

EE641000 Quantum Information and Computation

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Unit Two – Principles of Quantum Mechanics

Postulates of Quantum Mechanics

Postulate 1 – States

Associated to an *isolated* physical system is a Hilbert space \mathcal{H} (eg, a finite-dimensional complex inner product space). The system is completely described by its *state*, which is represented by a one-dimensional subspace of the Hilbert space \mathcal{H} .

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- A one-dimensional subspace of \mathcal{H} can be represented by a unit vector $|\psi\rangle$ in it.
- A state of the system can be represented by a unit vector $|\psi\rangle$ in the Hilbert space \mathcal{H} , where $|\psi\rangle$ is called a *state vector*.
 - This unit vector representation of a state is not unique since each of $|\psi\rangle$ and $e^{j\theta}|\psi\rangle$ spans the same one-dimensional subspace of \mathcal{H} .

A Quantum Bit (Qubit)

A quantum bit (qubit) is the state represented by unit vectors of a two-dimensional Hilbert space \mathcal{H} associated with a physical system.

- $\{|0\rangle, |1\rangle\}$: an orthonormal basis of \mathcal{H} .
- $|\psi\rangle = a|0\rangle + b|1\rangle$: a unit vector in \mathcal{H} where

$$|a|^2 + |b|^2 = 1.$$

- The unit vector $|\psi\rangle$ and each of $e^{j\theta}|\psi\rangle$ represent the same state of a qubit.

Postulate 2 - Time Evolution

The evolution of a *closed* quantum system is described by a *unitary operator*. That is, the state $|\psi\rangle$ of the system at time t_1 is related to the state $|\psi'\rangle$ of the system at time t_2 by a unitary operator U which depends only on the times t_1 and t_2 ,

$$|\psi'\rangle = U|\psi\rangle.$$

Postulate 2' – Time Evolution Revisited

The time evolution of the state of a *closed* quantum system is described by the *Schrödinger equation*,

$$i\hbar \frac{d|\psi\rangle}{dt} = H|\psi\rangle.$$

where

- \hbar : the Planck's constant
- H : a Hermitian operator known as the *Hamiltonian* of the closed system

Solution of Schrödinger Equation

$$|\psi(t)\rangle = e^{-i\frac{H}{\hbar}(t-t_0)}|\psi(t_0)\rangle = U(t;t_0)|\psi(t_0)\rangle$$

- H : a Hermitian operator
- $U(t;t_0) = e^{-i\frac{H}{\hbar}(t-t_0)}$: a unitary operator for given t and t_0 .

Postulate 3 – Quantum Measurements

A quantum measurement is described by a collection $\{M_m\}$ of *measurement operators*, acting on the Hilbert space associated to a quantum system being measured and satisfying the *completeness equation*

$$\sum_m M_m^\dagger M_m = I.$$

- m : the index which represents possible measurement outcomes.

If the pre-measurement state of the quantum system is $|\psi\rangle$, then the probability that a measurement result m occurs is given by

$$\mathcal{P}(m) = \langle\psi|M_m^\dagger M_m|\psi\rangle,$$

and the post-measurement state of the system is

$$\frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}}.$$

The completeness equation expresses the fact that probabilities sum to one

$$\sum_m \mathcal{P}(m) = \sum_m \langle\psi|M_m^\dagger M_m|\psi\rangle = \langle\psi|\left(\sum_m M_m^\dagger M_m\right)|\psi\rangle = \langle\psi|\psi\rangle = 1.$$

Measurement of a Qubit

- \mathcal{H} : a two-dimensional Hilbert space associated to a quantum system.
- $\{|0\rangle, |1\rangle\}$: an orthonormal basis of \mathcal{H} .
- $M_0 = |0\rangle\langle 0|$, $M_1 = |1\rangle\langle 1|$: measurement operators.
 - Hermitian operators.
 - $M_0^2 = M_0$ and $M_1^2 = M_1$.
 - Completeness equation is satisfied

$$M_0^\dagger M_0 + M_1^\dagger M_1 = M_0^2 + M_1^2 = M_0 + M_1 = I.$$

- $|\psi\rangle = a|0\rangle + b|1\rangle$: a qubit being measured.
 - $\mathcal{P}(0) = \langle\psi|M_0^\dagger M_0|\psi\rangle = \langle\psi|M_0|\psi\rangle = \langle\psi|0\rangle\langle 0|\psi\rangle = |a|^2$.
 - $\mathcal{P}(1) = \langle\psi|M_1^\dagger M_1|\psi\rangle = \langle\psi|M_1|\psi\rangle = \langle\psi|1\rangle\langle 1|\psi\rangle = |b|^2$.
 - State after measurement

$$\frac{M_0|\psi\rangle}{|a|} = \frac{a}{|a|}|0\rangle,$$
$$\frac{M_1|\psi\rangle}{|b|} = \frac{b}{|b|}|1\rangle.$$

Projective (von Neumann) Measurements

- M : a Hermitian operator on the Hilbert space, called an *observable*, with the spectral decomposition

$$M = \sum_m m P_m$$

where P_m is the projector onto the eigenspace of M associated with eigenvalue m .

- The projectors $\{P_m\}$ are measurement operators.

- * $P_m^\dagger = P_m$ and $P_m^2 = P_m$.

- Completeness equation :

$$\sum_m P_m^\dagger P_m = \sum_m P_m^2 = \sum_m P_m = I.$$

- m : possible outcomes of the measurement.

If the pre-measurement state of the quantum system is $|\psi\rangle$, then the probability that an outcome m occurs is given by

$$\mathcal{P}(m) = \langle\psi|P_m^\dagger P_m|\psi\rangle = \langle\psi|P_m|\psi\rangle,$$

and the post-measurement state of the system is

$$\frac{P_m|\psi\rangle}{\sqrt{\langle\psi|P_m|\psi\rangle}}.$$

The completeness relation expresses the fact that probabilities sum to one

$$\sum_m \mathcal{P}(m) = \sum_m \langle\psi|P_m|\psi\rangle = \langle\psi| \left(\sum_m P_m \right) |\psi\rangle = \langle\psi|\psi\rangle = 1.$$

Repeatability of a Projective Measurement M

- $|\psi\rangle$: pre-measurement state.
- $|\psi_m\rangle = P_m|\psi\rangle / \sqrt{\langle\psi|P_m|\psi\rangle}$: post-measurement state once the outcome m is measured, which occurs with probability $\langle\psi|P_m|\psi\rangle$.
- $P_m|\psi_m\rangle = P_m|\psi\rangle / \sqrt{\langle\psi|P_m|\psi\rangle}$: post-measurement state after repeating the same projective measurement M , which occurs with probability

$$\langle\psi_m|P_m|\psi_m\rangle = \frac{\langle\psi|P_m^\dagger P_m|\psi\rangle}{\langle\psi|P_m|\psi\rangle} = \frac{\langle\psi|P_m|\psi\rangle}{\langle\psi|P_m|\psi\rangle} = 1.$$

Not every measurement is a projective measurement!

