EE641000 Quantum Information and Computation

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Unit Three – Quantum Circuits

Visualization of Unitary Operations on a Qubit

Visualizing State of a Single Qubit on Bloch Sphere

- $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$: the state of a qubit

$$\alpha = ae^{j\gamma}, \beta = be^{j(\gamma+\varphi)}$$

with $a, b \geq 0$ and $0 \leq \gamma, \varphi < 2\pi$

Since $a^2 + b^2 = 1$, there is a unique θ, $0 \leq \theta \leq \pi$ such that

$$a = \cos\frac{\theta}{2}, b = \sin\frac{\theta}{2}$$

Thus

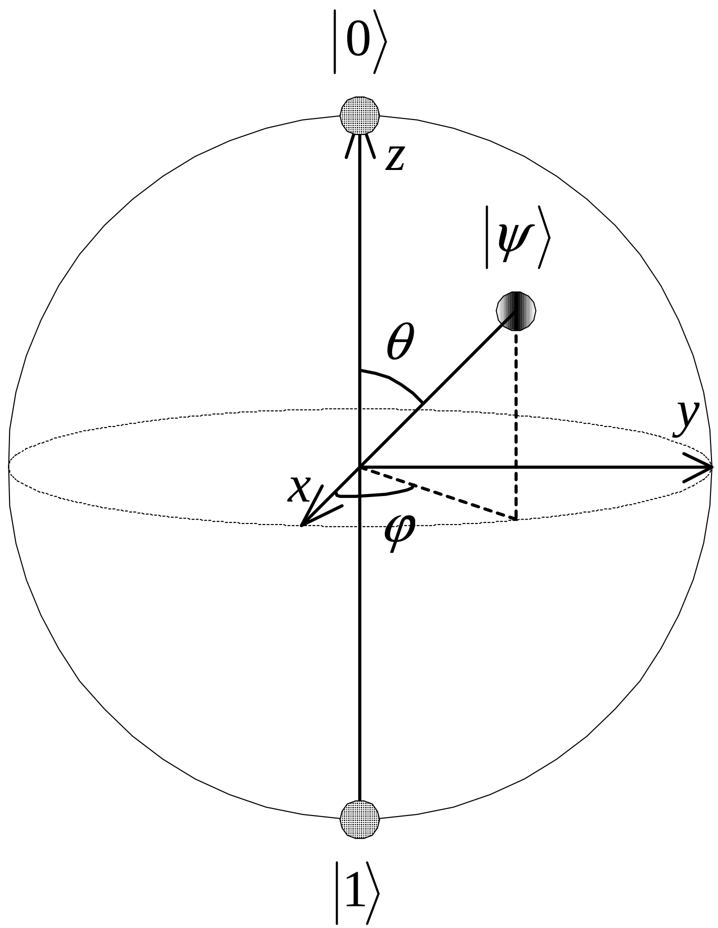
$$|\psi\rangle = e^{j\gamma}\left(\cos\frac{\theta}{2}|0\rangle + e^{j\varphi}\sin\frac{\theta}{2}|1\rangle\right)$$

and by ignoring the global factor $e^{j\gamma}$, we have

$$\boxed{|\psi\rangle = \cos\tfrac{\theta}{2}|0\rangle + e^{j\varphi}\sin\tfrac{\theta}{2}|1\rangle,\ 0 \leq \theta \leq \pi,\ 0 \leq \varphi < 2\pi}$$

Visualizing State of a Single Qubit on Bloch Sphere

- $(\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta)$: the Bloch vector



Complex Coordinate Vector Versus Real Bloch Vector

- $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = e^{j\gamma}\left(\cos\frac{\theta}{2}|0\rangle + e^{j\varphi}\sin\frac{\theta}{2}|1\rangle\right),\ 0 \leq \theta \leq \pi,\ 0 \leq \gamma, \varphi < 2\pi$: a qubit state where $|\alpha|^2 + |\beta|^2 = 1$
  - $\cos\frac{\theta}{2} = |\alpha|$ and then $\cos\theta = 2\alpha\bar{\alpha} - 1$
  - $e^{j\varphi}\sin\theta = 2\cos\frac{\theta}{2}e^{j\varphi}\sin\frac{\theta}{2} = 2\bar{\alpha}\beta$
- $(x, y, z) = (\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta)$: the Bloch vector of $|\psi\rangle$
  - $|\alpha| = \sqrt{\frac{1+\cos\theta}{2}} = \sqrt{\frac{1+z}{2}}$
  - $2\bar{\alpha}\beta = e^{j\varphi}\sin\theta = x + iy$

$$\boxed{\begin{array}{ccccc} x + iy & = & e^{i\varphi}\sin\theta & = & 2\bar{\alpha}\beta \\ z & = & \cos\theta & = & 2\alpha\bar{\alpha} - 1 \end{array}}$$

Elementary Unitary Operators on a Single Qubit

- $\mathcal{B} = \{|0\rangle, |1\rangle\}$: an orthonormal basis of state space of the qubit
- $\sigma_x, \sigma_y, \sigma_z$: Pauli operators with matrix representations $X = [\sigma_x]_{\mathcal{B}}, Y = [\sigma_y]_{\mathcal{B}}, Z = [\sigma_z]_{\mathcal{B}}$ respectively

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- Spectral decompositions of Pauli operators

$$\sigma_x = |+\rangle\langle+| - |-\rangle\langle-|, \sigma_y = |+'\rangle\langle+'| - |-'\rangle\langle-'|, \sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1$$

  - $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}, |-\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$
  - $|+'\rangle = (|0\rangle + i|1\rangle)/\sqrt{2}, |-'\rangle = (|0\rangle - i|1\rangle)/\sqrt{2}$

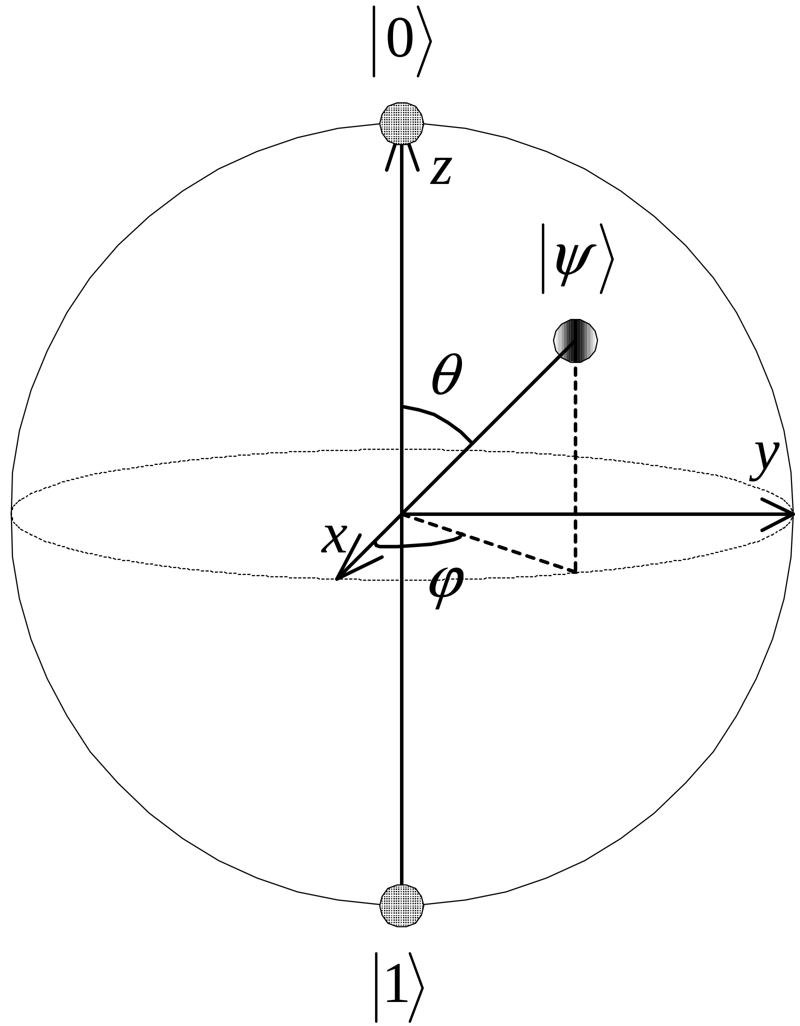
Elementary Unitary Operators on a Single Qubit (Cont')

- $\sigma_h, \sigma_s, \sigma_t$: Hadamard, phase, $\pi/8$ operators with matrix representations $H = [\sigma_h]_{\mathcal{B}}, S = [\sigma_s]_{\mathcal{B}}, T = [\sigma_t]_{\mathcal{B}}$ respectively

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{bmatrix}$$

- $\sigma_h, \sigma_x, \sigma_y, \sigma_z$: Hermitian operators

Visualizing Pauli Operators on Bloch Sphere



Δ° Rotation about $\hat{n}$ Axis of the Sphere

- A coordinate transformation on the sphere
- $\hat{n} = (n_x, n_y, n_z) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$: the rotation axis
- $\hat{x} = (1,0,0), \hat{y} = (0,1,0), \hat{z} = (0,0,1)$: standard right-hand orthogonal coordinate system $\mathcal{S}$ in R^3
- $\hat{x}' = (-\cos\theta\cos\varphi, -\cos\theta\sin\varphi, \sin\theta)$, $\hat{y}' = (\sin\varphi, -\cos\varphi, 0)$, $\hat{z}' = \hat{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$: another right-hand orthogonal coordinate system $\{\hat{n}\}$ in R^3

- $[\{\hat{n}\} \to \mathcal{S}], [\mathcal{S} \to \{\hat{n}\}]$: Coordinate transformations

$$[\{\hat{n}\} \to \mathcal{S}] = \begin{bmatrix} -\cos\theta\cos\varphi & \sin\varphi & \sin\theta\cos\varphi \\ -\cos\theta\sin\varphi & -\cos\varphi & \sin\theta\sin\varphi \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$[\mathcal{S} \to \{\hat{n}\}] = \begin{bmatrix} -\cos\theta\cos\varphi & -\cos\theta\sin\varphi & \sin\theta \\ \sin\varphi & -\cos\varphi & 0 \\ \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \end{bmatrix}$$

- $[\Delta^\circ]_{\{\hat{n}\}}$: Δ° Rotation about $\hat{n}$ axis in the $\{\hat{n}\}$ right-hand orthogonal coordinate system

$$[\Delta^\circ]_{\{\hat{n}\}} = \begin{bmatrix} \cos\Delta & -\sin\Delta & 0 \\ \sin\Delta & \cos\Delta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- $[\Delta^\circ - \hat{n}]$: Δ° Rotation about $\hat{n}$ axis in the standard right-hand orthogonal coordinate system

$$
\begin{aligned}
& [\Delta^\circ - \hat{n}] \\
= \quad & [\{\hat{n}\} \rightarrow \mathcal{S}][\Delta^\circ]_{\{\hat{n}\}}[\mathcal{S} \rightarrow \{\hat{n}\}] \\
= \quad & \begin{bmatrix}
(1-\cos\Delta)n_x^2 + \cos\Delta & (1-\cos\Delta)n_x n_y - \sin\Delta n_z & (1-\cos\Delta)n_z n_x + \sin\Delta n_y \\
(1-\cos\Delta)n_x n_y + \sin\Delta n_z & (1-\cos\Delta)n_y^2 + \cos\Delta & (1-\cos\Delta)n_y n_z - \sin\Delta n_x \\
(1-\cos\Delta)n_z n_x - \sin\Delta n_y & (1-\cos\Delta)n_y n_z + \sin\Delta n_x & (1-\cos\Delta)n_z^2 + \cos\Delta
\end{bmatrix}
\end{aligned}
$$

Special Examples

$$[\Delta^\circ - \hat{x}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Delta & -\sin\Delta \\ 0 & \sin\Delta & \cos\Delta \end{bmatrix}$$

$$[\Delta^\circ - \hat{y}] = \begin{bmatrix} \cos\Delta & 0 & \sin\Delta \\ 0 & 1 & 0 \\ -\sin\Delta & 0 & \cos\Delta \end{bmatrix}$$

$$[\Delta^\circ - \hat{z}] = \begin{bmatrix} \cos\Delta & -\sin\Delta & 0 \\ \sin\Delta & \cos\Delta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Visualizing Pauli-X Operator on Bloch Sphere

$$
\begin{array}{ccc}
\cos\frac{\theta}{2}|0\rangle + e^{j\varphi}\sin\frac{\theta}{2}|1\rangle & \xrightarrow{X} & \cos\frac{\pi-\theta}{2}|0\rangle + e^{-j\varphi}\sin\frac{\pi-\theta}{2}|1\rangle \\
\uparrow\downarrow & & \uparrow\downarrow \\
(\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta)^t & \xrightarrow{[\pi-\hat{x}]} & (\cos\varphi\sin\theta, -\sin\varphi\sin\theta, -\cos\theta)^t \\
 & & \| \\
 & & \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\pi & -\sin\pi \\ 0 & \sin\pi & \cos\pi \end{bmatrix} \begin{bmatrix} \cos\varphi\sin\theta \\ \sin\varphi\sin\theta \\ \cos\theta \end{bmatrix}
\end{array}
$$

