Supplemental Materials for EE203001 Students

I. A Primer of Functions

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March 2, 2003

Let X and Y be two sets. A function f from X to Y, denoted as $f : X \to Y$, is a rule to assign *each* element x of X to *exactly one* element, denoted as $f(x)$, of Y. Different ways of assignment will result in different functions.

The sets X and Y are called the domain and the codomain of the function f, respectively. The assigned element $f(x)$ of Y to an element x of X is called the value of the function f at the element x, or sometimes the image of x under f. Thus, the evaluation of the function f at x is just the value $f(x)$ of Y assigned to the element x.

If Y is the set R of all real numbers, the value $f(x)$ of f at $x \in X$ is a real value. We then call the function $f : X \to R$ a real-valued function on the set X. Similarly, if Y is the set C of all complex numbers, the value $f(x)$ of f at $x \in X$ is complex and the function $f : X \to C$ is called a complex-valued function on X. In vector calculus, we encounter the codomain Y of a function f to be the set R^n of all n-tuples of real numbers, i.e., all real n-vectors. In this case, we call the function f a real-vector-valued function on X or a vector-valued function on X for simplicity. Complex-vector-valued functions are often considered in physics, where the codomain of such a function is the set C^n of all n-tuples of complex numbers, i.e., all complex n-vectors.

In this course, the domain X of a function f considered is often the set R of all real numbers or a subset of R. Such a function f is usually called a function of one real variable. More generally, if the domain X of a function f is the set R^n of all real n-tuples or a subset of R^n, then f is called a function of n real variables. Functions of one or several complex variables, where the domain is either C or C^n or their subsets, will not be considered in this course.

Two functions f and g from X to Y are said to be equal, denoted as $f = g$, if their rules of assignment are the same, i.e.,

$$f(x) = g(x), \ \forall \ x \in X.$$

Thus in studying the real linear space V of all real-valued functions on a set X, we define the sum $f+g$ of two real-valued functions f and g on X to be a real-valued function on X by specifying the value $(f+g)(x)$ of $f+g$ at each $x \in X$ as

$$(f+g)(x) = f(x) + g(x). \tag{1}$$

And we define the multiplication αf of a real-valued function f on X by a real number α to be a real-valued function on X by specifying the value $(\alpha f)(x)$ of αf at each $x \in X$ as

$$(\alpha f)(x) = \alpha f(x).$$

Furthermore, to prove the Associative Law (Axiom 4) for the real linear space V, i.e., $f + (g+h) = (f+g)+h$, for any f, g, h in V, is equivalent to prove

$$(f + (g+h))(x) = ((f+g)+h)(x),\ \forall\ x \in X,$$

which by repeatedly applying (1) can be expressed as

$$f(x) + (g(x) + h(x)) = (f(x) + g(x)) + h(x),\ \forall\ x \in X,$$

which is valid by the Associative Law of the addition of real numbers.

Let f be a function from X to Y. Let A be a subset of X and B a subset of Y. The set

$$f(A) = \{y \in Y \mid y = f(x) \text{ for some } x \in A\}$$

is a subset of Y and is called the image of A under f. The set

$$f^{-1}(B) = \{x \in X \mid f(x) \in B\}$$

is a subset of X and is called the inverse image of B under f. If B consists of a single element $y \in Y$, i.e., $B = \{y\}$, we commonly write $f^{-1}(y)$ in place of $f^{-1}(\{y\})$.

A function $f : X \to Y$ is said to be injective (or one-to-one) if $f(x) = f(x')$ for two elements x, x' in X, then $x = x'$. It is said to be surjective (or onto) if $f(X) = Y$, i.e., if the image of the domain is equal to the codomain. And it is said to be bijective (or a bijection or a one-to-one correspondence) if it is both injective and surjective.

Let $f : X \to Y$ and $g : Y \to Z$ be two functions. The composite $g \circ f$ of f and g is the function from X to Z defined as

$$(g \circ f)(x) = g(f(x)),\ \forall\ x \in X.$$

Let 1_X and 1_Y be the identity functions on X and on Y, respectively, i.e.

$$1_X(x) = x,\ \forall\ x \in X \text{ and } 1_Y(y) = y,\ \forall\ y \in Y.$$

Then for any function $f : X \to Y$, we have

$$f \circ 1_X = f = 1_Y \circ f.$$

If $h : Z \to W$ is a third function, it is easy to verify that $h \circ (g \circ f) = (h \circ g) \circ f$.

Theorem 1 *Let $f : X \to Y$ be a function.*

1. *f is injective if and only if there is a function $g : Y \to X$ such that $g \circ f = 1_X$.*

2. *f is surjective if and only if there is a function $h : Y \to X$ such that $f \circ h = 1_Y$.*

Proof. If f is injective, then for each y in the image $f(X)$ of X under f there is a unique x in X with $f(x) = y$. Choose an arbitrarily fixed x_0 in X and define a function $g : Y \to X$ as

$$g(y) = \begin{cases} x, & \text{if } y \in f(X) \text{ and } f(x) = y, \\ x_0, & \text{if } y \notin f(X). \end{cases}$$

Then we have

$$(g \circ f)(x) = g(f(x)) = g(y) = x = 1_X(x), \ \forall\, x \in X,$$

i.e., $g \circ f = 1_X$. Conversely if $g \circ f = 1_X$, then for $x, x' \in X$ with $f(x) = f(x')$ we have

$$x = 1_X(x) = (g \circ f)(x) = g(f(x)) = g(f(x')) = (g \circ f)(x') = 1_X(x') = x',$$

i.e., f is injective. Now if f is surjective, then for each y in the codomain Y of f the inverse image $f^{-1}(y) = \{x \in X \mid f(x) = y\}$ of the singleton $\{y\}$ is nonempty. We can define a function $h : Y \to X$ by choosing an element x_y in $f^{-1}(y)$ for each $y \in Y$. Then we have

$$(f \circ h)(y) = f(h(y)) = f(x_y) = y = 1_Y(y), \ \forall\, y \in Y,$$

i.e., $f \circ h = 1_Y$. Conversely if $f \circ h = 1_Y$, then for each $y \in Y$ we have

$$y = 1_Y(y) = (f \circ h)(y) = f(h(y)) = f(x_y) \in f(X),$$

i.e., $Y = f(X)$ and f is surjective. □

The function $g : Y \to X$ in the above theorem is called a left inverse of $f : X \to Y$ if f is injective and the function $h : Y \to X$ is called a right inverse of f if f is surjective. Thus from the above theorem, f is bijective if and only if f has both a left inverse g and a right inverse h and in this case, we have

$$g = g \circ 1_Y = g \circ (f \circ h) = (g \circ f) \circ h = 1_X \circ h = h,$$

i.e., any left inverse of f and any right inverse of f coincide. Such a function is called a two-sided inverse of f, which is unique and denoted as f^{-1}.

Remark 2 *In general, the inverse operation f^{-1} for an arbitrary function $f : X \to Y$ maps a subset B of the codomain Y to a subset $f^{-1}(B)$ of the domain X. But if f is a bijection, the operation f^{-1} maps a singleton of Y to a singleton of X and can be regarded as a function from Y to X such that*

$$f^{-1} \circ f = 1_X \text{ and } f \circ f^{-1} = 1_Y$$

as we have verified in above.