Average Value of an Observable M

$$\begin{aligned}\mathcal{E}(M) &= \sum_m m \mathcal{P}(m) = \sum_m m \langle \psi | P_m | \psi \rangle \\ &= \langle \psi | \left(\sum_m m P_m \right) | \psi \rangle = \langle \psi | M | \psi \rangle.\end{aligned}$$

- $\langle M \rangle \equiv \langle \psi | M | \psi \rangle$.
- Variance of observable M

$$\sigma^2(M) = \langle (M - \langle M \rangle)^2 \rangle = \langle M^2 \rangle - \langle M \rangle^2.$$

Two Descriptions of Projective Measurements

- A complete set of orthogonal projectors $\{P_m\}$

$$\sum_m P_m = I \quad \text{and} \quad P_m P_{m'} = \delta_{mm'} P_m$$

- Observable : $M = \sum_m m P_m$
- m : real numbers

- An orthonormal basis $\{|m\rangle\}$

$$P_m = |m\rangle\langle m|$$

- Observable : $M = \sum_m m |m\rangle\langle m|$
- m : real numbers

Observable Z on a Qubit

- The observable $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ has eigenvalues $+1$ and -1 with eigenvectors $|0\rangle$ and $|1\rangle$ respectively
- $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$: spectral decomposition
- $|\psi\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$: a qubit.

$$\mathcal{P}(+1) = \langle \psi | 0 \rangle \langle 0 | \psi \rangle = 1/2$$

$$\mathcal{P}(-1) = \langle \psi | 1 \rangle \langle 1 | \psi \rangle = 1/2$$

- $\langle Z \rangle = 0$

Heisenberg Uncertainty Principle

Commutator and Anti-commutator

- A and B : two operators.
- Commutator : $[A, B] \equiv AB - BA$
 - $[A, B] = 0$: A commutes with B .
- Anti-commutator : $\{A, B\} \equiv AB + BA$.
 - $\{A, B\} = 0$: A anti-commutes with B .

Pauli Matrices (Pauli Operators)

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

- Hermitian and unitary.
- $[X, Y] = 2iZ$, $[Y, Z] = 2iX$ and $[Z, X] = 2iY$.

Simultaneous Diagonalization of Two Normal Operators

Let A and B be two normal operators. Then $[A, B] = 0$ if and only if there exists an orthonormal basis $\{|\psi_i\rangle\}$ such that A and B are diagonalizable with respect to that basis, i.e.,

$$A = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|,$$

$$B = \sum_i \mu_i |\psi_i\rangle\langle\psi_i|.$$

$$|\langle \psi | [A, B] | \psi \rangle|^2 \leq 4 \langle \psi | A^2 | \psi \rangle \langle \psi | B^2 | \psi \rangle$$

- A and B : two Hermitian operators.
- With $\langle \psi | AB | \psi \rangle = x + iy$ where x, y real numbers, we have $\langle \psi | BA | \psi \rangle = (\langle \psi | AB | \psi \rangle)^\dagger = x - iy$ and then

$$\langle \psi | [A, B] | \psi \rangle = 2iy \quad \text{and} \quad \langle \psi | \{A, B\} | \psi \rangle = 2x.$$

- $|\langle \psi | [A, B] | \psi \rangle|^2 + |\langle \psi | \{A, B\} | \psi \rangle|^2 = 4 |\langle \psi | AB | \psi \rangle|^2$.
- Schwarz inequality :

$$|\langle \psi | AB | \psi \rangle|^2 \leq \langle \psi | A^2 | \psi \rangle \langle \psi | B^2 | \psi \rangle.$$

Thus we have

$$|\langle \psi | [A, B] | \psi \rangle|^2 \leq 4 |\langle \psi | AB | \psi \rangle|^2 \leq 4 \langle \psi | A^2 | \psi \rangle \langle \psi | B^2 | \psi \rangle.$$

Heisenberg Uncertainty Principle

$$\delta(C)\delta(D) \geq \frac{|\langle\psi|[C, D]|\psi\rangle|}{2}.$$

- C and D : two observables.
- With $A = C - \langle C \rangle$ and $B = D - \langle D \rangle$, we have

$$[A, B] = [C, D].$$

- $\delta^2(C) = \langle (C - \langle C \rangle)^2 \rangle = \langle A^2 \rangle = \langle \psi | A^2 | \psi \rangle.$
- $\delta^2(D) = \langle (D - \langle D \rangle)^2 \rangle = \langle B^2 \rangle = \langle \psi | B^2 | \psi \rangle.$

Now we have

$$\delta^2(C)\delta^2(D) = \langle \psi | A^2 | \psi \rangle \langle \psi | B^2 | \psi \rangle \geq \frac{|\langle \psi | [A, B] | \psi \rangle|^2}{4} = \frac{|\langle \psi | [C, D] | \psi \rangle|^2}{4}.$$

Heisenberg Uncertainty Principle

If we prepare a large number of quantum systems in identical states, $|\psi\rangle$, and then perform measurements of C on some of those systems, and of D on others, then the standard deviation $\delta(C)$ of all measurement results of C times the standard deviation $\delta(D)$ of all measurement results of D will satisfy the inequality

$$\delta(C)\delta(D) \geq \frac{|\langle\psi|[C,D]|\psi\rangle|}{2}.$$

An Example

- X and Y : Pauli observables.
- $[X, Y] = 2iZ$.
- $|\psi\rangle = |0\rangle$: quantum system state.
- $\delta(X)\delta(Y) \geq \langle 0|Z|0\rangle = 1$.

Positive Operator-Valued Measure (POVM) Measurements

- $\{M_m\}$: a collection of *measurement operators* with

$$\sum_m M_m^\dagger M_m = I.$$

- $\mathcal{P}(m) = \langle \psi | M_m^\dagger M_m | \psi \rangle$.
- $E_m \equiv M_m^\dagger M_m$: positive operators, called POVM elements

$$\sum_m E_m = I \quad \text{and} \quad \mathcal{P}(m) = \langle \psi | E_m | \psi \rangle.$$

- $\{E_m\}$: a POVM.
- Useful when only the measurement statistics matter.

For a projective measurement $\{P_m\}$, all the POVM elements are the same as the measurement operators since

$$E_m = P_m^\dagger P_m = P_m^2 = P_m.$$

What Are POVMs ?

- A collection of positive operators $\{E_m\}$.
- Satisfying the completeness relation

$$\sum_m E_m = I.$$

The corresponding measurement operators can be chosen as $\{\sqrt{E_m}\}$.

Postulate 4 – Composite Systems

- \mathcal{Q}_i : i th quantum system.
- \mathcal{H}_i : the Hilbert space associated to the quantum system \mathcal{Q}_i .
- $\mathcal{H} = \otimes_i \mathcal{H}_i$: the Hilbert space associated to the composite system of \mathcal{Q}_i 's.
- $|\psi_i\rangle$: a state of quantum system \mathcal{Q}_i .
- $|\psi\rangle = \otimes_i |\psi_i\rangle$: the joint state of the composite system.