- Pauli-X operator corresponds to a 180° rotation along the $\hat{x}$-axis on the Bloch sphere, i.e., $[180^\circ - \hat{x}]$

Visualizing Pauli-Y Operator on Bloch Sphere

$$
\begin{array}{ccc}
\cos\dfrac{\theta}{2}|0\rangle + e^{j\varphi}\sin\dfrac{\theta}{2}|1\rangle & \xrightarrow{Y} & \cos\dfrac{\pi-\theta}{2}|0\rangle + e^{j(\pi-\varphi)}\sin\dfrac{\pi-\theta}{2}|1\rangle \\
\uparrow\downarrow & & \uparrow\downarrow \\
(\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta)^t & \xrightarrow{[\pi-\hat{y}]} & (-\cos\varphi\sin\theta, \sin\varphi\sin\theta, -\cos\theta)^t \\
 & & || \\
 & & \begin{bmatrix} \cos\pi & 0 & \sin\pi \\ 0 & 1 & 0 \\ -\sin\pi & 0 & \cos\pi \end{bmatrix} \begin{bmatrix} \cos\varphi\sin\theta \\ \sin\varphi\sin\theta \\ \cos\theta \end{bmatrix}
\end{array}
$$

- Pauli-Y operator corresponds to a 180° rotation along the $\hat{y}$-axis on the Bloch sphere, i.e., $[180^\circ - \hat{y}]$

Visualizing Pauli-Z Operator on Bloch Sphere

$$
\begin{array}{ccc}
\cos\frac{\theta}{2}|0\rangle + e^{j\varphi}\sin\frac{\theta}{2}|1\rangle & \xrightarrow{Z} & \cos\frac{\theta}{2}|0\rangle + e^{j(\pi+\varphi)}\sin\frac{\theta}{2}|1\rangle \\
\uparrow\downarrow & & \uparrow\downarrow \\
(\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta)^t & \xrightarrow{[\pi-\hat{z}]} & (-\cos\varphi\sin\theta, -\sin\varphi\sin\theta, \cos\theta)^t \\
 & & \| \\
 & & \begin{bmatrix} \cos\pi & -\sin\pi & 0 \\ \sin\pi & \cos\pi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\varphi\sin\theta \\ \sin\varphi\sin\theta \\ \cos\theta \end{bmatrix}
\end{array}
$$

- Pauli-Z operator corresponds to a 180° rotation along the $\hat{z}$-axis on the Bloch sphere, i.e., $[180^\circ - \hat{z}]$

Special Rotation Operators

$$
\begin{aligned}
R_x(\Delta) &= e^{-i\frac{\Delta}{2}\sigma_x} = e^{-i\frac{\Delta}{2}}|+\rangle\langle+| + e^{i\frac{\Delta}{2}}|-\rangle\langle-| \\
&= \cos\frac{\Delta}{2}(|+\rangle\langle+| + |-\rangle\langle-|) - i\sin\frac{\Delta}{2}(|+\rangle\langle+| - |-\rangle\langle-|) \\
&= \cos\frac{\Delta}{2}I - i\sin\frac{\Delta}{2}\sigma_x \Longleftrightarrow \begin{bmatrix} \cos\frac{\Delta}{2} & -i\sin\frac{\Delta}{2} \\ -i\sin\frac{\Delta}{2} & \cos\frac{\Delta}{2} \end{bmatrix} \\
R_y(\Delta) &= e^{-i\frac{\Delta}{2}\sigma_y} = e^{-i\frac{\Delta}{2}}|+'\rangle\langle+'| + e^{i\frac{\Delta}{2}}|-'\rangle\langle-'| \\
&= \cos\frac{\Delta}{2}(|+'\rangle\langle+'| + |-'\rangle\langle-'|) - i\sin\frac{\Delta}{2}(|+'\rangle\langle+'| - |-'\rangle\langle-'|) \\
&= \cos\frac{\Delta}{2}I - i\sin\frac{\Delta}{2}\sigma_y \Longleftrightarrow \begin{bmatrix} \cos\frac{\Delta}{2} & -\sin\frac{\Delta}{2} \\ \sin\frac{\Delta}{2} & \cos\frac{\Delta}{2} \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
R_z(\Delta) &= e^{-i\frac{\Delta}{2}\sigma_z} = e^{-i\frac{\Delta}{2}}|0\rangle\langle 0| + e^{i\frac{\Delta}{2}}|1\rangle\langle 1| \\
&= \cos\frac{\Delta}{2} I - i\sin\frac{\Delta}{2}\sigma_z \Longleftrightarrow \begin{bmatrix} e^{-i\Delta/2} & 0 \\ 0 & e^{i\Delta/2} \end{bmatrix}
\end{aligned}
$$

Generic Rotation Operators

- $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$: the vector of Pauli operators
  - Anti-commutation relations of Pauli operators

$$\{\sigma_x, \sigma_y\} = \sigma_x\sigma_y + \sigma_y\sigma_x = 0$$
$$\{\sigma_y, \sigma_z\} = \sigma_y\sigma_z + \sigma_z\sigma_y = 0$$
$$\{\sigma_z, \sigma_x\} = \sigma_z\sigma_x + \sigma_x\sigma_z = 0$$

- $\hat{n} = (n_x, n_y, n_z)$: a real unit vector in three dimensions
- $\hat{n} \cdot \vec{\sigma} \equiv n_x\sigma_x + n_y\sigma_y + n_z\sigma_z$: a Hermitian and unitary operator on the state space of a qubit such that

$$\begin{aligned}
(\hat{n} \cdot \vec{\sigma})^2 &= n_x^2\sigma_x^2 + n_y^2\sigma_y^2 + n_z^2\sigma_z^2 \\
&\quad + n_x n_y\{\sigma_x\sigma_y\} + n_y n_z\{\sigma_y\sigma_z\} + n_z n_x\{\sigma_z\sigma_x\} \\
&= I
\end{aligned}$$

- $\hat{n} \cdot \vec{\sigma} = |\psi_1\rangle\langle\psi_1| - |\psi_2\rangle\langle\psi_2|$: spectral decomposition of $\hat{n} \cdot \vec{\sigma}$

We define

$$\begin{aligned}
R_{\hat{n}}(\Delta) &\triangleq e^{-i\frac{\Delta}{2}\hat{n}\cdot\vec{\sigma}} \\
&= e^{-i\frac{\Delta}{2}}|\psi_1\rangle\langle\psi_1| + e^{i\frac{\Delta}{2}}|\psi_2\rangle\langle\psi_2| \\
&= \cos\frac{\Delta}{2}(|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|) - i\sin\frac{\Delta}{2}(|\psi_1\rangle\langle\psi_1| - |\psi_2\rangle\langle\psi_2|) \\
&= \cos\frac{\Delta}{2}I - i\sin\frac{\Delta}{2}\hat{n}\cdot\vec{\sigma}
\end{aligned}$$

Visualizing a Generic Rotation Operator on Bloch Sphere

$$|\psi\rangle = \cos\tfrac{\theta}{2}|0\rangle + e^{i\varphi}\sin\tfrac{\theta}{2}|1\rangle$$

$$\downarrow R_{\hat{n}}(\Delta)$$

$$\begin{aligned}
&|\psi'\rangle \\
= \;& \cos\tfrac{\Delta}{2}\left(\cos\tfrac{\theta}{2}|0\rangle + e^{i\varphi}\sin\tfrac{\theta}{2}|1\rangle\right) - i\sin\tfrac{\Delta}{2}\left(n_x\left(e^{i\varphi}\sin\tfrac{\theta}{2}|0\rangle + \cos\tfrac{\theta}{2}|1\rangle\right)\right. \\
& \left. +n_y\left(-ie^{i\varphi}\sin\tfrac{\theta}{2}|0\rangle + i\cos\tfrac{\theta}{2}|1\rangle\right) + n_z\left(\cos\tfrac{\theta}{2}|0\rangle - e^{i\varphi}\sin\tfrac{\theta}{2}|1\rangle\right)\right) \\
= \;& \left(\cos\tfrac{\Delta}{2}\cos\tfrac{\theta}{2} - i\sin\tfrac{\Delta}{2}\left(n_x e^{i\varphi}\sin\tfrac{\theta}{2} - in_y e^{i\varphi}\sin\tfrac{\theta}{2} + n_z\cos\tfrac{\theta}{2}\right)\right)|0\rangle \\
& + \left(\cos\tfrac{\Delta}{2}e^{i\varphi}\sin\tfrac{\theta}{2} - i\sin\tfrac{\Delta}{2}\left(n_x\cos\tfrac{\theta}{2} + in_y\cos\tfrac{\theta}{2} - n_z e^{i\varphi}\sin\tfrac{\theta}{2}\right)\right)|1\rangle \\
= \;& \left(\cos\tfrac{\Delta}{2}\cos\tfrac{\theta}{2} + \sin\tfrac{\Delta}{2}\left(n_x\sin\varphi\sin\tfrac{\theta}{2} - n_y\cos\varphi\sin\tfrac{\theta}{2}\right)\right. \\
& \left. +i\sin\tfrac{\Delta}{2}\left(n_x\cos\varphi\sin\tfrac{\theta}{2} + n_y\sin\varphi\sin\tfrac{\theta}{2} + n_z\cos\tfrac{\theta}{2}\right)\right)|0\rangle \\
& + \left(\cos\tfrac{\Delta}{2}\cos\varphi\sin\tfrac{\theta}{2} + \sin\tfrac{\Delta}{2}\left(n_y\cos\tfrac{\theta}{2} - n_z\sin\varphi\sin\tfrac{\theta}{2}\right)\right. \\
& \left. +i\left(\cos\tfrac{\Delta}{2}\sin\varphi\sin\tfrac{\theta}{2} - \sin\tfrac{\Delta}{2}\left(n_x\cos\tfrac{\theta}{2} - n_z\cos\varphi\sin\tfrac{\theta}{2}\right)\right)\right)|1\rangle
\end{aligned}$$

$$(x,y,z)^t = (\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta)^t$$

$$\downarrow [\Delta - \hat{n}]$$

$$\begin{aligned}
& (x',y',z')^t \\
= \quad & (((1-\cos\Delta)n_x^2 + \cos\Delta)x + ((1-\cos\Delta)n_x n_y - \sin\Delta n_z)y \\
& +((1-\cos\Delta)n_z n_x + \sin\Delta n_y)z, ((1-\cos\Delta)n_x n_y + \sin\Delta n_z)x \\
& +((1-\cos\Delta)n_y^2 + \cos\Delta)y + ((1-\cos\Delta)n_y n_z - \sin\Delta n_x)z, \\
& ((1-\cos\Delta)n_z n_x - \sin\Delta n_y)x + ((1-\cos\Delta)n_y n_z + \sin\Delta n_x)y \\
& +((1-\cos\Delta)n_z^2 + \cos\Delta)z)^t \\
= \quad & [\Delta - \hat{n}] \begin{bmatrix} x \\ y \\ z \end{bmatrix}
\end{aligned}$$

Δ° Rotation along $\hat{x}$ Axis

- $\hat{n} = \hat{x} = (1, 0, 0)$

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$$
$$\downarrow R_{\hat{x}}(\Delta)$$

$$\begin{aligned} & |\psi'\rangle \\ = \; & \left(\cos\frac{\Delta}{2}\cos\frac{\theta}{2} + \sin\frac{\Delta}{2}\sin\varphi\sin\frac{\theta}{2} + i\sin\frac{\Delta}{2}\cos\varphi\sin\frac{\theta}{2}\right)|0\rangle \\ & + \left(\cos\frac{\Delta}{2}\cos\varphi\sin\frac{\theta}{2} + i\left(\cos\frac{\Delta}{2}\sin\varphi\sin\frac{\theta}{2} - \sin\frac{\Delta}{2}\cos\frac{\theta}{2}\right)\right)|1\rangle \end{aligned}$$

$$(x, y, z)^t = (\cos\varphi \sin\theta, \sin\varphi \sin\theta, \cos\theta)^t$$

$$\downarrow [\Delta - \hat{x}]$$

$$\begin{aligned} & (x', y', z')^t \\ = \; & (x, y\cos\Delta - z\sin\Delta, y\sin\Delta + z\cos\Delta)^t \\ = \; & \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Delta & -\sin\Delta \\ 0 & \sin\Delta & \cos\Delta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \end{aligned}$$

