EE 641000
Homework Assignment #1

Spring Semester, 2006

Due Date: March 14, 2006

We let $\mathcal{H}, \mathcal{H}_1, \mathcal{H}_2$ be finite-dimensional complex inner product spaces.

1. (a) Exercise 2.6 on Page 66 of the textbook.
   (b) Exercise 2.9 on Page 68 of the textbook.
   (c) Exercise 2.10 on Page 68 of the textbook.
   (d) Exercise 2.11 on Page 69 of the textbook.

2. (a) Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two finite-dimensional complex inner product spaces. For a linear transformation T from $\mathcal{H}_1$ to $\mathcal{H}_2$, prove that the adjoint $T^\dagger$ of T is a linear transformation from $\mathcal{H}_2$ to $\mathcal{H}_1$.
   (b) Let T_1, T_2 be in $L(\mathcal{H}_1, \mathcal{H}_2)$ and α, β in C. Prove $(\alpha T_1 + \beta T_2)^\dagger = \bar{\alpha} T_1^\dagger + \bar{\beta} T_2^\dagger$.

3. (a) Please show that P is a projector on a finite-dimensional complex inner product space $\mathcal{H}$ if and only if $P^\dagger = P$ and $P^2 = P$.
   (b) Let W be a subspace of $\mathcal{H}$. Let $\{|\phi_k\rangle\}$ be an orthonormal basis of W. Please show that the projector P_W onto W can be represented as

$$P_W = \sum_k |\phi_k\rangle\langle\phi_k|.$$

4. Please show that the identity operator I has

$$I = \sum_i |\psi_i\rangle\langle\psi_i|$$

   for any orthonormal basis $\{|\psi_i\rangle\}$ of $\mathcal{H}$.

5. Please show that a linear operator T on $\mathcal{H}$ is a normal operator if and only if there is an orthonormal basis $|\psi_i\rangle$ of $\mathcal{H}$ such that $T = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$, where λ_i are complex numbers.

6. Please show that U is a unitary operator on $\mathcal{H}$ if and only if U is a normal operator on $\mathcal{H}$ and eigenvalues of U have unit modulus.

7. Please show that T is a positive operator on $\mathcal{H}$ if and only if T is normal and has non-negative eigenvalues only.

8. Let T be a Hermitian operator on $\mathcal{H}$. Please show that the eigenspaces E_1 and E_2 corresponding to distinct eigenvalues λ_1 and λ_2 respectively are orthogonal, i.e., for any $|\psi_1\rangle \in E_1$ and any $|\psi_2\rangle \in E_2$, we have $\langle\psi_1|\psi_2\rangle = 0$.

9. Please show that if T and S are two commuting normal operators on $\mathcal{H}$, then $e^{T+S} = e^T e^S$.

10. Please use spectral decomposition to show that

    (a) If U is a unitary operator on $\mathcal{H}$, then $K = -i \log U$ is a Hermitian operator on $\mathcal{H}$.

    (b) If K is a Hermitian operator on $\mathcal{H}$, then $U = \exp(iK)$ is a unitary operator on $\mathcal{H}$.