Entangled States

- States in a composite quantum system.
- Not a direct product of states of component systems.
- $(|00\rangle + |01\rangle)/\sqrt{2}$ is not an entangled state since

$$\frac{|00\rangle + |01\rangle}{\sqrt{2}} = |0\rangle \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \right).$$

- Bell states in a two-qubit system are entangled states

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}, \frac{|00\rangle - |11\rangle}{\sqrt{2}}, \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \frac{|01\rangle - |10\rangle}{\sqrt{2}}.$$

A Proof

Suppose that

$$\begin{aligned}\frac{|00\rangle + |11\rangle}{\sqrt{2}} &= (a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle) \\ &= ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle,\end{aligned}$$

where $|a|^2 + |b|^2 = |c|^2 + |d|^2 = 1$. Then we have

$$ad = bc = 0.$$

- $a = c = 0 \Rightarrow \frac{|00\rangle + |11\rangle}{\sqrt{2}} = e^{j\theta} |11\rangle$, a contradiction.
- $b = d = 0 \Rightarrow \frac{|00\rangle + |11\rangle}{\sqrt{2}} = e^{j\theta'} |00\rangle$, a contradiction.

The Density Operator Formulation of Quantum Mechanics

- A convenient means for describing quantum systems whose states is *not completely* known.
- A convenient tool for the description of *individual* subsystems of a composite quantum system.

An Ensemble of Quantum Pure States $\{p_i, |\psi_i\rangle\}$

- $|\psi_i\rangle$: states of a quantum system, called *pure* states.
- p_i : the probability that the quantum system is in pure state $|\psi_i\rangle$,

$$\sum_i p_i = 1.$$

- The *density operator* or *density matrix* which represents this ensemble is

$$\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|.$$

- Not necessary a spectral decomposition of ρ since $\{|\psi_i\rangle\}$ may not be an orthonormal set.

Evolution of a Density Operator

- U : a unitary operator, describing the evolution of a closed quantum system during a time interval.
- ρ : a density operator, representing an ensemble $\{p_i, |\psi_i\rangle\}$ of pure states, which describes the initial state of the system.
- $U\rho U^\dagger$: density operator, describing the final state of the system.

$$\begin{aligned} |\psi_i\rangle &\xrightarrow{U} U|\psi_i\rangle \\ \rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| &\xrightarrow{U} \rho' = \sum_i p_i U|\psi_i\rangle\langle\psi_i|U^\dagger = U\rho U^\dagger. \end{aligned}$$

Measurement Effect on a Density Operator

- $\{M_m\}$: a collection of *measurement operators*, acting on the Hilbert space associated to the system being measured and satisfying the *completeness equation*

$$\sum_m M_m^\dagger M_m = I.$$

- m : index which represents possible measurement outcomes.
- ρ : a density operator, representing an ensemble $\{p_i, |\psi_i\rangle\}$ of pure states.

If the pre-measurement state of the quantum system is $|\psi_i\rangle$, then the probability of getting result m is

$$\mathcal{P}(m|i) = \langle \psi_i | M_m^\dagger M_m | \psi_i \rangle = \text{tr}(M_m^\dagger M_m |\psi_i\rangle \langle \psi_i|),$$

and the post-measurement state of the system is

$$|\psi_i^{(m)}\rangle = \frac{M_m |\psi_i\rangle}{\sqrt{\langle \psi_i | M_m^\dagger M_m | \psi_i \rangle}}.$$

The total probability of getting result m is

$$\begin{aligned} \mathcal{P}(m) &= \sum_i p_i \mathcal{P}(m|i) = \sum_i p_i \text{tr}(M_m^\dagger M_m |\psi_i\rangle \langle \psi_i|) \\ &= \text{tr} \left(M_m^\dagger M_m \left(\sum_i p_i |\psi_i\rangle \langle \psi_i| \right) \right) = \text{tr}(M_m^\dagger M_m \rho) = \text{tr}(M_m \rho M_m^\dagger). \end{aligned}$$

After a measurement which yields the result m , we have

- $\{\mathcal{P}(i|m), |\psi_i^{(m)}\rangle\}$: an ensemble of pure states
- $\mathcal{P}(i|m)$: the probability that the quantum system is in pure state $|\psi_i^{(m)}\rangle$ given that outcome m is measured

$$\mathcal{P}(i|m) = \frac{p_i \mathcal{P}(m|i)}{\mathcal{P}(m)}$$

- $\rho^{(m)}$: density operator, describing the state of the quantum system after the outcome m is measured

$$\begin{aligned} \rho^{(m)} &= \sum_i \mathcal{P}(i|m) |\psi_i^{(m)}\rangle \langle \psi_i^{(m)}| = \sum_i \mathcal{P}(i|m) \frac{M_m |\psi_i\rangle \langle \psi_i| M_m^\dagger}{\langle \psi_i| M_m^\dagger M_m |\psi_i\rangle} \\ &= \frac{\sum_i p_i M_m |\psi_i\rangle \langle \psi_i| M_m^\dagger}{\mathcal{P}(m)} = \frac{M_m \rho M_m^\dagger}{\text{tr}(M_m^\dagger M_m \rho)} = \frac{M_m \rho M_m^\dagger}{\text{tr}(M_m \rho M_m^\dagger)}. \end{aligned}$$

Pure States vs Mixed States

- Pure state $|\psi\rangle$: a quantum system whose state is exactly known as $|\psi\rangle$ and can be described by the density operator

$$\rho = |\psi\rangle\langle\psi|.$$

- Mixed state ρ : a quantum system whose state is not completely known and is described by the density operator

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|.$$

- A pure state can be regarded as a very special mixed state.

Characterization of Density Operators

ρ is a density operator associated with an ensemble $\{p_i, |\psi_i\rangle\}$ if and only if

- Unit trace condition : $\text{tr}(\rho) = 1$.
- Positivity condition : ρ is a positive operator.

Proof \Rightarrow

- $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|.$
- $\text{tr}(\rho) = \sum_i p_i \text{tr}(|\psi_i\rangle\langle\psi_i|) = \sum_i p_i \langle\psi_i|\psi_i\rangle = \sum_i p_i = 1.$
- $\langle\varphi|\rho|\varphi\rangle = \sum_i p_i \langle\varphi|\psi_i\rangle\langle\psi_i|\varphi\rangle = \sum_i p_i |\langle\varphi|\psi_i\rangle|^2 \geq 0.$

Proof \Leftarrow

- ρ is positive with a spectral decomposition

$$\rho = \sum_j \lambda_j |\psi_j\rangle\langle\psi_j|.$$

- λ_j : non-negative eigenvalues.
- $|\psi_j\rangle$: eigenvectors.
- $1 = \text{tr}(\rho) = \sum_j \lambda_j$.
- $\{\lambda_j, |\psi_j\rangle\}$: an ensemble of pure states giving rise to the density operator ρ .

A Criterion of Pure States

A density operator ρ is in a pure state if and only if

$$\text{tr}(\rho^2) = 1.$$

- For a mixed (not a pure) state ρ , we have $\text{tr}(\rho^2) < 1$.