Δ° Rotation along $\hat{y}$ Axis

- $\hat{n} = \hat{y} = (0, 1, 0)$

$$|\psi\rangle = \cos\tfrac{\theta}{2}|0\rangle + e^{i\varphi}\sin\tfrac{\theta}{2}|1\rangle$$

$$\downarrow R_{\hat{y}}(\Delta)$$

$$\begin{aligned} & |\psi'\rangle \\ = \quad & \left(\cos\tfrac{\Delta}{2}\cos\tfrac{\theta}{2} - \sin\tfrac{\Delta}{2}\cos\varphi\sin\tfrac{\theta}{2} + i\sin\tfrac{\Delta}{2}\sin\varphi\sin\tfrac{\theta}{2}\right)|0\rangle \\ & + \left(\cos\tfrac{\Delta}{2}\cos\varphi\sin\tfrac{\theta}{2} + \sin\tfrac{\Delta}{2}\cos\tfrac{\theta}{2} + i\cos\tfrac{\Delta}{2}\sin\varphi\sin\tfrac{\theta}{2}\right)|1\rangle \end{aligned}$$

$$(x, y, z)^t = (\cos\varphi \sin\theta, \sin\varphi \sin\theta, \cos\theta)^t$$

$$\downarrow [\Delta - \hat{y}]$$

$$\begin{aligned} & (x', y', z')^t \\ = \quad & (z \sin\Delta + x \cos\Delta, y, z \cos\Delta - x \sin\Delta)^t \\ = \quad & \begin{bmatrix} \cos\Delta & 0 & \sin\Delta \\ 0 & 1 & 0 \\ -\sin\Delta & 0 & \cos\Delta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \end{aligned}$$

Δ° Rotation along $\hat{z}$ Axis

- $\hat{n} = \hat{z} = (0, 0, 1)$

$$|\psi\rangle = \cos\tfrac{\theta}{2}|0\rangle + e^{i\varphi}\sin\tfrac{\theta}{2}|1\rangle$$

$$\downarrow R_{\hat{z}}(\Delta)$$

$$\begin{aligned}
& |\psi'\rangle \\
= \quad & \left(\cos\tfrac{\Delta}{2}\cos\tfrac{\theta}{2} + i\sin\tfrac{\Delta}{2}\cos\tfrac{\theta}{2}\right)|0\rangle \\
& + \left(\cos\tfrac{\Delta}{2}\cos\varphi\sin\tfrac{\theta}{2} - \sin\tfrac{\Delta}{2}\sin\varphi\sin\tfrac{\theta}{2}\right. \\
& \left. +i\left(\cos\tfrac{\Delta}{2}\sin\varphi\sin\tfrac{\theta}{2} + \sin\tfrac{\Delta}{2}\cos\varphi\sin\tfrac{\theta}{2}\right)\right)|1\rangle
\end{aligned}$$

$$(x, y, z)^t = (\cos\varphi \sin\theta, \sin\varphi \sin\theta, \cos\theta)^t$$

$$\downarrow [\Delta - \hat{z}]$$

$$\begin{aligned}
& (x', y', z')^t \\
= \quad & (x\cos\Delta - y\sin\Delta, x\sin\Delta + y\cos\Delta, z)^t \\
= \quad & \begin{bmatrix} \cos\Delta & -\sin\Delta & 0 \\ \sin\Delta & \cos\Delta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}
\end{aligned}$$

Single Qubit Operations

Algebraic Relations on Elementary Unitary Operators

- $\sigma_h = (\sigma_x + \sigma_z)/\sqrt{2}$
- $\sigma_z = \sigma_s^2$ and $\sigma_s = \sigma_t^2$
- $\sigma_h^2 = \sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I$
- $\sigma_x\sigma_y = -\sigma_y\sigma_x$, $\sigma_x\sigma_z = -\sigma_z\sigma_x$, $\sigma_y\sigma_z = -\sigma_z\sigma_y$
- $\sigma_x\sigma_y\sigma_x = -\sigma_y$, $\sigma_x\sigma_z\sigma_x = -\sigma_z$, $\sigma_y\sigma_x\sigma_y = -\sigma_x$, $\sigma_y\sigma_z\sigma_y = -\sigma_z$, $\sigma_z\sigma_x\sigma_z = -\sigma_x$, $\sigma_z\sigma_y\sigma_z = -\sigma_y$
- $\sigma_h\sigma_x = \sigma_z\sigma_h$, $\sigma_h\sigma_y = -\sigma_y\sigma_h$, $\sigma_h\sigma_z = \sigma_x\sigma_h$
- $\sigma_h\sigma_x\sigma_h = \sigma_z$, $\sigma_h\sigma_y\sigma_h = -\sigma_y$, $\sigma_h\sigma_z\sigma_h = \sigma_x$

Relation Between Elementary and Rotation Operators

- $\sigma_x = e^{i\pi/2} R_x(\pi)$
- $\sigma_y = e^{i\pi/2} R_y(\pi)$
- $\sigma_z = e^{i\pi/2} R_z(\pi)$
- $\sigma_s = e^{i\pi/4} R_z(\pi/2)$
- $\sigma_t = e^{i\pi/8} R_z(\pi/4)$
- $\sigma_h = e^{i\pi/2} R_y(\pi/2) R_z(\pi)$

$$
\begin{aligned}
H &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
&= e^{i\pi/2} [R_y(\pi/2)][R_z(\pi)]
\end{aligned}
$$

- $\sigma_x R_y(\theta)\sigma_x = R_y(-\theta)$, $\sigma_x R_z(\theta)\sigma_x = R_z(-\theta)$,
  $\sigma_y R_x(\theta)\sigma_y = R_x(-\theta)$, $\sigma_y R_z(\theta)\sigma_y = R_z(-\theta)$,
  $\sigma_z R_x(\theta)\sigma_z = R_x(-\theta)$, $\sigma_z R_y(\theta)\sigma_z = R_y(-\theta)$
- $\sigma_h R_x(\theta)\sigma_h = R_z(\theta)$, $\sigma_h R_y(\theta)\sigma_h = R_y(-\theta)$, $\sigma_h R_z(\theta)\sigma_h = R_x(\theta)$
- $\sigma_h R_{\hat{n}=(n_x,n_y,n_z)}(\theta)\sigma_h = R_{\hat{m}=(n_z,-n_y,n_x)}(\theta)$
- $\sigma_h \sigma_t \sigma_h = e^{i\pi/8}\sigma_h R_z(\pi/4)\sigma_h = e^{i\pi/8}R_x(\pi/4)$
- $\sigma_h \sigma_s \sigma_h = e^{i\pi/4}\sigma_h R_z(\pi/2)\sigma_h = e^{i\pi/4}R_x(\pi/2)$

Every Unitary Operator on a Qubit Is a Rotation Operator

$$\boxed{U = e^{i\alpha} R_{\hat{n}}(\theta)}$$

- The proof is an exercise

$Z - Y$ Decomposition of Unitary Operators on a Qubit

- U : a unitary operator on a single qubit

There exist real numbers $\alpha, \beta, \gamma, \delta$ such that

$$\boxed{U = e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta)}$$

Proof

- Every row and column of U is a unit vector : there exist real numbers $\theta_1, \theta_2, \theta_3, \theta_4$ and $0 \leq \theta \leq \pi/2$ such that

$$U = \begin{bmatrix} e^{i\theta_1}\cos\theta & e^{i\theta_2}\sin\theta \\ e^{i\theta_3}\sin\theta & e^{i\theta_4}\cos\theta \end{bmatrix}$$

- The two rows (columns) of U are orthogonal : for $0 < \theta < \pi/2$

$$\begin{aligned} e^{i(\theta_1-\theta_3)} + e^{i(\theta_2-\theta_4)} &= 0 \\ e^{i(\theta_1-\theta_2)} + e^{i(\theta_3-\theta_4)} &= 0 \end{aligned}$$

Thus we have

$$\theta_1 - \theta_3 \equiv \theta_2 - \theta_4 + \pi \pmod{2\pi}, \text{i.e.}, \theta_4 \equiv -\theta_1 + \theta_2 + \theta_3 + \pi \pmod{2\pi}$$

and then

$$U = \begin{bmatrix} e^{i\theta_1}\cos\theta & e^{i\theta_2}\sin\theta \\ e^{i\theta_3}\sin\theta & e^{i(-\theta_1+\theta_2+\theta_3+\pi)}\cos\theta \end{bmatrix}$$

- $\theta_1 = \alpha - \beta/2 - \delta/2$
- $\theta_2 = \alpha - \beta/2 + \delta/2 + \pi$
- $\theta_3 = \alpha + \beta/2 - \delta/2$
- $\theta = \gamma/2$

$$\begin{aligned} U &= \begin{bmatrix} e^{i(\alpha-\beta/2-\delta/2)}\cos\frac{\gamma}{2} & -e^{i(\alpha-\beta/2+\delta/2)}\sin\frac{\gamma}{2} \\ e^{i(\alpha+\beta/2-\delta/2)}\sin\frac{\gamma}{2} & e^{i(\alpha+\beta/2+\delta/2)}\cos\frac{\gamma}{2} \end{bmatrix} \\ &= e^{i\alpha}\begin{bmatrix} e^{-i\beta/2} & 0 \\ 0 & e^{i\beta/2} \end{bmatrix}\begin{bmatrix} \cos\frac{\gamma}{2} & -\sin\frac{\gamma}{2} \\ \sin\frac{\gamma}{2} & \cos\frac{\gamma}{2} \end{bmatrix}\begin{bmatrix} e^{-i\delta/2} & 0 \\ 0 & e^{i\delta/2} \end{bmatrix} \end{aligned}$$

$$= \quad e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta)$$

A Corollary

- U : a unitary operator on a qubit

There exist unitary operators A, B, C on the qubit such that $ABC = I$ and

$$\boxed{U = e^{i\alpha} A \sigma_x B \sigma_x C}$$

where α is some global phase factor.

Proof

- $U = e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta)$: $Z - Y$ decomposition of U
- $A \equiv R_z(\beta) R_y(\gamma/2)$, $B \equiv R_y(-\gamma/2) R_z(-(\delta+\beta)/2)$, $C \equiv R_z((\delta-\beta)/2)$: three unitary operators on the qubit

$$ABC = I$$

Since $\sigma_x^2 = I$, we have

$$\sigma_x B \sigma_x = \sigma_x R_y(-\gamma/2)\sigma_x \sigma_x R_z(-(\delta{+}\beta)/2\sigma_x) = R_y(\gamma/2) R_z((\delta{+}\beta)/2)$$

and then

$$\begin{aligned} & e^{i\alpha} A \sigma_x B \sigma_x C \\ = \;& e^{i\alpha} R_z(\beta) R_y(\gamma/2) R_y(\gamma/2) R_z((\delta+\beta)/2) R_z((\delta-\beta)/2) \\ = \;& e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta) = U \end{aligned}$$

An Example – σ_h

- $\sigma_h = e^{i\pi/2} R_y(\pi/2) R_z(\pi) = e^{i\pi/2} R_z(0) R_y(\pi/2) R_z(\pi)$
- $A = R_z(\beta) R_y(\gamma/2) = R_y(\pi/4)$
- $B = R_y(-\gamma/2) R_z(-(\delta+\beta)/2) = R_y(-\pi/4) R_z(-\pi/2)$
- $C = R_z((\delta-\beta)/2) = R_z(\pi/2)$
- $\alpha = \pi/2$

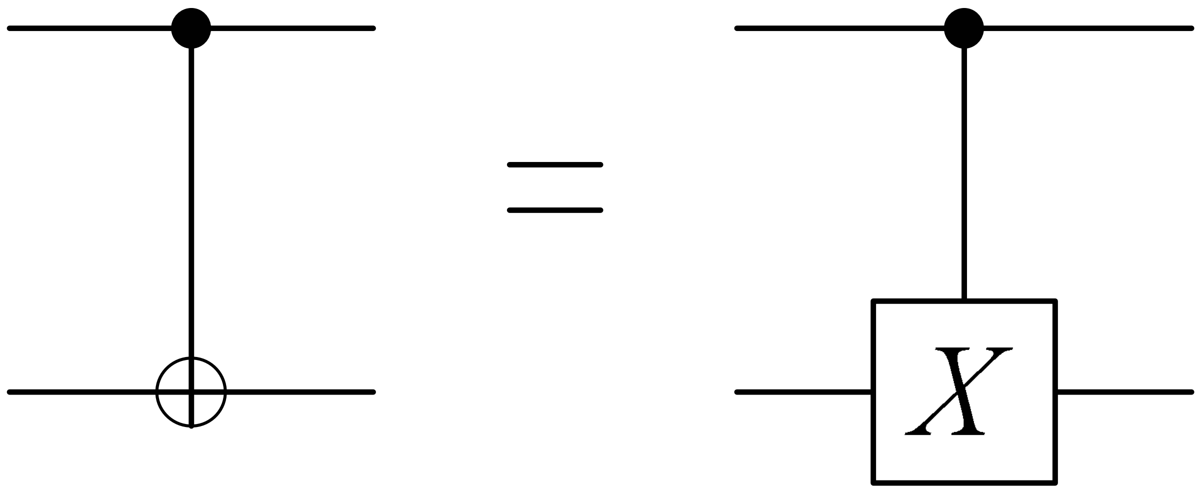
Controlled Operations

The CNOT Gate

- Prototypical control operation
- $|c\rangle|t\rangle \rightarrow |c\rangle|t \oplus c\rangle$: a unitary operation on two-qubit system, where c for the control qubit and t for the target qubit
- Matrix representation : realtive to a computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$

