EE203001 Linear Algebra
Solutions to Quiz #9 04/29/2003

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1. (a).

$$\begin{cases} T(\boldsymbol{i}+2\boldsymbol{j}) &= 4\boldsymbol{i}+\boldsymbol{j} \\ T(\boldsymbol{j}) &= \boldsymbol{i}+\boldsymbol{j} \end{cases} \Rightarrow \begin{cases} T(\boldsymbol{i})+2T(\boldsymbol{j}) &= 4\boldsymbol{i}+\boldsymbol{j} \\ T(\boldsymbol{j}) &= \boldsymbol{i}+\boldsymbol{j} \end{cases} \Rightarrow \begin{cases} T(\boldsymbol{i}) &= 2\boldsymbol{i}-\boldsymbol{j} \\ T(\boldsymbol{j}) &= \boldsymbol{i}+\boldsymbol{j} \end{cases}$$

$T(4\boldsymbol{i}-3\boldsymbol{j}) = 4T(\boldsymbol{i}) - 3T(\boldsymbol{j}) = 4(2\boldsymbol{i}-\boldsymbol{j}) - 3(\boldsymbol{i}+\boldsymbol{j}) = 5\boldsymbol{i} - 7\boldsymbol{j}$ and
$T^2(4\boldsymbol{i}-3\boldsymbol{j}) = T(5\boldsymbol{i}-7\boldsymbol{j}) = 5T(\boldsymbol{i}) - 7T(\boldsymbol{j}) = 5(2\boldsymbol{i}-\boldsymbol{j}) - 7(\boldsymbol{i}+\boldsymbol{j}) = 3\boldsymbol{i} - 12\boldsymbol{j}$.

(b). The matrix of T is $\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$.

2. We first show that $N(T)$ is trivial.

$$\begin{aligned} v = \alpha\boldsymbol{i} + \beta\boldsymbol{j} \in N(T) &\Leftrightarrow T(v) = T(\alpha\boldsymbol{i}+\beta\boldsymbol{j}) = (0,0,0) \\ &\Leftrightarrow \alpha(1,1,0) + \beta(-1,0,1) = (0,0,0) \\ &\Leftrightarrow (\alpha-\beta, \alpha, \beta) = (0,0,0) \\ &\Leftrightarrow \alpha = \beta = 0, \end{aligned}$$

so $N(T) = \{O\}$.

Thus we choose $v_1 = \boldsymbol{i}$ and $v_2 = \boldsymbol{j}$ to form a basis (v_1, v_2) for $\mathbb{R}^2$. Then $w_1 = T(v_1) = T(\boldsymbol{i}) = (1,1,0)$ and $w_2 = T(v_2) = T(\boldsymbol{j}) = (-1,0,1)$ are linearly independent in $\mathbb{R}^3$. We add $w_3 = (0,0,1)$. Then $\{w_1, w_2, w_3\}$ is a basis for $\mathbb{R}^3$. Since $T(v_1) = w_1$ and $T(v_2) = w_2$, the matrix representation relative to these two new bases is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$.

3. $D(e^x) = e^x = 1 \cdot e^x + 0 \cdot xe^x$,
$D(xe^x) = e^x + xe^x = 1 \cdot e^x + 1 \cdot xe^x$,
$D^2(e^x) = D(e^x) = 1 \cdot e^x + 0 \cdot xe^x$,
$D^2(xe^x) = D(e^x + xe^x) = 1 \cdot e^x + e^x + xe^x = 2 \cdot e^x + 1 \cdot xe^x$.
Thus the matrix representations of D and D^2 relative to $\{e^x, xe^x\}$ are

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{respectively.}$$

4. Please refer to the solutions of Homework 10.