Proof

Let ρ be a density operator with spectral decomposition

$$\rho = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|,$$

where $\lambda_i \geq 0$ and $\text{tr}(\rho) = \sum_i \lambda_i = 1$. Since

$$\rho^2 = \sum_i \lambda_i^2 |\psi_i\rangle\langle\psi_i|,$$

we have

$$\text{tr}(\rho^2) = \sum_i \lambda_i^2 \leq \sum_i \lambda_i^2 + 2 \sum_{i < j} \lambda_i \lambda_j = \left(\sum_i \lambda_i \right)^2 = 1,$$

where equality holds if and only if only one λ_i is non-zero and is equal to one, i.e., $\rho = |\psi_i\rangle\langle\psi_i|$, a pure state.

Mixture of Mixed States

$$\rho = \sum_i p_i \rho_i.$$

- ρ_i : density operator corresponding to an ensemble $\{p_{ij}, |\psi_{ij}\rangle\}$

$$\rho_i = \sum_j p_{ij} |\psi_{ij}\rangle \langle \psi_{ij}|.$$

- p_i : probability that the state of the quantum system is prepared in ρ_i .

The probability of being in the pure state $|\psi_{ij}\rangle$ is $p_i p_{ij}$ and the overall density operator to describe the state of the quantum system is

$$\rho = \sum_{ij} p_i p_{ij} |\psi_{ij}\rangle \langle \psi_{ij}| = \sum_i p_i \sum_j p_{ij} |\psi_{ij}\rangle \langle \psi_{ij}| = \sum_i p_i \rho_i.$$

Density Operator After Unspecified Measurement $\{M_m\}$

$$\rho' = \sum_m \mathcal{P}(m) \rho^{(m)} = \sum_m \text{tr}(M_m \rho M_m^\dagger) \frac{M_m \rho M_m^\dagger}{\text{tr}(M_m \rho M_m^\dagger)} = \sum_m M_m \rho M_m^\dagger.$$

Average for Projective Measurement

- ρ : density operator for a quantum system
- M : an observable for the quantum system with spectral decomposition

$$M = \sum_m m P_m$$

- $\mathcal{P}(m) = \text{tr}(P_m \rho P_m) = \text{tr}(P_m^2 \rho) = \text{tr}(P_m \rho)$: the probability that outcome m occurs
- $\langle M \rangle$: the average measurement value

$$\langle M \rangle = \sum_m m \mathcal{P}(m) = \sum_m m \text{tr}(P_m \rho) = \text{tr}(M \rho).$$

What Class of Ensembles Gives Rise to a Particular ρ ?

- $\rho = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|$ (spectral decomposition).
- $|a\rangle = \sqrt{\frac{3}{4}}|0\rangle + \sqrt{\frac{1}{4}}|1\rangle$, $|b\rangle = \sqrt{\frac{3}{4}}|0\rangle - \sqrt{\frac{1}{4}}|1\rangle$.

$$\frac{1}{2}|a\rangle\langle a| + \frac{1}{2}|b\rangle\langle b| = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1| = \rho.$$

- A lesson : the collection of eigenstates of a density operator is not an especially privileged ensemble.

Unitary Freedom in the Ensemble for Density Operators

Two ensembles $\{p_i, |\psi_i\rangle\}$ and $\{q_j, |\varphi_j\rangle\}$ give rise to the same density operator ρ , i.e.,

$$\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|, \rho = \sum_j q_j |\varphi_j\rangle \langle \varphi_j|$$

if and only if

$$\sqrt{p_i} |\psi_i\rangle = \sum_j z_{ij} \sqrt{q_j} |\varphi_j\rangle$$

where z_{ij} is a unitary matrix of complex numbers and pure states with zero probability are padded to the smaller ensemble to have the same size as the larger one.

Proof \Leftarrow

- $|v_i\rangle \equiv \sqrt{p_i}|\psi_i\rangle$, $|w_j\rangle \equiv \sqrt{q_j}|\varphi_j\rangle$.

Since

$$|v_i\rangle = \sum_j z_{ij}|w_j\rangle,$$

we have

$$\begin{aligned} \sum_i p_i |\psi_i\rangle \langle \psi_i| &= \sum_i |v_i\rangle \langle v_i| = \sum_i \sum_{jk} z_{ij} z_{ik}^* |w_j\rangle \langle w_k| \\ &= \sum_{jk} \left(\sum_i z_{ij} z_{ik}^* \right) |w_j\rangle \langle w_k| \\ &= \sum_j |w_j\rangle \langle w_j| \\ &= \sum_j q_j |\varphi_j\rangle \langle \varphi_j|. \end{aligned}$$

Proof \Rightarrow

By spectral decomposition of ρ , we have

$$\rho = \sum_k \lambda_k |k\rangle\langle k| = \sum_k |k'\rangle\langle k'|,$$

where λ_k are positive, $|k\rangle$ are orthonormal and $|k'\rangle = \sqrt{\lambda_k} |k\rangle$.

- $|u\rangle$: a vector in the orthogonal complement $\text{Span}\{|k'\rangle\}^\perp$ of $\text{Span}\{|k'\rangle\}$.

Then

$$0 = \sum_k \langle u|k'\rangle\langle k'|u\rangle = \langle u|\rho|u\rangle = \sum_i \langle u|v_i\rangle\langle v_i|u\rangle = \sum_i |\langle u|v_i\rangle|^2$$

which implies that

$$|u\rangle \in \text{Span}\{|v_i\rangle\}^\perp.$$

Thus

$$\text{Span}\{|k'\rangle\}^\perp \subseteq \text{Span}\{|v_i\rangle\}^\perp \text{ and then } \text{Span}\{|v_i\rangle\} \subseteq \text{Span}\{|k'\rangle\}.$$

For each $|v_i\rangle$, we have

$$|v_i\rangle = \sum_k c_{ik} |k'\rangle$$

Then

$$\rho = \sum_k |k'\rangle\langle k'| = \sum_i |v_i\rangle\langle v_i| = \sum_{kl} \left(\sum_i c_{ik} c_{il}^* \right) |k'\rangle\langle l'|$$

Since the operators $|k'\rangle\langle l'|$ are linearly independent, we have

$$\sum_i c_{ik} c_{il}^* = \delta_{kl}$$

By appending more columns to the matrix $C = [c_{ik}]$, we obtain a

unitary matrix $T = [t_{ik}]$ such that

$$|v_i\rangle = \sum_k t_{ik} |k'\rangle$$

where some zero vectors are padded into the list of $|k'\rangle$. Similarly, there is a unitary matrix $S = [s_{jk}]$ such that

$$|w_j\rangle = \sum_k s_{jk} |k'\rangle$$

Then with $Z = TS^\dagger$ a unitary matrix and $Z = [z_{ij}]$, we have

$$|v_i\rangle = \sum_j z_{ij} |w_j\rangle$$

since

$$\begin{aligned}
\sum_j z_{ij} |w_j\rangle &= \sum_j \sum_k t_{ik} s_{jk}^* \sum_l s_{jl} |l'\rangle \\
&= \sum_{kl} t_{ik} |l'\rangle \sum_j s_{jk}^* s_{jl} \\
&= \sum_k t_{ik} |k'\rangle \\
&= |v_i\rangle
\end{aligned}$$