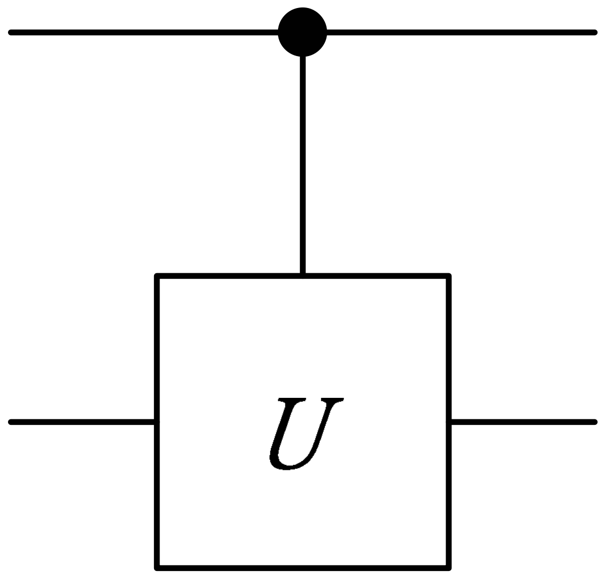
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Circuit Representation of CNOT Gate

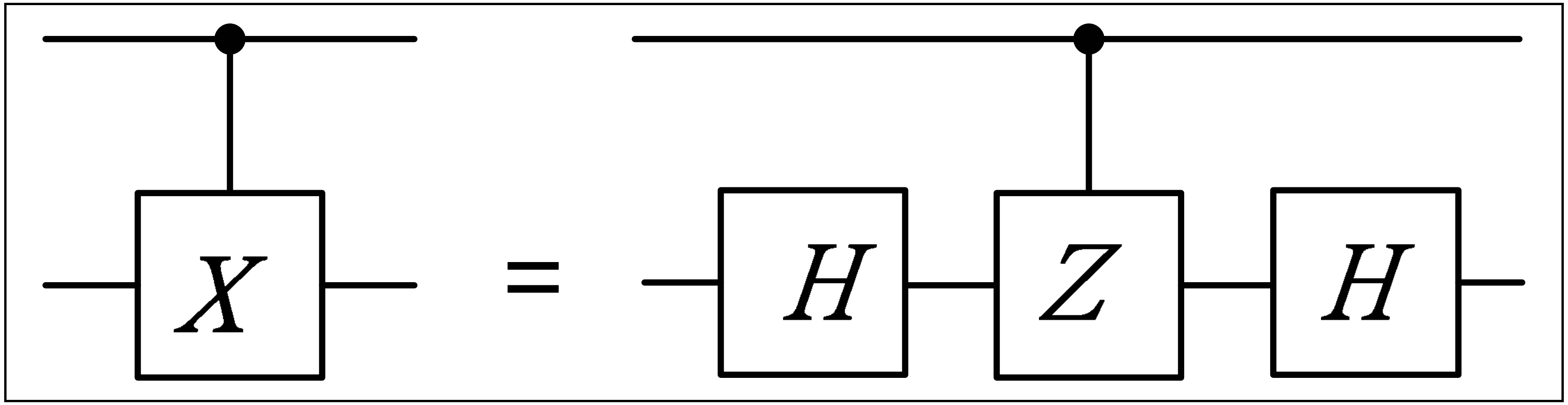


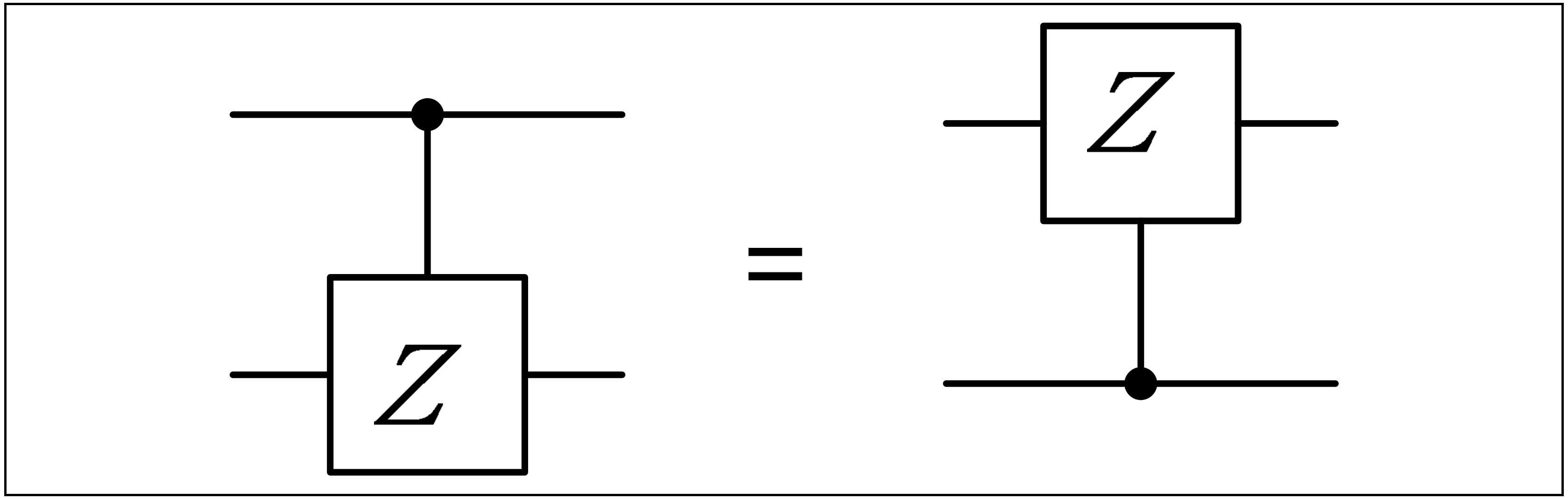
The Controlled-U Gate

- U : a unitary operator on a qubit
- $|c\rangle|t\rangle \rightarrow |c\rangle U^c|t\rangle$: if the controlled qubit is in state $|1\rangle$, then the single qubit operation U will operate on the target qubit; otherwise, no action
- Circuit representation

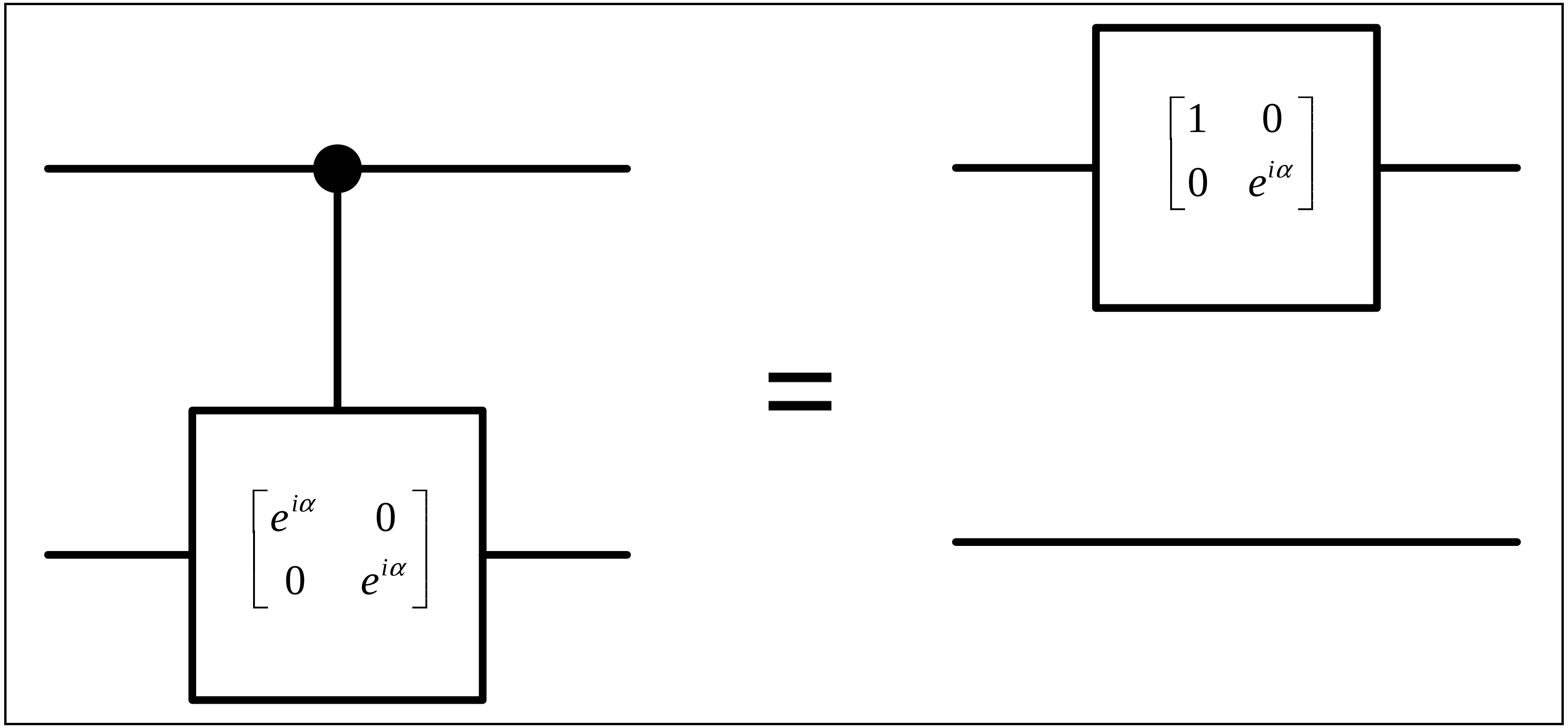


Some Circuit Identities of Gates



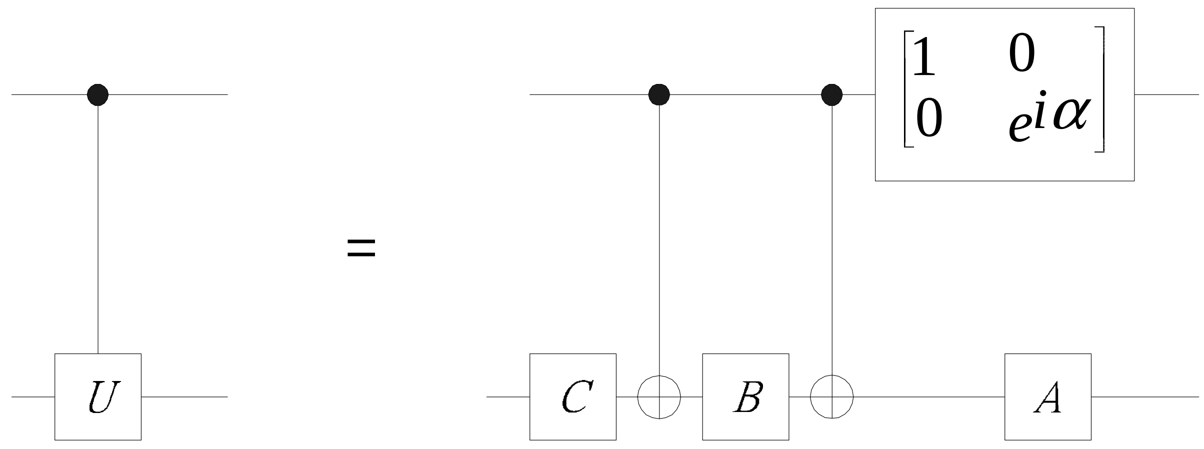


- Controlled phase shift $e^{i\alpha}$



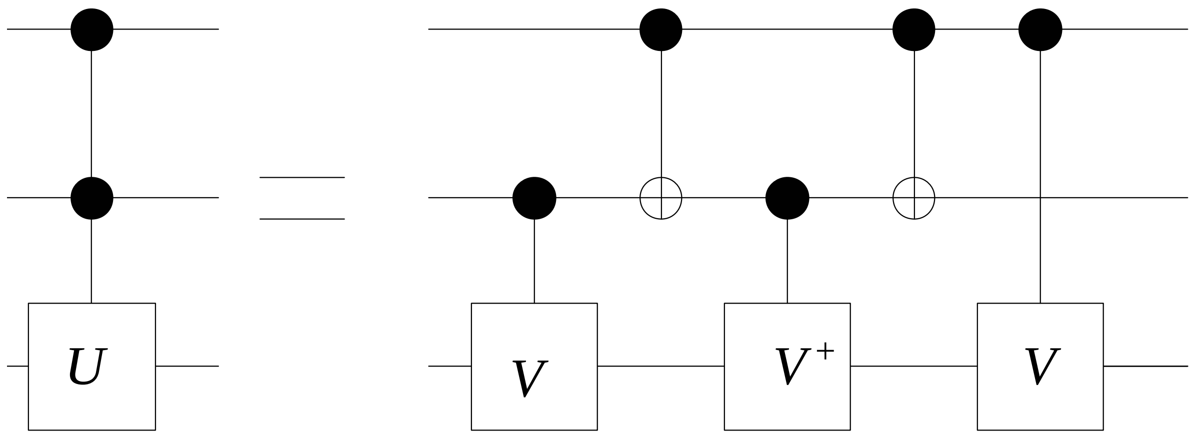
Circuit Implementation of Controlled-U Operations

