EE 641000
Homework Assignment #2

Spring Semester, 2006

Due Date: March 28, 2006

We let $\mathcal{H}, \mathcal{H}_1, \mathcal{H}_2$ be finite-dimensional complex inner product spaces.

1. (a) Exercise 2.27 on Page 74 of the textbook.
   (b) Exercise 2.33 on Page 74 of the textbook.

2. Let $\mathcal{H}$ be a Hilbert space of dimension n and $L(\mathcal{H})$ the space of all linear operators on $\mathcal{H}$. Define a complex-valued function $(\cdot, \cdot)$ on $L(\mathcal{H}) \times L(\mathcal{H})$

$$(T, S) \triangleq \text{tr}(A^\dagger S).$$

   Please show that

   (a) $(\cdot, \cdot)$ is an inner product on $L(\mathcal{H})$. Thus $L(\mathcal{H})$ is an Hilbert space with this trace inner product. What is the dimension of $L(\mathcal{H})$?

   (b) Find an orthonormal basis of Hermitian operators for the Hilbert space $L(\mathcal{H})$.

   (c) Let $L^H(\mathcal{H})$ be the set of all Hermitian operators on $\mathcal{H}$. Please show that $L^H(\mathcal{H})$ is a real inner product space with trace inner product $(T, S) = \text{tr}(TS)$ for any two Hermitian operators T, S in $L^H(\mathcal{H})$. What is the dimension of $L^H(\mathcal{H})$? Find an orthonormal basis of $L^H(\mathcal{H})$.

3. (Polar decomposition) Let T be a linear operator on a vector space V. Please show that there exist unitary operator U and positive operators J and K such that

$$T = UJ = KU.$$

   Are the two positive operators J, K unique ? Why and if yes, what are they? Is the unitary operator unique ? Why and if not, when is it unique? (Please refer to page 78.)

4. Please show that if $\mathcal{B} = \{|i\rangle\}$ and $\mathcal{C} = \{|j\rangle\}$ are orthonormal bases of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, then $\mathcal{B} \otimes \mathcal{C} = \{|i\rangle \otimes |j\rangle\}$ is an orthonormal basis of the tensor product space $\mathcal{H}_1 \otimes \mathcal{H}_2$.

5. Please show that the trace function tr on $L(\mathcal{H})$ is

   (a) Linear : $\text{tr}(\alpha T + \beta S) = \alpha\ \text{tr}(T) + \beta\ \text{tr}(S)$ for all complex numbers α, β and for all operators T, S in $L(\mathcal{H})$.

   (b) Cyclic : $\text{tr}(TS) = \text{tr}(ST)$ for all operators T, S.

6. (a) Let $|\psi\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ be a state in $\mathcal{H}$ of two dimension. Please write out $|\psi\rangle \otimes |\psi\rangle$ and $|\psi\rangle \otimes |\psi\rangle \otimes |\psi\rangle$ explicitly in terms of a basis of $\mathcal{H} \otimes \mathcal{H}$ and a basis of $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$, respectively.

   (b) Please determine if the state $|\varphi\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$ is an entangled state or not.

   (c) Please determine if the density operator ρ is a pure state or not

$$\rho = \begin{bmatrix} 1/2 & (-1-i)/4 \\ (-1+i)/4 & 1/2 \end{bmatrix}.$$

   (d) Please find the square root and logarithm of the linear operator $\rho = 2|0\rangle\langle 0| + (1+i)|0\rangle\langle 1| + (1-i)|1\rangle\langle 0| + 2|1\rangle\langle 1|$ on $\mathcal{H}$.

7. Let $\sigma_1, \sigma_2, \sigma_3$ be Pauli operators on a two-dimensional Hilbert space $\mathcal{H}$ with matrix representation with respective to an orthonormal basis $|0\rangle, |1\rangle\}$ of $\mathcal{H}$ as follows:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \ \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

   (a) Please show that for any real number θ, we have

$$\exp(i\theta\vec{v}\cdot\vec{\sigma}) = \cos\theta I + i\sin\theta\vec{v}\cdot\vec{\sigma},$$

   where $\vec{v}\cdot\vec{\sigma} = v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3$.

   (b) Please show that $\vec{v}\cdot\vec{\sigma}$ has eigenvalues ± 1, and that the projectors onto the corresponding eigenspaces are given by $P_\pm = (I \pm \vec{v}\cdot\vec{\sigma})/2$.

   (c) For the quantum measurement corresponding to the observable $\vec{v}\cdot\vec{\sigma}$, please calculate the probability of obtaining the result $+1$, given that the pre-measurement state is $|0\rangle$. What is the post-measurement state if $+1$ is obtained ?

   (d) Please show that the commutators are $[\sigma_j, \sigma_k] = 2i\sum_{l=1}^{3}\epsilon_{jkl}\sigma_l$, where $\epsilon_{jkl} = 0$ except for $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$ and $\epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1$.

   (e) Please show that the anti-commutators $\{\sigma_i, \sigma_j\} = 0$ for all $i \neq j$.

8. (a) Exercise 2.57 on Page 86 of the textbook.

   (b) Exercise 2.58 on Page 88 of the textbook.

9. (a) Consider a two-qubit system AB with state

$$|\psi\rangle = \frac{|00\rangle + |10\rangle + |11\rangle}{\sqrt{3}}.$$

   Please find the states ρ^A and ρ^B of qubits A and B, respectively.

   (b) Consider a three-qubit system ABC with state

$$|\psi\rangle = \frac{|000\rangle + |111\rangle}{\sqrt{2}}.$$

   Please find the states ρ^A, ρ^B and ρ^C of qubits A, B and C, respectively.

10. (a) Exercise 2.61 on Page 90 of the textbook.
    (b) Exercise 2.63 on Page 92 of the textbook.