Postulates of Quantum Mechanics

– Density Operator Version

Postulate 1 – States

Associated to an *isolated* physical system is a Hilbert space \mathcal{H} (eg, a finite-dimensional complex inner product space). The state of the system is completely described by its *density operator*, which is a positive operator with trace one acting on the Hilbert space \mathcal{H} . If the quantum system is in the state ρ_i with probability p_i , then the density operator for this system is

$$\rho = \sum_i p_i \rho_i.$$

Postulate 2 - Time Evolution

The evolution of a *closed* quantum system is described by a *unitary operator*. That is, the state ρ of the system at time t_1 is related to the state ρ' of the system at time t_2 by a unitary operator U which depends only on the times t_1 and t_2 ,

$$\rho' = U\rho U^\dagger.$$

Postulate 3 – Quantum Measurements

- $\{M_m\}$: a collection of *measurement operators*, acting on the Hilbert space associated to the system being measured and satisfying the *completeness equation*

$$\sum_m M_m^\dagger M_m = I.$$

- m : measurement outcomes that may occur in the experiment.

If the pre-measurement state of the quantum system is ρ , then the probability that result m occurs is given by

$$\mathcal{P}(m) = \text{tr}(M_m \rho M_m^\dagger),$$

and the post-measurement state of the system is

$$\frac{M_m \rho M_m^\dagger}{\text{tr}(M_m \rho M_m^\dagger)}.$$

The completeness equation expresses the fact that probabilities sum to one

$$\begin{aligned} \sum_m \mathcal{P}(m) &= \sum_m \text{tr}(M_m \rho M_m^\dagger) = \sum_m \text{tr}(M_m^\dagger M_m \rho) \\ &= \text{tr} \left(\left(\sum_m M_m^\dagger M_m \right) \rho \right) = \text{tr}(\rho) = 1. \end{aligned}$$

Postulate 4 – Composite Systems

- \mathcal{Q}_i : i th quantum system.
- \mathcal{H}_i : the Hilbert space associated to the quantum system \mathcal{Q}_i .
- $\mathcal{H} = \bigotimes_i \mathcal{H}_i$: the Hilbert space associated to the composite system of \mathcal{Q}_i 's.
- ρ_i : the state in which the quantum system \mathcal{Q}_i is prepared.
- $\rho = \bigotimes_i \rho_i$: the joint state of the composite system.

Reduced Density Operator

Definition

$$\rho^A \triangleq \text{tr}_B(\rho^{AB}).$$

- ρ^{AB} : density operators for composite quantum system AB .
- $\rho^A \triangleq \text{tr}_B(\rho^{AB})$: reduced density operator for subsystem A .
 - A description for the state of subsystem A : justification needed.

A Simple Justification

- $\rho^{AB} = \rho \otimes \sigma$: a direct product density operator for composite quantum system AB .
- $\rho^A = \text{tr}_B(\rho^{AB}) = \rho \text{tr}(\sigma) = \rho$: correct description of system A .
- $\rho^B = \text{tr}_A(\rho^{AB}) = \text{tr}(\rho)\sigma = \sigma$: correct description of system B .

A Further Justification

Local and Global Observables

- M : the observable on subsystem A for a measurement carrying out on subsystem A , a Hermitian operator with spectral decomposition

$$M = \sum_m m P_m.$$

- $M \otimes I$: the corresponding observable on the composite system AB for the same measurement carrying out on subsystem A , a Hermitian operator with spectral decomposition

$$M \otimes I = \sum_m m (P_m \otimes I).$$

- $|m\rangle$ is an eigenstate of the observable M and $|\psi\rangle$ is any state of subsystem $B \iff |m\rangle \otimes |\psi\rangle$ is an eigenstate of $M \otimes I$.

When System AB Is Prepared With State $|m\rangle \otimes |\psi\rangle$

- m : the outcome which occurs with probability one by the observable M on subsystem A .
- m : the outcome which occurs with probability one by the observable $M \otimes I$ on the composite system AB .
- Consistency.

When System AB Is in a Mixed State ρ^{AB}

- $f(\rho^{AB})$: a density operator on subsystem A as a function of the density operator on system AB , serving as an appropriate description of the state of subsystem A .
- Measurement statistics must be consistent between the local observable M on subsystem A and the global observable $M \otimes I$ on system AB

$$\text{tr}(M f(\rho^{AB})) = \langle M \rangle = \langle M \otimes I \rangle = \text{tr}((M \otimes I) \rho^{AB}).$$

Existence : $f(\rho^{AB}) = \text{tr}_B(\rho^{AB})$

- $\rho^{AB} = \sum_i \alpha_i T_i^A \otimes T_i^B$: a linear operator on the state space of the composite system AB .

$$\begin{aligned}
 & \text{tr}((M \otimes I)\rho^{AB}) \\
 = & \text{tr}((M \otimes I)(\sum_i \alpha_i T_i^A \otimes T_i^B)) = \text{tr}(\sum_i \alpha_i (MT_i^A) \otimes T_i^B) \\
 = & \text{tr}(\text{tr}_B(\sum_i \alpha_i (MT_i^A) \otimes T_i^B)) = \text{tr}(\sum_i \alpha_i (MT_i^A) \text{tr}(T_i^B)) \\
 = & \text{tr}(M(\sum_i \alpha_i T_i^A \text{tr}(T_i^B))) = \text{tr}(M \text{tr}_B(\sum_i \alpha_i T_i^A \otimes T_i^B)) \\
 = & \text{tr}(M \text{tr}_B(\rho^{AB})).
 \end{aligned}$$

Uniqueness

- \mathcal{H} : the Hilbert space associated to the quantum system A .
- $L^H(\mathcal{H})$: the real inner product space of all Hermitian operators on \mathcal{H} with trace inner product.
- $\{M_i\}$: an orthonormal basis of $L^H(\mathcal{H})$.
- $f(\rho^{AB}) = \sum_i M_i \operatorname{tr}(M_i f(\rho^{AB}))$: the expansion of $f(\rho^{AB})$ by the orthonormal basis $\{M_i\}$.

Since

$$\operatorname{tr}(M_i f(\rho^{AB})) = \operatorname{tr}((M_i \otimes I) \rho^{AB}) \quad \forall i,$$

we have

$$f(\rho^{AB}) = \sum_i M_i \operatorname{tr}((M_i \otimes I) \rho^{AB})$$

which uniquely specifies the function f .

An Example

- Suppose a two-qubit system is in a pure Bell state $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$ with density operator

$$\begin{aligned}\rho^{12} &= \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}} \right) \left(\frac{\langle 00| + \langle 11|}{\sqrt{2}} \right) \\ &= \frac{|00\rangle\langle 00| + |11\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 11|}{2}.\end{aligned}$$

- ρ^1 : the reduced density operator of the first qubit

$$\begin{aligned}
 \rho^1 &= \text{tr}_2(\rho^{12}) \\
 &= \frac{\text{tr}_2(|00\rangle\langle 00|) + \text{tr}_2(|11\rangle\langle 00|) + \text{tr}_2(|00\rangle\langle 11|) + \text{tr}_2(|11\rangle\langle 11|)}{2} \\
 &= \frac{|0\rangle\langle 0|\langle 0|0\rangle + |1\rangle\langle 0|\langle 0|1\rangle + |0\rangle\langle 1|\langle 1|0\rangle + |1\rangle\langle 1|\langle 1|1\rangle}{2} \\
 &= \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} = \frac{I}{2}.
 \end{aligned}$$

- Reduced density operator ρ^1 for the first qubit is in a *mixed* state while the two-qubit system is in a *pure* state.