- $U = e^{i\alpha} A \sigma_x B \sigma_x C$: decomposition of a single qubit operation
  - A, B, C : single qubit operations
  - $ABC = I$



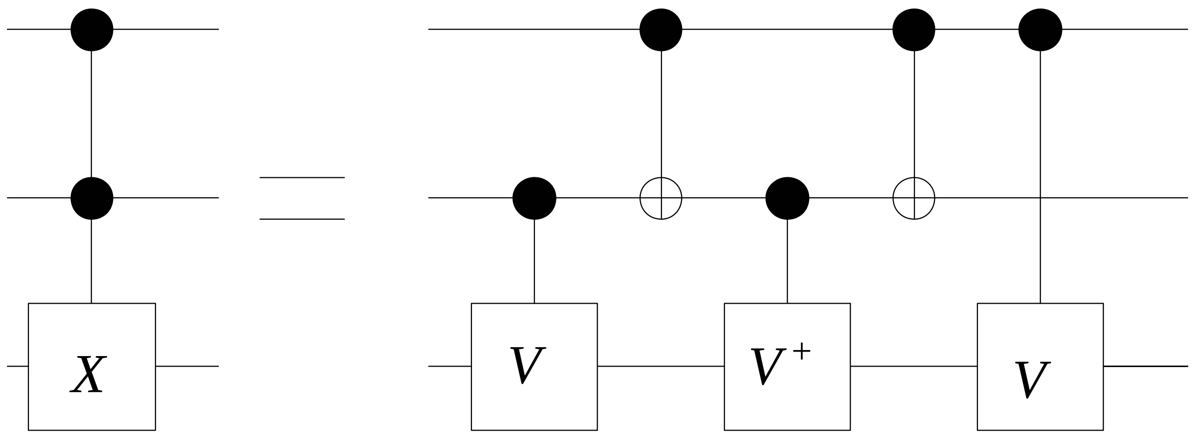
Implementation of two-qubit controlled-U operations

- U : a single qubit operation
- $C^2(U)|c_1c_2\rangle|t\rangle = |c_1c_2\rangle U^{c_1c_2}|t\rangle$: if the states of both control qubits are in $|1\rangle$, i.e., $c_1 = c_2 = 1$, then U will operate on the target qubit; otherwise, no action
- V : a unitary operator on a single qubit such that $V^2 = U$



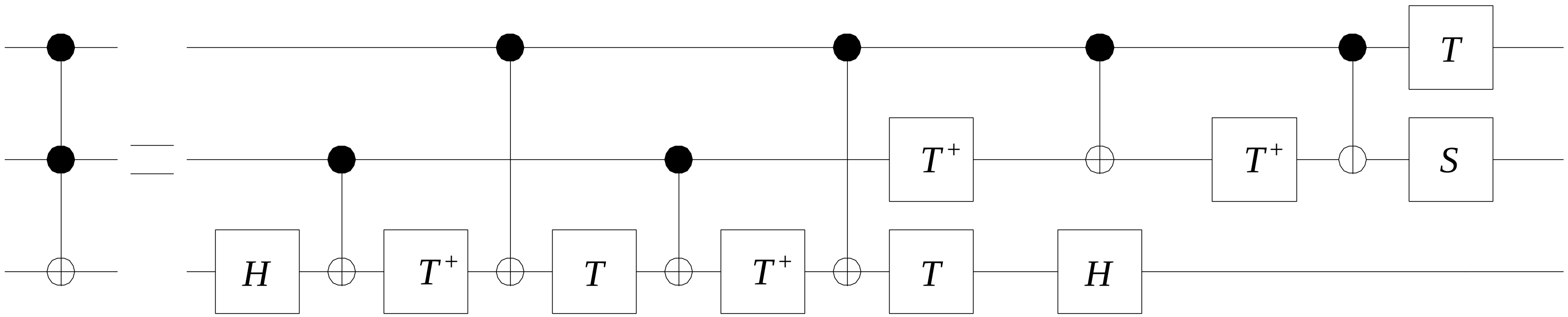
An Example – the Toffoli Gate

- $V = (1-i)(I+iX)/2 : V^2 = X$



The Toffoli Gate

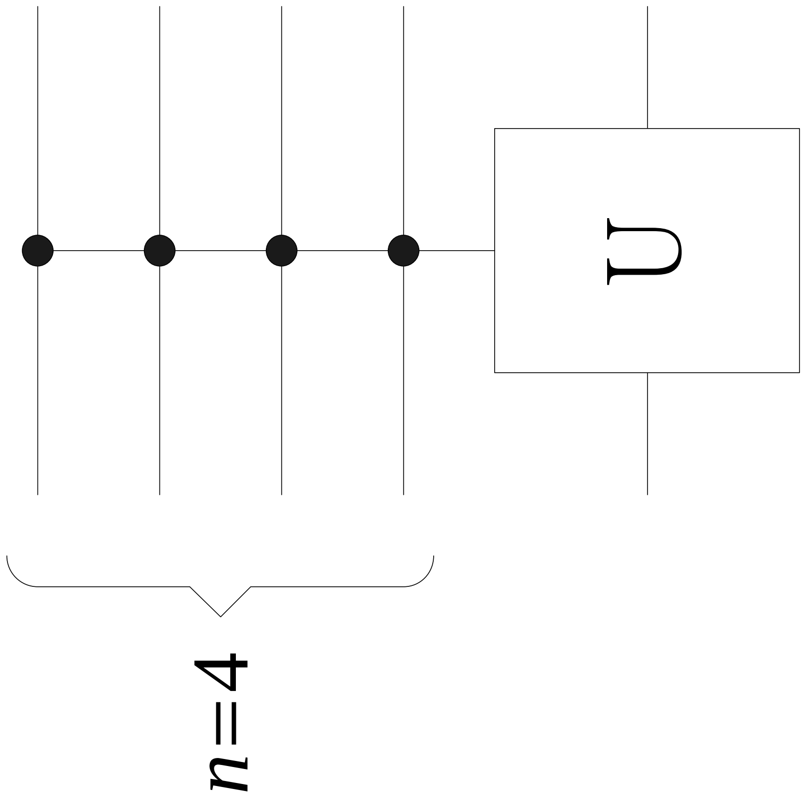
- Built by a universal set of gates : CNOT, Hadamard, phase and $\pi/8$ gates



- $XTX = X(e^{i\pi/8}R_z(\pi/4))X = e^{i\pi/8}R_z(-\pi/4)$
- $XT^\dagger X = X(e^{-i\pi/8}R_z(-\pi/4))X = e^{-i\pi/8}R_z(\pi/4)$
- $\sigma_h\sigma_t\sigma_h = e^{i\pi/8}\sigma_h R_z(\pi/4)\sigma_h = e^{i\pi/8}R_x(\pi/4)$
- $\sigma_x = e^{i\pi/2}R_x(\pi)$

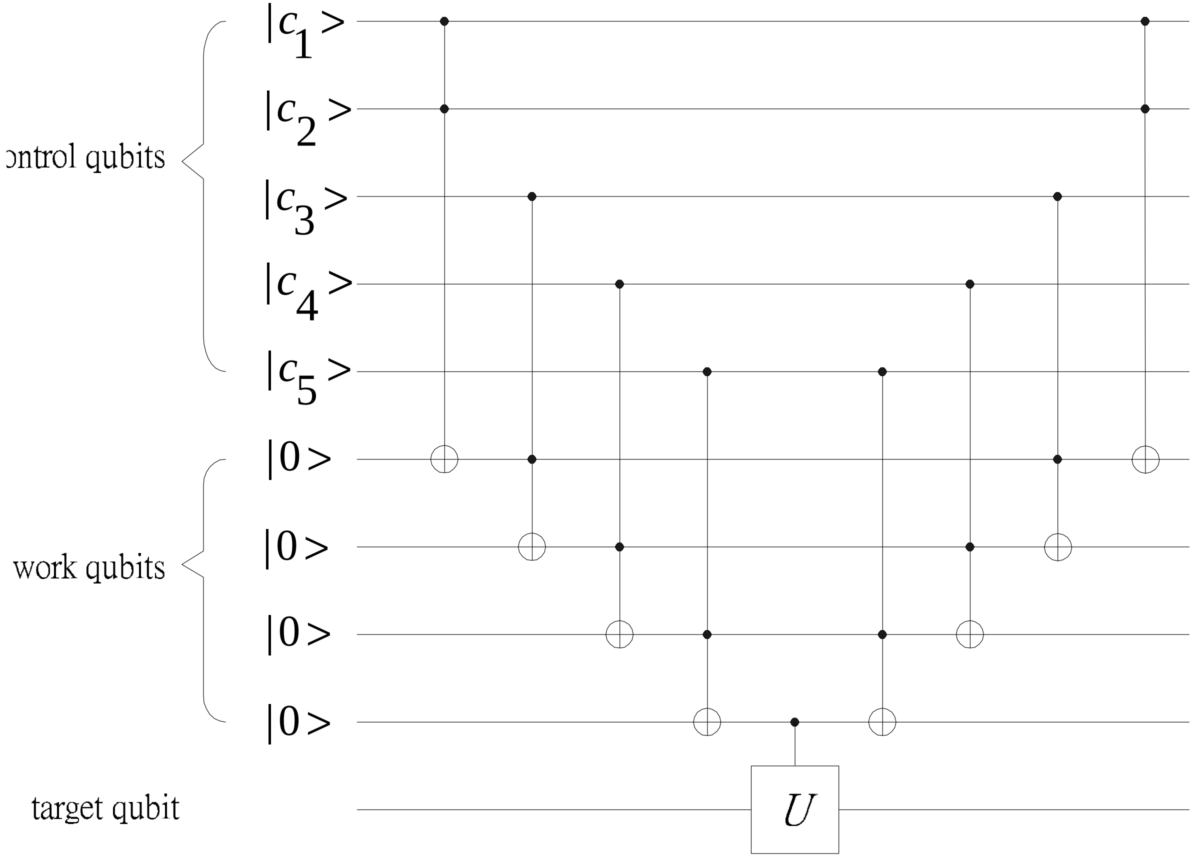
n-qubit controlled-U operations

- U : a single qubit operation
- $C^n(U)|c_1c_2\cdots c_n\rangle|t\rangle = |c_1c_2\cdots c_n\rangle U^{c_1c_2\cdots c_n}|t\rangle$: if the states of all n control qubits are in $|1\rangle$, i.e., $c_1 = c_2 = \cdots = c_n = 1$, then U will operate on the target qubit; otherwise no action

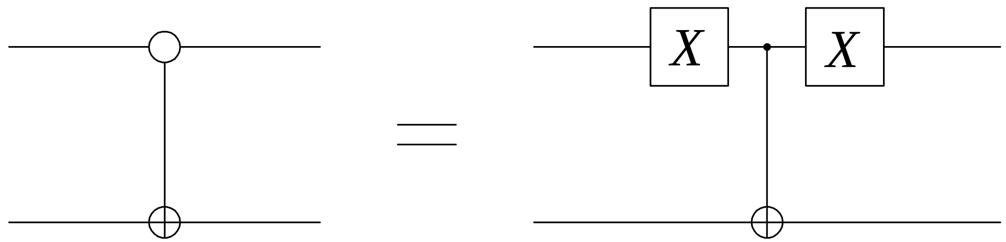

U
$n=4$

An Implementation of n-qubit controlled-U operations

- Need $(n-1)$ work qubits, $2(n-1)$ Toffoli gates and one single-qubit controlled-U gate

