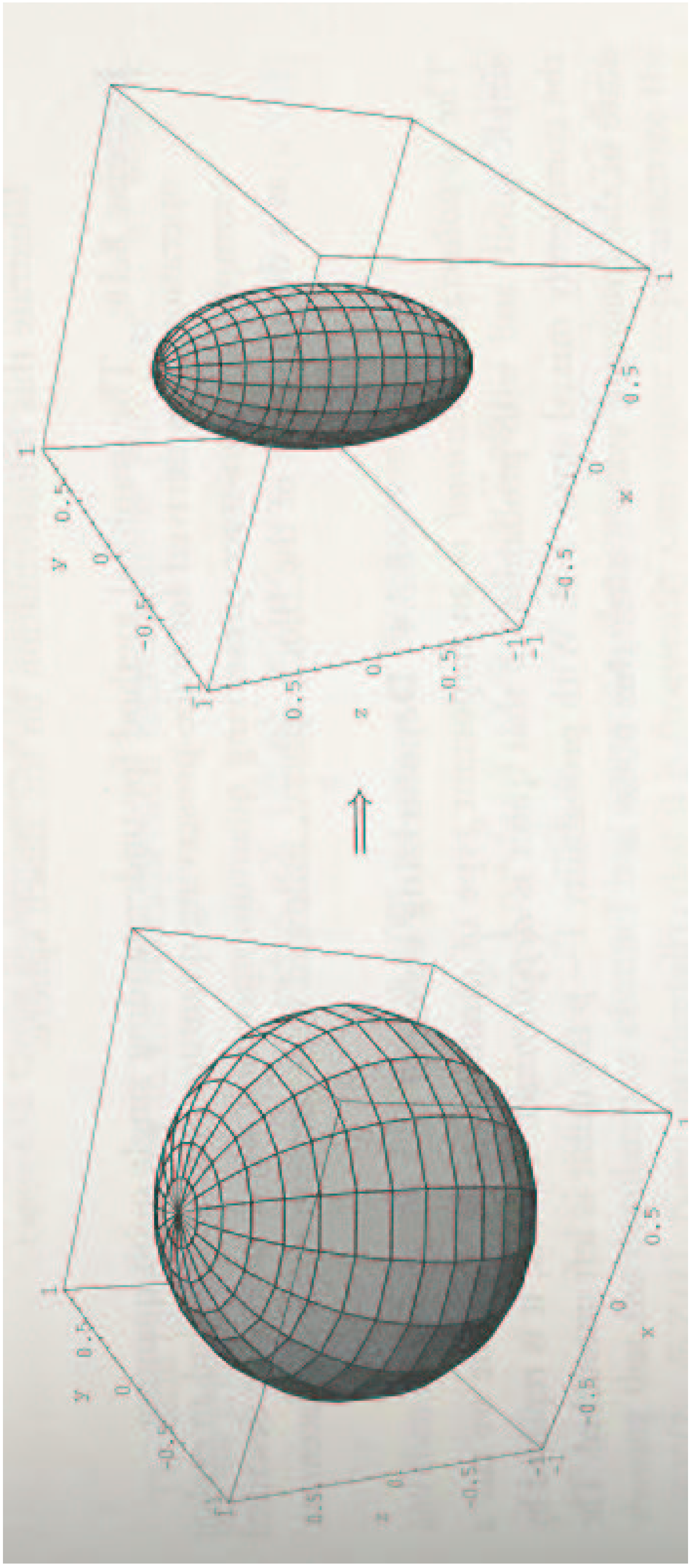
- The phase damping channel is the same as the phase flip channel
- $\mathcal{E}(\rho) = E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger$
  - ρ : the density operator of single qubit
  - $E_0 = \sqrt{1-p} I = \sqrt{1-p} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
  - $E_1 = \sqrt{p} \sigma_z = \sqrt{p} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
- Trace-preserving : $E_0^\dagger E_0 + E_1^\dagger E_1 = I$

Visualization of the Phase Damping Channel

- $\rho = (I + \vec{r} \cdot \vec{\sigma})/2$ and $\mathcal{E}(\rho) = (I + \vec{r'} \cdot \vec{\sigma})/2$

$$\vec{r'} = \begin{bmatrix} 1-2p & 0 & 0 \\ 0 & 1-2p & 0 \\ 0 & 0 & 1 \end{bmatrix} \vec{r}$$

- When describing $p = (1 - e^{-t/T_1})/2$ as a time-varying function, the effect of phase damping is as a flow on the Bloch sphere, which perpendicularly moves every points in the unit sphere to the z-axis of the unit sphere



y
x
z
y
x
z