Schmidt Decomposition and Purification

Schmidt Decomposition

For each pure state $|\psi\rangle$ in a composite quantum system AB , there exist a set $\{|i_A\rangle\}$ of orthonormal states for subsystem A and a set $\{|i_B\rangle\}$ of orthonormal states for subsystem B of the same size such that

$$|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle$$

where λ_i are non-negative real numbers with

$$\sum_i \lambda_i^2 = 1.$$

- λ_i : Schmidt coefficients.
- $\{|i_A\rangle\}$ and $\{|i_B\rangle\}$: Schmidt “bases” for A and B respectively.
 - Dependent on $|\psi\rangle$.
- # of non-zero values λ_i : Schmidt number for $|\psi\rangle$.

Proof

- $\{|j\rangle\}, \{|k\rangle\}$: given orthonormal bases of the Hilbert spaces of subsystems A and B respectively

$$|\psi\rangle = \sum_{jk} c_{jk} |j\rangle |k\rangle.$$

- $C = UDV$: singular value decomposition

$$C = [c_{jk}], U = [u_{ji}], D = \text{diag}(d_{ii}), V = [v_{ik}],$$

$$c_{jk} = \sum_i u_{ji} d_{ii} v_{ik}.$$

- U and V : unitary matrices.
- D : a diagonal matrix, not necessarily square.

$$\begin{aligned}
|\psi\rangle &= \sum_{jk} \sum_i u_{ji} d_{ii} v_{ik} |j\rangle |k\rangle \\
&= \sum_i d_{ii} \left(\sum_j u_{ji} |j\rangle \right) \left(\sum_k v_{ik} |k\rangle \right) = \sum_i \lambda_i |i_A\rangle |i_B\rangle.
\end{aligned}$$

- $|i_A\rangle = \sum_j u_{ji} |j\rangle$: orthonormal states of subsystem A

$$\langle i_A | i'_A \rangle = \sum_{jj'} u_{ji}^* u_{j'i'} \langle j | j' \rangle = \sum_j u_{ji}^* u_{ji'} = \delta_{ii'}.$$

- $|i_B\rangle = \sum_k v_{ik} |k\rangle$: orthonormal states of subsystem B

$$\langle i_B | i'_B \rangle = \sum_{kk'} v_{ik}^* v_{i'k'} \langle k | k' \rangle = \sum_k v_{ik}^* v_{i'k} = \delta_{ii'}.$$

- $\lambda_i = d_{ii}$: non-negative real numbers

$$1 = \langle \psi | \psi \rangle = \sum_{ii'} \lambda_i \lambda_{i'} \langle i_A | i'_A \rangle \langle i_B | i'_B \rangle = \sum_i \lambda_i^2.$$

Schmidt Number for State $|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle$

- "Amount" of entanglement between systems A and B when the composite system AB is in state $|\psi\rangle$.
- Invariance under unitary transformations on subsystem A or subsystem B alone.
 - U : a unitary operator on subsystem A .
 - $U|i_A\rangle$: orthonormal states of subsystem A .

$$(U \otimes I)|\psi\rangle = \sum_i \lambda_i (U \otimes I)(|i_A\rangle \otimes |i_B\rangle) = \sum_i \lambda_i U|i_A\rangle |i_B\rangle.$$

Purification

- ρ_A : a density operator for system A with ensemble $\{p_i, |i_A\rangle\}$

$$\rho_A = \sum_i p_i |i_A\rangle\langle i_A|.$$

- R : a reference system.
- $\{|i_R\rangle\}$: an orthonormal basis of the Hilbert space associated to system R , having the same cardinality as that of $\{|i_A\rangle\}$.
- $|AR\rangle$: a pure state of the composite system AR with

$$|AR\rangle \triangleq \sum_i \sqrt{p_i} |i_A\rangle |i_R\rangle.$$

$$\begin{aligned}
\text{tr}_R(|AR\rangle\langle AR|) &= \sum_{ij} \sqrt{p_i p_j} \text{tr}_R(|i_A\rangle\langle j_A| \otimes |i_R\rangle\langle j_R|) \\
&= \sum_{ij} \sqrt{p_i p_j} |i_A\rangle\langle j_A| \text{tr}(|i_R\rangle\langle j_R|) \\
&= \sum_i p_i |i_A\rangle\langle i_A| = \rho_A.
\end{aligned}$$

- A mixed state of a local system is a local view of a pure state in a global composite system.

Applications

Non-orthogonal States Cannot Be Distinguished

- $\{M_j\}$: measurement operators
- $|\psi_1\rangle$ and $|\psi_2\rangle$: two non-orthogonal states to be distinguished and

$$|\psi_2\rangle = \alpha|\psi_1\rangle + \beta|\psi\rangle,$$

where $|\psi_1\rangle$ and $|\psi\rangle$ are orthonormal. Note that $|\alpha|^2 + |\beta|^2 = 1$ and then $|\beta| < 1$.

- $f(\cdot)$: a rule to guess which state vector is observed based on the outcome of the measurement, i.e., either $f(j) = 1$ or $f(j) = 2$.

Suppose that $|\psi_1\rangle$ and $|\psi_2\rangle$ can be distinguished reliably, i.e.,

$$\sum_{j:f(j)=1} \langle \psi_1 | M_j^\dagger M_j | \psi_1 \rangle = \langle \psi_1 | \left(\sum_{j:f(j)=1} M_j^\dagger M_j \right) | \psi_1 \rangle = \langle \psi_1 | G_1 | \psi_1 \rangle = 1$$

$$\sum_{j:f(j)=2} \langle \psi_2 | M_j^\dagger M_j | \psi_2 \rangle = \langle \psi_2 | \left(\sum_{j:f(j)=2} M_j^\dagger M_j \right) | \psi_2 \rangle = \langle \psi_2 | G_2 | \psi_2 \rangle = 1$$

where $G_i = \sum_{j:f(j)=i} M_j^\dagger M_j$, for $i = 1, 2$.

Since $G_1 + G_2 = I$, we have $\langle \psi_1 | (G_1 + G_2) | \psi_1 \rangle = 1$ and then

$$\langle \psi_1 | G_2 | \psi_1 \rangle = 0 \Rightarrow \sqrt{G_2} | \psi_1 \rangle = 0 \Rightarrow \sqrt{G_2} | \psi_2 \rangle = \beta \sqrt{G_2} | \psi \rangle$$

Thus a contradiction is obtained as follows

$$\langle \psi_2 | G_2 | \psi_2 \rangle = |\beta|^2 \langle \psi | G_2 | \psi \rangle \leq |\beta|^2 < 1$$

since $\langle \psi | G_2 | \psi \rangle \leq \langle \psi | (G_1 + G_2) | \psi \rangle = \langle \psi | \psi \rangle = 1$.