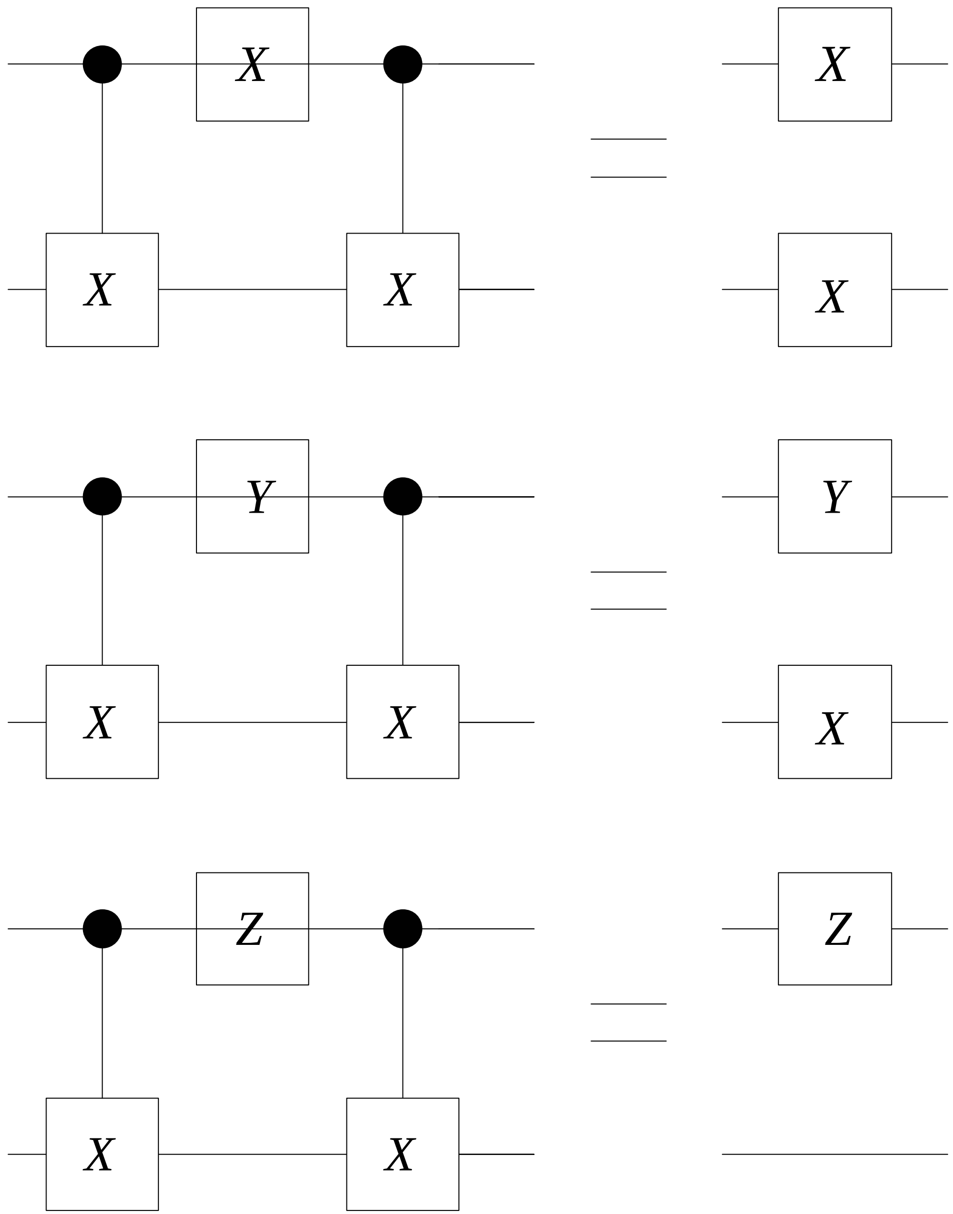


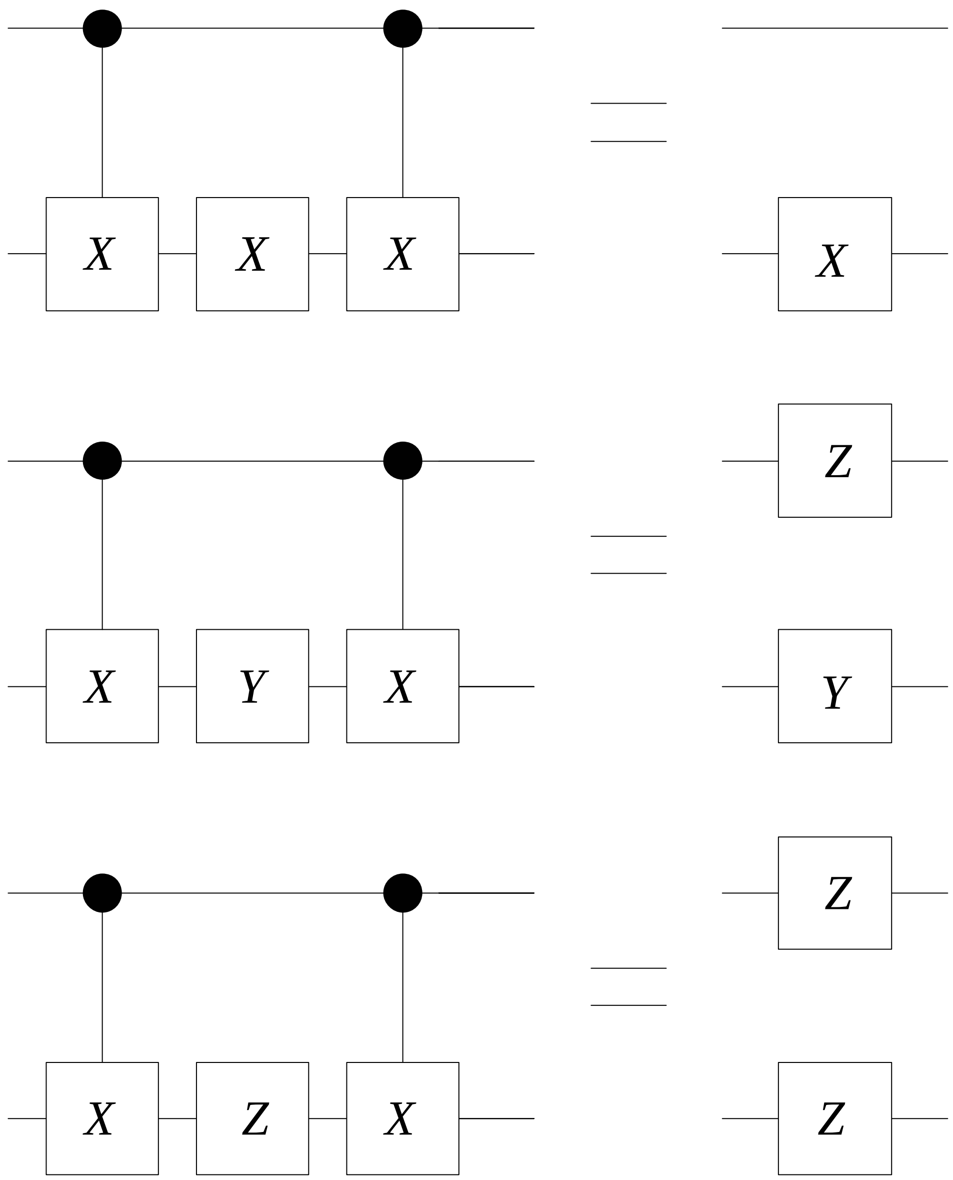
CNOT Gate Controlled by Setting Control Qubit to Zero



Some Circuit Identities

- C : a CNOT with qubit 1 the control qubit and qubit 2 the target qubit
- $C\sigma_{x,1}C = \sigma_{x,1}\sigma_{x,2}$, $C\sigma_{y,1}C = \sigma_{y,1}\sigma_{x,2}$, $C\sigma_{z,1}C = \sigma_{z,1}$
- $C\sigma_{x,2}C = \sigma_{x,2}$, $C\sigma_{y,2}C = \sigma_{z,1}\sigma_{y,2}$, $C\sigma_{z,2}C = \sigma_{z,1}\sigma_{z,2}$
- $R_{z,1}(\theta)C = CR_{z,1}(\theta)$
- $R_{x,2}(\theta)C = CR_{x,2}(\theta)$


X
X
X
X
X
Y
Y
X
X
X
Z
Z
X
X


X
X
X
X
Z
X
Y
X
Z
Y
Z
X
Z
X
Z

Measurement in Quantum Circuits

General Measurements Through Projective Measurements

- $\{M_m\}$: a collection of general measurement operators on the state space H of a quantum system
  - $\mathcal{P}(m) = \langle\psi|M_m^\dagger M_m|\psi\rangle$: the probability that result m occurs, given the pre-measurement state $|\psi\rangle$
  - $\frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}}$: the post-measurement state of the quantum system
- $\{|m\rangle\}$: an orthonormal basis of the state space G of an ancilla quantum system
- $|0\rangle$: any fixed state vector of G
- $E \equiv \mathrm{Span}(|0\rangle)$: the subspace of G generated by $|0\rangle$

- U : a linear transformation from $H \otimes E$ into $H \otimes G$ defined as

$$U|\psi\rangle|0\rangle \equiv \sum_m M_m|\psi\rangle|m\rangle$$

  - U preserves inner product

$$\langle\varphi|\langle 0|U^\dagger U|\psi\rangle|0\rangle = \sum_{m,m'} \langle\varphi|M_m^\dagger M_{m'}|\psi\rangle\langle m|m'\rangle$$

$$= \sum_m \langle\varphi|M_m^\dagger M_m|\psi\rangle = \langle\varphi|\psi\rangle = \langle\varphi 0|\psi 0\rangle$$

  - U can be extended to be a unitary operator on $H \otimes G$

- $\{P_m = I_H \otimes |m\rangle\langle m|\}$: projective measurements on the state space $H \otimes G$ of the composite system
  - $\mathcal{P}'(m) = \langle\psi|\langle 0|U^\dagger P_m U|\psi\rangle|0\rangle$: the probability that result m occurs, given the pre-measurement state $U|\psi\rangle|0\rangle$ of the composite quantum system

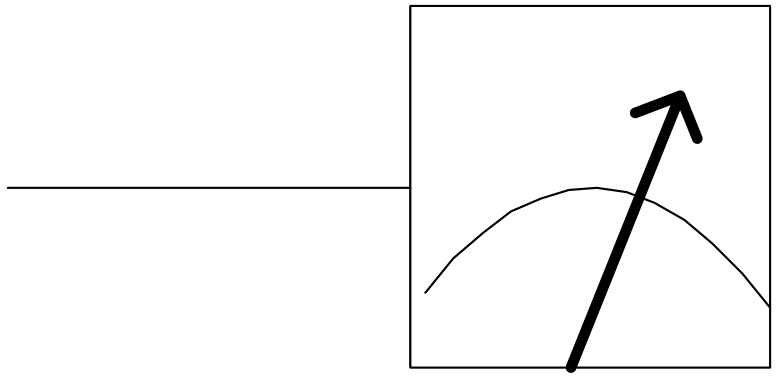
$$
\begin{aligned}
\mathcal{P}'(m) &= \sum_{m',''} \langle\psi|M^\dagger_{m'}\langle m'|(I_H \otimes |m\rangle\langle m|)M_{m''}|\psi\rangle|m''\rangle \\
&= \langle\psi|M^\dagger_m M_m|\psi\rangle = \mathcal{P}(m)
\end{aligned}
$$

  - $\frac{P_m U|\psi\rangle|0\rangle}{\langle\psi|\langle 0|U^\dagger P_m U|\psi\rangle|0\rangle} = \frac{M_m|\psi\rangle|m\rangle}{\sqrt{\langle\psi|M^\dagger_m M_m|\psi\rangle}} = \frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M^\dagger_m M_m|\psi\rangle}} \otimes |m\rangle$: the post-measurement state of the composite system

- A general measurement on a quantum system can be implemented by a projective measurement on a composite quantum system of the original quantum system and an ancilla quantum system after applying a unitary operator on the composite system

The Meter Symbol in Quantum Circuits

$$|\psi\rangle$$



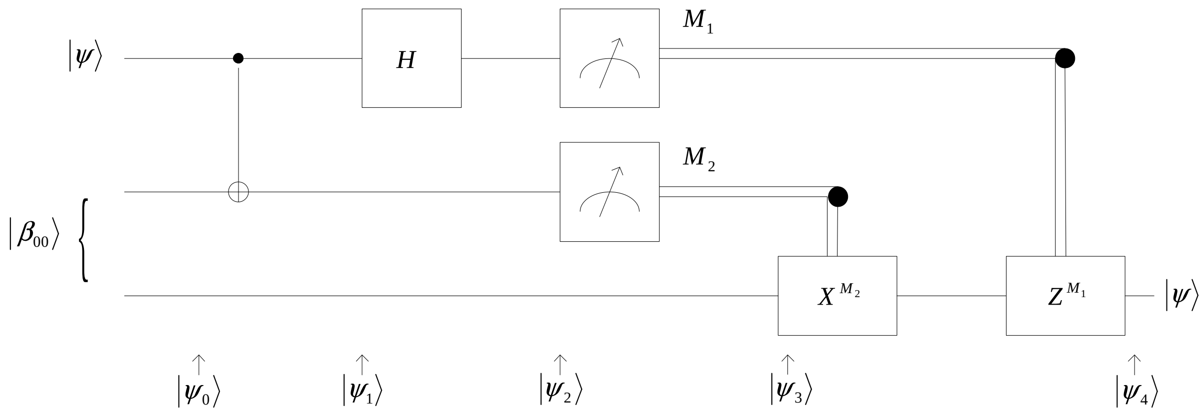
- $\{|m\rangle\langle m|\}$: projective measurement in the computational basis

Two Principles

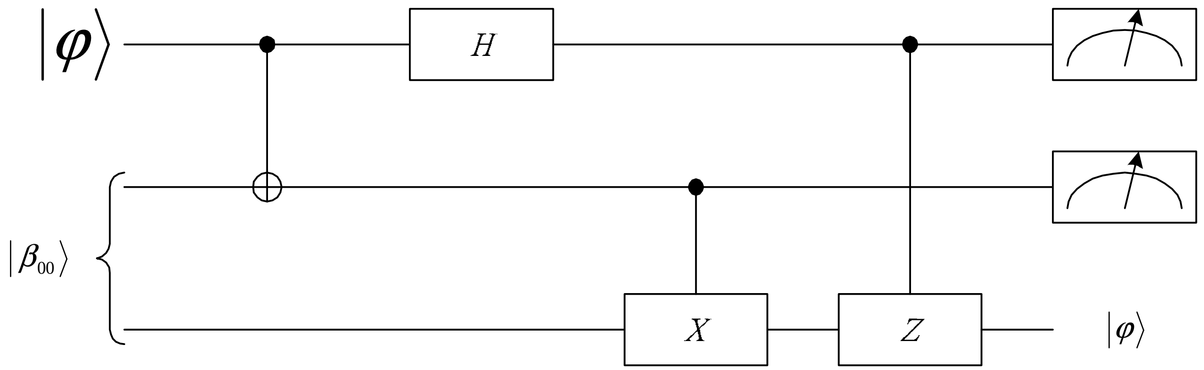
- Principle of deferred measurement : measurements can always be moved from an intermediate stage of a quantum circuit to the end of the circuit and if the measurement results are used at any stage of the circuit then the classically controlled operations can be replaced by conditional quantum operations
- Principle of implicit measurement : without loss of generality, any unterminated quantum wires (qubits which are not measured) at the end of a quantum circuit may be assumed to be measured

An Illustration of the Principle of Deferred Measurement

- Quantum teleportation circuit



- Deferred measurement in quantum teleportation circuit



An Illustration of the Principle of Implicit Measurement

- $\rho = \sum_i \alpha_i T_i \otimes S_i$: density operator describing a two-qubit system
- $\{P_0 = I \otimes |0\rangle\langle 0|, P_1 = I \otimes |1\rangle\langle 1|\}$: projective measurement in the computational basis of the 2nd qubit
- ρ' : density operator after the measurement

  $$\rho' = P_0 \rho P_0 + P_1 \rho P_1 = \sum_i \alpha_i T_i \otimes (|0\rangle\langle 0|S_i|0\rangle\langle 0| + |1\rangle\langle 1|S_i|1\rangle\langle 1|)$$

- $\mathrm{tr}_2(\rho) = \mathrm{tr}_2(\rho')$: the reduced density operators for the first qubit are the same no matter whether the second qubit is measured or not

$$
\begin{aligned}
\mathrm{tr}_2(\rho) &= \sum_i \alpha_i T_i \mathrm{tr}(S_i) \\
\mathrm{tr}_2(\rho') &= \sum_i \alpha_i T_i \mathrm{tr}(|0\rangle\langle 0|S_i|0\rangle\langle 0| + |1\rangle\langle 1|S_i|1\rangle\langle 1|) \\
&= \sum_i \alpha_i T_i (\langle 0|S_i|0\rangle + \langle 1|S_i|1\rangle) \\
&= \sum_i \alpha_i T_i \mathrm{tr}(S_i)
\end{aligned}
$$

Universal Quantum Gates

A Universal Set of Gates for Quantum Computation

- Any unitary operation on a quantum system can be approximated to arbitrary accuracy by a quantum circuit involving only gates from this set

Construction of a Universal Set of Gates

- Step 1 : any unitary operator can be expressed as a product of unitary operators which act non-trivially only on a subspace spanned by two computational basis states, called two-level unitary operators
- Step 2 : any unitary operator can be implemented exactly by single-qubit and CNOT gates
  - Any two-level unitary operator can be implemented exactly by single-qubit and CNOT gates
- Step 3 : any unitary operator can be approximated to arbitrary accuracy by using Hadamard, phase, $\pi/8$ and CNOT gates
  - Any single-qubit gate can be approximated to arbitrary accuracy by using Hadamard, phase and $\pi/8$ gates

Two-Level Unitary Operators

- H : a finite-dimensional (≥ 2) complex inner product space
- U : a unitary operator on H

U is called a *two-level* unitary operator if there is a 2-dimensional subspace G of H such that the restriction U_G of U on G is a unitary operator on G and the restriction $U_{G^\perp}$ of U on the orthogonal complement $G^\perp$ is the identity operator on $G^\perp$.

- $\{|\psi_i\rangle\}$: an ordered orthonormal basis of H with $G = \mathrm{Span}(|\psi_{i_1}\rangle, |\psi_{i_2}\rangle)$ for some i_1 and i_2
- $A = [a_{ij}]$: the matrix representation of U relative to the ordered basis $\{|\psi_i\rangle\}$. Then we have
  - $A' = \begin{bmatrix} a_{i_1 i_1} & a_{i_1 i_2} \\ a_{i_2 i_1} & a_{i_2 i_2} \end{bmatrix}$: a 2×2 unitary matrix
  - $a_{ij} = \delta_{ij}$: either $i \neq i_1, i_2$ or $j \neq i_1, i_2$
  - A is called a two-level unitary matrix

Decomposition of 3×3 Unitary Matrices into Two-Level Unitary Matrices

$$U = U_1 U_2 U_3$$

- U : 3×3 unitary matrix

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{bmatrix}$$

- U_1, U_2, U_3 : 3×3 *two-level* unitary matrices

Step 1

- If $u_{21} = 0$, let $A_1 = I_{3\times 3}$, a two-level unitary matrix
- If $u_{21} \neq 0$, let A_1 be the following two-level unitary matrix

$$A_1 = \begin{bmatrix} \frac{\bar{u}_{11}}{\sqrt{|u_{11}|^2+|u_{21}|^2}} & \frac{\bar{u}_{21}}{\sqrt{|u_{11}|^2+|u_{21}|^2}} & 0 \\ \frac{u_{21}}{\sqrt{|u_{11}|^2+|u_{21}|^2}} & \frac{-u_{11}}{\sqrt{|u_{11}|^2+|u_{21}|^2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Eliminating u_{21} of U by A_1, we have a unitary matrix

$$A_1 U = \begin{bmatrix} u'_{11} & u'_{12} & u'_{13} \\ 0 & u'_{22} & u'_{23} \\ u'_{31} & u'_{32} & u'_{33} \end{bmatrix}$$

Step 2

- If $u_{31} = 0$, let $A_2 = I_{3\times 3}$, a two-level unitary matrix
- If $u_{31} \neq 0$, let A_2 be the following two-level unitary matrix

$$A_1 = \begin{bmatrix} \frac{\bar{u}'_{11}}{\sqrt{|u'_{11}|^2+|u'_{31}|^2}} & 0 & \frac{\bar{u}'_{31}}{\sqrt{|u'_{11}|^2+|u'_{31}|^2}} \\ 0 & 1 & 0 \\ \frac{u'_{31}}{\sqrt{|u'_{11}|^2+|u'_{31}|^2}} & 0 & \frac{-u'_{11}}{\sqrt{|u'_{11}|^2+|u'_{31}|^2}} \end{bmatrix}$$

- Eliminating u'_{31} of A_1U by A_2, we have a unitary matrix

$$A_2A_1U = \begin{bmatrix} u''_{11} & u''_{12} & u''_{13} \\ 0 & u''_{22} & u''_{23} \\ 0 & u''_{32} & u''_{33} \end{bmatrix}$$

  - $u''_{11} = 1$, $u''_{12} = u''_{13} = 0$

Step 3

- Let A_3 be the following two-level unitary matrix

$$A_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \bar{u}''_{22} & \bar{u}''_{32} \\ 0 & \bar{u}''_{23} & \bar{u}''_{33} \end{bmatrix}$$

- $A_3 A_2 A_1 U = I$ and then $U = A_1^\dagger A_2^\dagger A_3^\dagger$
- $U_1 = A_1^\dagger, U_2 = A_2^\dagger, U_3 = A_3^\dagger$: two-level unitary matrices

A Theorem

Every $n \times n$ unitary matrix U can be decomposed as a product of two-level $n \times n$ unitary matrices $U_1, U_2, \ldots, U_k$,

$$U = U_1 U_2 \cdots U_k$$

Proof

- Step 1 : using $(n-1)$ $n \times n$ two-level unitary matrices $A_1, A_2, \ldots, A_{n-1}$, some of them are the identity matrix $I_{n\times n}$, to eliminate the entries of the first column of U except the toppest one in a top-down manner

$$A_{n-1}\cdots A_2 A_1 U = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & u_{n2} & \cdots & u_{nn} \end{bmatrix}$$

  - $u_{11} = 1$ and $u_{12} = \cdots = u_{1n} = 0$

- Step 2 : by induction on $(n-1)\times(n-1)$ unitary matrices, we have

$$A_k \cdots A_2 A_1 U = I \Rightarrow U = A_1^\dagger A_2^\dagger \cdots A_k^\dagger$$

  – $k \leq (n-1)+(n-2)+\cdots+1 = n(n-1)/2$

  – $A_i^\dagger$ are two-level unitary matrices

An Application

Every unitary operator on an n-qubit system can be decomposed as a product of at most $2^{n-1}(2^n - 1)$ two-level unitary operators, which can be represented by two-level unitary matrices relative to the same orthonormal basis

Implementation of a Two-Level Unitary Operator

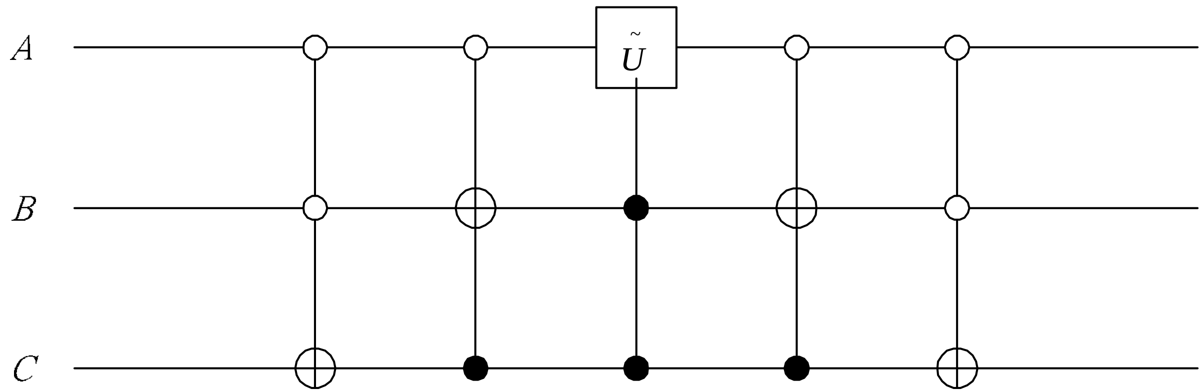
- U : two-level unitary matrix representation of a two-level unitary operator relative to a computational basis $\{|i_1 i_2 \cdots i_n\rangle\}$, $i_j = 0, 1$
- Span$(|s\rangle, |t\rangle)$: the two-dimensional state subspace where U has a non-trivial action, $s = s_1 s_2 \cdots s_n$ and $t = t_1 t_2 \cdots t_n$
- $\tilde{U}$: the 2×2 submatrix of U which acts on Span$(|s\rangle, |t\rangle)$, regarded to represent a unitary operator on a qubit
- A Gray code : connecting s to t