Superdense Coding

- Goal : Alice wants to send two classical bits of information to Bob by transmitting only one qubit to Bob
- Initialization : preparing a pair of qubits in a Bell State

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Alice held the first qubit and Bob held the second qubit before apart (may send by a third party)

- Alice takes action on her qubit according the two bits of information she wants to send

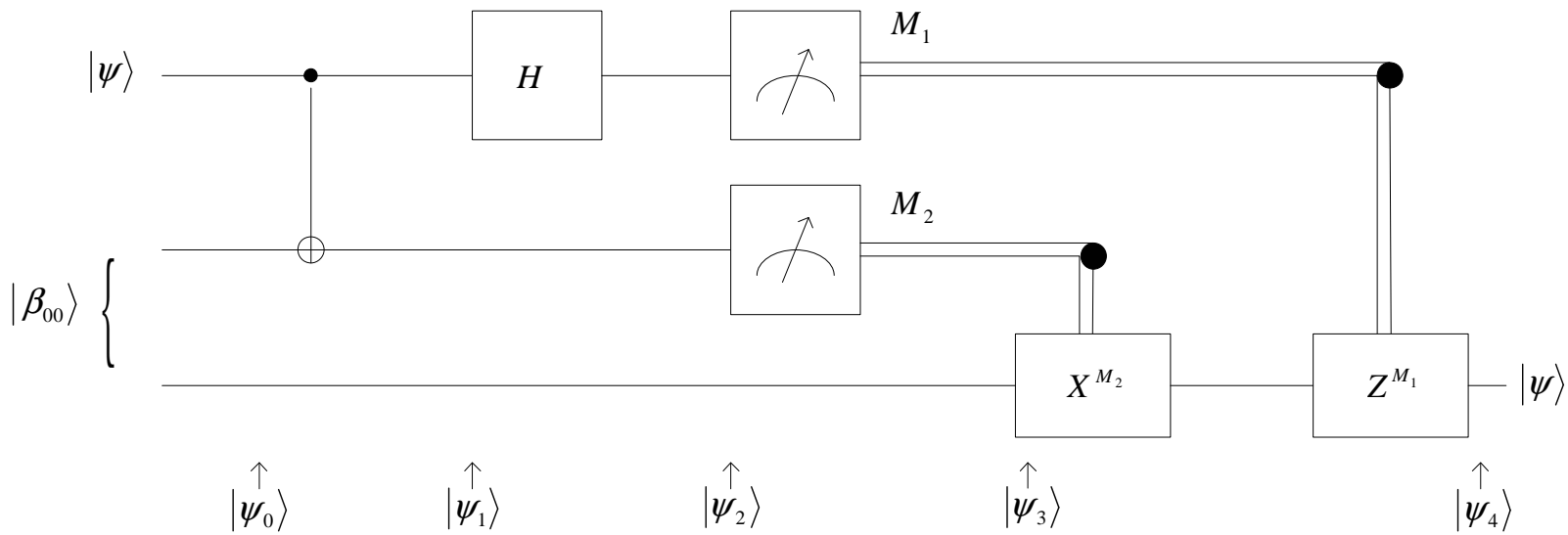
$$00 : |\psi\rangle \rightarrow (I \otimes I)|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

$$01 : |\psi\rangle \rightarrow (Z \otimes I)|\psi\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

$$\begin{aligned}
10 : |\psi\rangle &\rightarrow (X \otimes I)|\psi\rangle = \frac{|10\rangle + |01\rangle}{\sqrt{2}} \\
11 : |\psi\rangle &\rightarrow (iY \otimes I)|\psi\rangle = \frac{|10\rangle - |01\rangle}{\sqrt{2}}
\end{aligned}$$

- Alice sends her qubit to Bob
- The four Bell states form an orthonormal basis of the two-qubit system and can form a projective measurement
- With two qubits together, Bob makes the projective measurement

Quantum Teleportation



Quantum Teleportation

- $|\psi_2\rangle$: state of the three-qubit system before Alice makes her measurement

$$|\psi_2\rangle = \frac{1}{2} (|00\rangle(\alpha|0\rangle + \beta|1\rangle) + |01\rangle(\alpha|1\rangle + \beta|0\rangle) + |10\rangle(\alpha|0\rangle - \beta|1\rangle) + |11\rangle(\alpha|1\rangle - \beta|0\rangle))$$

- $\{|00\rangle\langle 00|, |01\rangle\langle 01|, |10\rangle\langle 10|, |11\rangle\langle 11|\}$: a POVM measurement made by Alice on her two qubits
- $\rho = |\psi_2\rangle\langle\psi_2|$: density operator for the three-qubit system before the measurement
- ρ' : density operator for the three-qubit system after the unspecified (from Bob's point of view) measurement

$$\rho' = \sum_m M_m \rho M_m^\dagger = \sum_m M_m |\psi_2\rangle\langle\psi_2| M_m^\dagger$$

- $|00\rangle\langle 00|\psi_2\rangle = (1/2)|00\rangle(\alpha|0\rangle + \beta|1\rangle)$
- $|01\rangle\langle 01|\psi_2\rangle = (1/2)|01\rangle(\alpha|1\rangle + \beta|0\rangle)$
- $|10\rangle\langle 10|\psi_2\rangle = (1/2)|10\rangle(\alpha|0\rangle - \beta|1\rangle)$
- $|11\rangle\langle 11|\psi_2\rangle = (1/2)|11\rangle(\alpha|1\rangle - \beta|0\rangle)$
- ρ^B : the reduced density operator of Bob's qubit

$$\begin{aligned}
\rho^B &= \text{tr}_A(\rho') = \sum_m \text{tr}_A(M_m|\psi_2\rangle\langle\psi_2|M_m^\dagger) \\
&= \frac{1}{4}((\alpha|0\rangle + \beta|1\rangle)(\alpha|0\rangle + \beta|1\rangle)^\dagger + (\alpha|1\rangle + \beta|0\rangle)(\alpha|1\rangle + \beta|0\rangle)^\dagger \\
&\quad + (\alpha|0\rangle - \beta|1\rangle)(\alpha|0\rangle - \beta|1\rangle)^\dagger + (\alpha|1\rangle - \beta|0\rangle)(\alpha|1\rangle - \beta|0\rangle)^\dagger) \\
&= \frac{2(|\alpha|^2 + |\beta|^2)|0\rangle\langle 0| + 2(|\alpha|^2 + |\beta|^2)|1\rangle\langle 1|}{4} \\
&= \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} = \frac{I}{2}
\end{aligned}$$

- Bob does not have any information about the state $|\psi\rangle$ if Alice does not send him her measurement result, preventing Alice from using teleportation to transmit information to Bob faster than light