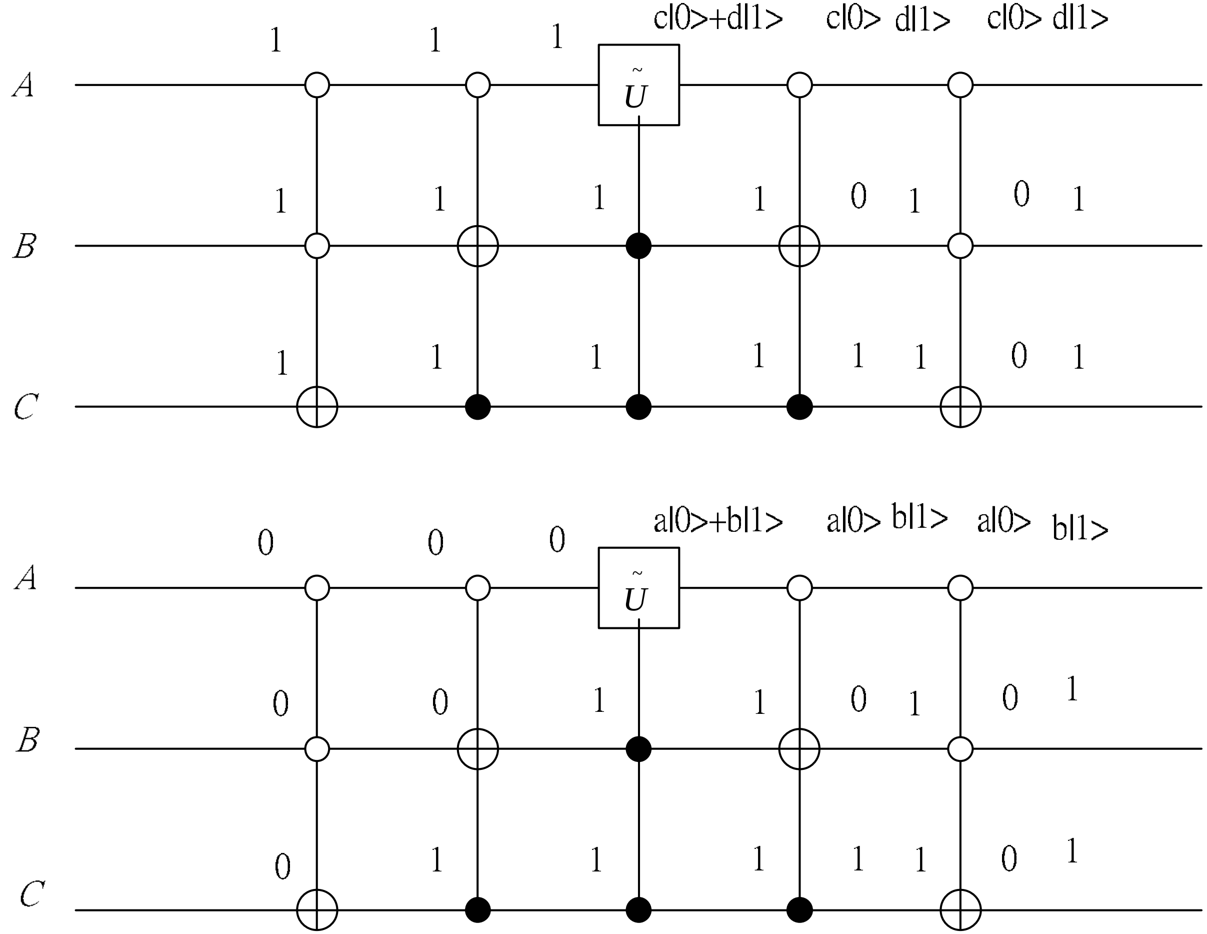
$$\begin{array}{ccccccc} s & = & 0 & 0 & 0 & = & g_1 \\ & & 1 & 0 & 0 & = & g_2 \\ & & 1 & 1 & 0 & = & g_3 \\ t & = & 1 & 1 & 1 & = & g_4 \end{array}$$

- A two-level unitary matrix

$$U_{2^3 \times 2^3} = \begin{bmatrix} a & 0 & c \\ 0 & I_{6\times 6} & 0 \\ b & 0 & d \end{bmatrix}$$

- A circuit implementation by $(n-1)$-qubit CNOT gates and a $C^{n-1}(\tilde{U})$ gate





A
B
C
1 1 1
1 1 1 1
1 1 1 1
Ũ
c|0>+d|1>
c|0> d|1>
c|0> d|1>
0 1
1 1
0 1
0 1
A
B
C
0 0 0
0 0 1 1
0 1 1 1
Ũ
a|0>+b|1>
a|0> b|1>
a|0> b|1>
0 1
1 1
0 1
0 1

Complexity of Implementing a Two-Level Unitary Operator

- Swapping : at most $2(n-1)$ $(n-1)$-qubit CNOT gates
  - Each $(n-1)$-qubit CNOT gate needs $O(n)$ single-qubit and CNOT gates
- Unitary action : one $(n-1)$-qubit controlled-$\tilde{U}$ gates
  - Each $(n-1)$-qubit controlled-$\tilde{U}$ gate needs $O(n)$ single-qubit and CNOT gates
- Total complexity : $O(n^2)$ single-qubit and CNOT gates

Complexity of Implementing an n-qubit Unitary Operator

- $O(2^{2n})$ two-level unitary operators : for the implementation of a n-quibit unitary operator
- $O(n^2)$ single-qubit and CNOT gates : for the implementation of a two-level unitary operator
- Total complexity : $O(n^2 4^n)$ single-qubit and CNOT gates

Approximating Unitary Operators

- U : the target unitary operator to be approximated
- V : the unitary operator actually implemented
- Approximation error :

$$E(U,V) \triangleq \max_{|\psi\rangle} \|(U-V)|\psi\rangle\|$$

  - $|\psi\rangle$: the state of the quantum system
  - $\{E_m\}$: a POVM measurement
  - $P_U(m)$ $(P_V(m))$: the probability that the measurement result is m after applying U (V) on the state $|\psi\rangle$

$$\boxed{|P_U(m) - P_V(m)| \leq 2E(U,V)}$$

Proof :

$$\begin{aligned}
& |P_U(m) - P_V(m)| \\
= \;& |\langle\psi|U^\dagger E_m U|\psi\rangle - \langle\psi|V^\dagger E_m V|\psi\rangle| \\
= \;& |\langle\psi|U^\dagger E_m U|\psi\rangle - \langle\psi|U^\dagger E_m V|\psi\rangle \\
& +\langle\psi|U^\dagger E_m V|\psi\rangle - \langle\psi|V^\dagger E_m V|\psi\rangle| \\
= \;& |\langle\psi|U^\dagger E_m|\Delta\rangle + \langle\Delta|E_m V|\psi\rangle| \\
\leq \;& |\langle\psi|U^\dagger E_m|\Delta\rangle| + |\langle\Delta|E_m V|\psi\rangle| \\
\leq \;& \sqrt{\langle\psi|U^\dagger U|\psi\rangle\langle\Delta|E_m|\Delta\rangle} + \sqrt{\langle\psi|V^\dagger V|\psi\rangle\langle\Delta|E_m|\Delta\rangle} \\
\leq \;& \||\Delta\rangle\| + \||\Delta\rangle\| \\
\leq \;& 2E(U,V)
\end{aligned}$$

since $\langle\Delta|E_m|\Delta\rangle \leq \sum_m \langle\Delta|E_m|\Delta\rangle = \langle\Delta|\left(\sum_m E_m\right)|\Delta\rangle = \langle\Delta|\Delta\rangle$

- U_i, $1 \leq i \leq m$: target unitary operators
- V_i, $1 \leq i \leq m$: actual unitary operators implemented for U_i
- $E(U_m U_{m-1} \cdots U_1, V_m V_{m-1} \cdots V_1)$: overall accumulated approximation error

$$\boxed{E(U_m U_{m-1} \cdots U_1, V_m V_{m-1} \cdots V_1) \leq \sum_{j=1}^{m} E(U_j, V_j)}$$

  – Proof for $m = 2$:

$$\begin{aligned}
& E(U_2 U_1, V_2 V_1) \\
= \quad & \max_{|\psi\rangle} \|(U_2 U_1 - V_2 V_1)|\psi\rangle\| \\
= \quad & \max_{|\psi\rangle} \|(U_2 U_1 - V_2 U_1)|\psi\rangle + (V_2 U_1 - V_2 V_1)|\psi\rangle\| \\
\leq \quad & \max_{|\psi\rangle} \|(U_2 - V_2) U_1 |\psi\rangle\| + \max_{|\psi\rangle} \|V_2 (U_1 - V_1)|\psi\rangle\| \\
\leq \quad & \max_{|\psi'\rangle} \|(U_2 - V_2)|\psi'\rangle\| + \max_{|\psi\rangle} \|(U_1 - V_1)|\psi\rangle\| \\
= \quad & E(U_2, V_2) + E(U_1, V_1)
\end{aligned}$$

A Universal Approximation Theorem

Any single qubit gate can be approximated to arbitrary accuracy by Haddamard and $\pi/8$ gates

- $\sigma_t = e^{i\pi/8} R_z(\pi/4)$
- $\sigma_h \sigma_t \sigma_h = e^{i\pi/8} R_x(\pi/4)$
- Compositions of σ_t and $\sigma_h \sigma_t \sigma_h$
  - $\sigma_t(\sigma_h \sigma_t \sigma_h)$
  - $(\sigma_h \sigma_t \sigma_h)\sigma_t$

$$\begin{aligned} & \sigma_t(\sigma_h \sigma_t \sigma_h) \\ = \; & e^{i\pi/4} R_z(\pi/4) R_x(\pi/4) \\ = \; & e^{i\pi/4} \left(\cos\frac{\pi}{8} I - i \sin\frac{\pi}{8}\sigma_z\right)\left(\cos\frac{\pi}{8} I - i \sin\frac{\pi}{8}\sigma_x\right) \end{aligned}$$

$$= \quad e^{i\pi/4}\left(\cos^2\frac{\pi}{8}I - i\sin\frac{\pi}{8}\left(\cos\frac{\pi}{8}(\sigma_x+\sigma_z)+\sin\frac{\pi}{8}(-i\sigma_z\sigma_x)\right)\right)$$

$$= \quad e^{i\pi/4}\left(\cos^2\frac{\pi}{8}I - i\sin\frac{\pi}{8}\left(\cos\frac{\pi}{8}(\sigma_x+\sigma_z)+\sin\frac{\pi}{8}\sigma_y\right)\right)$$

which is a rotation along the axis

$$\hat{n} = \frac{1}{\sqrt{1+\cos^2\frac{\pi}{8}}}\left(\cos\frac{\pi}{8}, \sin\frac{\pi}{8}, \cos\frac{\pi}{8}\right)$$

with an angle Δ such that

$$\cos\frac{\Delta}{2} = \cos^2\frac{\pi}{8} \quad \text{and} \quad \sin\frac{\Delta}{2} = \sin\frac{\pi}{8}\sqrt{1+\cos^2\frac{\pi}{8}}$$

– Δ is an irrational multiple of π, see P.O. Boykin, T. Mor, M. Pulver, V. Roychowdhury, and F. Vatan, “On universal and fault-tolerant quantum computing,” *arXive e-print quant-ph/9906054*, 1999.

– $\boxed{\sigma_t(\sigma_h\sigma_t\sigma_h) = e^{i\pi/4}R_{\hat{n}}(\Delta)}$

- $\boxed{(\sigma_h \sigma_t \sigma_h)\sigma_t = e^{i\pi/4} R_{\hat{m}}(\Delta)}$
  - $(\sigma_h \sigma_t \sigma_h)\sigma_t = \sigma_h(\sigma_t(\sigma_h \sigma_t \sigma_h))\sigma_h$
  - $\hat{m} = \frac{1}{\sqrt{1+\cos^2 \frac{\pi}{8}}} \left(\cos \frac{\pi}{8}, -\sin \frac{\pi}{8}, \cos \frac{\pi}{8}\right)$

- Approximating $R_{\hat{n}}(\alpha)$, $0 \leq \alpha < 2\pi$, to arbitrary accuracy by repeatedly using $R_{\hat{n}}(\Delta)$
  - ϵ : the desired accuracy
  - N : a poitive integer such that $2\pi/N < \epsilon$
  - $\Delta_k \stackrel{\triangle}{=} k\Delta \pmod{2\pi}$: k is a non-negative integer
    * No two distinct k, j such that $\Delta_k = \Delta_j$ since Δ is not a rational multiple of π
    * Pigeonhole principle : there exist two distinct k, j, $0 \leq j < k \leq N$, such that $|\Delta_k - \Delta_j| \leq 2\pi/N < \epsilon$ and then $0 < \Delta_l < \epsilon$ where $l = k - j$
  - m : $0 \leq \alpha - m\Delta_l < \Delta_l < \epsilon$

$$0 \leq \alpha - \Delta_{ml} < \epsilon$$

  - $\boxed{E(R_{\hat{n}}(\alpha), R_{\hat{n}}(\alpha+\beta)) = |1 - e^{i\beta/2}| = 2\sin\frac{\beta}{4}, 0 \leq \alpha, \beta < 2\pi}$

$*$ Proof :

$$
\begin{aligned}
& \left\| \left(e^{-i\alpha(\hat{n}\cdot\vec{\sigma})/2} - e^{-i(\alpha+\beta)(\hat{n}\cdot\vec{\sigma})/2} \right) |\psi\rangle \right\| \\
= \quad & \left\| \left(I - e^{-i\beta(\hat{n}\cdot\vec{\sigma})/2} \right) |\psi\rangle \right\| \\
= \quad & \left\| U^\dagger \begin{bmatrix} 1 - e^{-i\beta/2} & 0 \\ 0 & 1 - e^{i\beta/2} \end{bmatrix} U|\psi\rangle \right\| \\
= \quad & \left\| \begin{bmatrix} 1 - e^{-i\beta/2} & 0 \\ 0 & 1 - e^{i\beta/2} \end{bmatrix} |\psi'\rangle \right\| \\
= \quad & |1 - e^{i\beta/2}| \left\| \begin{bmatrix} \frac{1-e^{-i\beta/2}}{|1-e^{i\beta/2}|} & 0 \\ 0 & \frac{1-e^{i\beta/2}}{|1-e^{i\beta/2}|} \end{bmatrix} |\psi'\rangle \right\| \\
= \quad & |1 - e^{i\beta/2}| = 2\sin\frac{\beta}{4}
\end{aligned}
$$

since $|1 - e^{i\beta/2}| = |1 - e^{-i\beta/2}|$ and the last matrix is unitary

- $\sin\theta \leq 2\theta$ for all $\theta \geq 0$
- $E(R_{\hat{n}}(\alpha), R_{\hat{n}}(\Delta)^{ml}) = 2\sin\frac{\alpha-\Delta_{ml}}{4} \leq \alpha - \Delta_{ml} < \epsilon$