Anti-correlations in the EPR Experiment

- V : the state space of a qubit
- $\mathcal{B} = \{|0\rangle, |1\rangle\}$: an orthonormal basis of V
- $M = \alpha\sigma_x + \beta\sigma_y + \gamma\sigma_z$: an observable on V

$$|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1, X = [\sigma_x]_{\mathcal{B}}, Y = [\sigma_y]_{\mathcal{B}}, Z = [\sigma_z]_{\mathcal{B}}$$

- ± 1 : eigenvalues of M
- $\mathcal{B}' = \{|a\rangle, |b\rangle\}$: unit eigenvectors of M

$$|0\rangle = \alpha|a\rangle + \beta|b\rangle$$

$$|1\rangle = \gamma|a\rangle + \delta|b\rangle$$

with coordinate transformation matrix U , which is unitary

$$U = [\mathcal{B}' \rightarrow \mathcal{B}] = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, |\det(U)| = |\alpha\delta - \beta\gamma| = 1$$

- $|\psi\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$: a Bell state prepared on a two-qubit quantum system

$$\frac{|01\rangle - |10\rangle}{\sqrt{2}} = (\alpha\delta - \beta\gamma) \frac{|ab\rangle - |ba\rangle}{\sqrt{2}}$$

- $M \otimes M$: observable on the two-qubit system

$$M \otimes M = (I \otimes M)(M \otimes I)$$

with spectral decomposition

$$M \otimes I = (|a\rangle\langle a| \otimes I) - (|b\rangle\langle b| \otimes I)$$

$$I \otimes M = (I \otimes |a\rangle\langle a|) - (I \otimes |b\rangle\langle b|)$$

$$\begin{array}{lcl}
\frac{|ab\rangle - |ba\rangle}{\sqrt{2}} & \xrightarrow{M \otimes I} & \begin{array}{l} +1 : |ab\rangle \text{ with prob. } \frac{1}{2} \\ -1 : |ba\rangle \text{ with prob. } \frac{1}{2} \end{array} \\
& & \xrightarrow{I \otimes M} \begin{array}{l} -1 : |ab\rangle \text{ with prob. } 1 \\ +1 : |ba\rangle \text{ with prob. } 1 \end{array}
\end{array}$$

The Argument of Einstein, Podolsky and Rosen

- Any "element of reality" must be represented in any complete physical theory
- It is sufficient to say a physical property to be an element of reality if it is possible to predict with certainty the value that property will have, immediately before measurement
- As in the anti-correlation experiment on a Bell state, once Alice gets her measurement result $+1$ (-1), she can predict with certainty that Bob will measure -1 ($+1$) on his qubit
- The physical property revealed by various observables M on Bob's qubit is an element of reality of Bob's qubit

- Quantum mechanics only tell one how to calculate the probability of the respective measurement outcomes if M is measured, it does not include any fundamental element intended to represent such a physical property
- Quantum mechanics is not a complete physical theory

Bell's Inequality

- A compelling example which illustrates an essential difference between quantum and classical physics

The Setup

- Charlie prepares two particles
 - He is capable of repeating the experimental procedure
- Charlie sends one particle to Alice and another particle to Bob

Derivation of Bell's Inequality in Classical Scenario

- P_Q, P_R : two physical properties of Alice's particle
- Q, R : values of P_Q, P_R respectively
 - Assumed to exist *independent of* measurement, i.e., assumed to be *objective properties* of Alice's particle
 - Merely revealed by measurement apparatuses
 - Variables each taking +1 or -1
- P_S, P_T : two physical properties of Bob's particle
- S, T : values of P_S, P_T respectively
 - Assumed to exist *independent of* measurement, i.e., assumed to be *objective properties* of Bob's particle
 - Merely revealed by measurement apparatuses
 - Variables taking +1 or -1

- Alice and Bob do their measurements at the same time, which are assumed *uncorrelatedly*
 - Alice performing her measurement does not disturb the result of Bob's measurement
 - Bob performing his measurement does not disturb the result of Alice's measurement
- $QS + RS + RT - QT$: a derived quantity

$$QS + RS + RT - QT = (Q + R)S + (R - Q)T = \pm 2,$$

since $Q, R = \pm 1$ implies that either $(Q + R)S = 0$ or $(R - Q)T = 0$

- $\mathcal{P}(q, r, s, t)$: probability that, before the measurements are performed, the system is in a state where $Q = q, R = r, S = s, T = t$
 - Dependent on how Charlie prepares the two particles
 - Dependent on experimental noise
- Bell's inequality :

$$\begin{aligned}
 & \mathcal{E}(QS) + \mathcal{E}(RS) + \mathcal{E}(RT) - \mathcal{E}(QT) \\
 = & \mathcal{E}(QS + RS + RT - QT) \\
 = & \sum_{q,r,s,t} \mathcal{P}(q, r, s, t)(qs + rs + rt - qt) \\
 \leq & \sum_{q,r,s,t} \mathcal{P}(q, r, s, t) \cdot 2 \\
 = & 2
 \end{aligned}$$

- It doesn't matter how Charlie prepares the particles

A Quantum Mechanical Scenario

- Charlie prepares a quantum system of two qubits in the Bell state

$$|\psi\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

- Q, R : observables performed by Alice on her qubit

$$Q = Z_1, R = X_1$$

- S, T : observables performed by Bob on his qubit

$$S = \frac{-Z_2 - X_2}{\sqrt{2}}, R = \frac{Z_2 - X_2}{\sqrt{2}}$$

A Quantum Mechanical Scenario (Cont')

- $\langle QS \rangle, \langle RS \rangle, \langle RT \rangle, \langle QT \rangle$: average values of measurements

$$\langle QS \rangle = \langle \psi | (Z_1 \otimes \frac{-Z_2 - X_2}{\sqrt{2}}) | \psi \rangle = \frac{1}{\sqrt{2}}$$

$$\langle RS \rangle = \langle \psi | (X_1 \otimes \frac{-Z_2 - X_2}{\sqrt{2}}) | \psi \rangle = \frac{1}{\sqrt{2}}$$

$$\langle RT \rangle = \langle \psi | (X_1 \otimes \frac{Z_2 - X_2}{\sqrt{2}}) | \psi \rangle = \frac{1}{\sqrt{2}}$$

$$\langle QT \rangle = \langle \psi | (Z_1 \otimes \frac{Z_2 - X_2}{\sqrt{2}}) | \psi \rangle = -\frac{1}{\sqrt{2}}$$

$$\langle QS \rangle + \langle RS \rangle + \langle RT \rangle - \langle QT \rangle = 2\sqrt{2}$$

- But the Bell's inequality says that the above quantity cannot exceed two

Experimental results were in favor of the prediction of quantum mechanics. The Bell's inequality is not obeyed by Nature!

Two Questionable Assumptions Made Classically

- Assumption of *realism* : the physical properties P_Q, P_R, P_S, P_T have definite values Q, R, S, T which exist independent of observation
- Assumption of *locality* : Alice performing her measurement does not influence the result of Bob's measurement

The Lesson

Bell's inequality together with substantial experimental evidence convinces ourselves that either or both of realism and locality must be dropped in order to develop a correct intuitive understanding of quantum mechanics.