EE203001 Linear Algebra
Solutions to Quiz #7 Spring Semester, 2003

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1. To find the projection of x on S, we have to find an orthonormal basis for S, firstly. Applying the Gram-Schmidt process, we have

$$\begin{aligned}
y_1 &= x_1 = (1,0,1,0), \\
y_2 &= x_2 - \frac{(x_2,y_1)}{(y_1,y_1)}y_1 = (2,3,1,1) - \frac{3}{2}(1,0,1,0) = (\frac{1}{2},3,-\frac{1}{2},1), \\
y_3 &= x_3 - \frac{(x_3,y_1)}{(y_1,y_1)}y_1 - \frac{(x_3,y_2)}{(y_2,y_2)}y_2 \\
&= (1,-3,2,-1) - \frac{3}{2}(1,0,1,0) - \frac{-\frac{21}{2}}{\frac{21}{2}}(\frac{1}{2},3,-\frac{1}{2},1) \\
&= 0.
\end{aligned}$$

Thus $\{y_1, y_2\}$ is an orthogonal basis for S.

We normalize y_1 and y_2 to obtain an orthonormal basis $\{e_1, e_2\}$, where

$$e_1 = \frac{y_1}{||y_1||} = \frac{1}{\sqrt{2}}(1,0,1,0), \quad \text{and} \quad e_2 = \frac{y_2}{||y_2||} = \frac{\sqrt{2}}{\sqrt{21}}(\frac{1}{2},3,-\frac{1}{2},1).$$

Thus, the projection of x on S is

$$\begin{aligned}
p(x) &= (x,e_1)e_1 + (x,e_2)e_2 \\
&= (2,0,2,0) + (\frac{1}{21},\frac{6}{21},\frac{-1}{21},\frac{2}{21}) \\
&= \frac{1}{21}(43,6,41,2).
\end{aligned}$$

2. In Homework #7, Exercise 9, we know that the vectors

$$e_0(x) = \frac{1}{\sqrt{2\pi}}, \quad e_1(x) = \frac{\cos x}{\sqrt{\pi}}, \quad \text{and} \quad e_2(x) = \frac{\sin x}{\sqrt{\pi}}$$

forms an orthonormal basis for the subspace S spanned by $\{1, \sin x, \cos x\}$. And the projection $p_0(x)$ of x on the subspace is $\pi - 2\sin x$.

Since $1 \in S$, the projection $p_1(x)$ of 1 on S is just 1 itself.

Thus the projection $p(x)$ of $2x+1$ on S, i.e., the trigometric polynomial nearest to $f(x) = 2x+1$ is

$$2p_0(x) + p_1(x) = 2(\pi - 2\sin x) + 1.$$

Note. In a Euclidean space V, let S be a subspace with an orthonormal basis $\{e_1, \ldots, e_n\}$. If $x, y \in V$, p_x and p_y are the projections of x and y on a subspace S of V, respectively, then the projection of $ax+y$ on S is $ap_x + p_y$, since

$$\sum_{i=1}^{n}(ax+y,e_i)e_i = \sum_{i=1}^{n} a(x,e_i)e_i + (y,e_i)e_i = a\sum_{i=1}^{n}(x,e_i)e_i + \sum_{i=1}^{n}(y,e_i)e_i = ap_x + p_y.$$

3. Firstly, we need some integral results.

$$
\begin{aligned}
\int \sin \pi x \, dx &= \frac{-\cos \pi x}{\pi}, \\
\int x \sin \pi x \, dx &= x\frac{-\cos \pi x}{\pi} - \int \frac{-\cos \pi x}{\pi} \, dx \\
& \quad (u = x, dv = \sin \pi x \, dx \Rightarrow du = dx, v = \frac{-\cos \pi x}{\pi}) \\
&= -x\frac{\cos \pi x}{\pi} + \frac{\sin \pi x}{\pi^2} + C, \\
\int x \cos \pi x \, dx &= x\frac{\sin \pi x}{\pi} - \int \frac{\sin \pi x}{\pi} \, dx \\
& \quad (u = x, dv = \cos \pi x \, dx \Rightarrow du = dx, v = \frac{\sin \pi x}{\pi}) \\
&= x\frac{\sin \pi x}{\pi} + \frac{\cos \pi x}{\pi^2} + C, \\
\int x^2 \sin \pi x \, dx &= x^2\frac{-\cos \pi x}{\pi} - \int \frac{-\cos \pi x}{\pi} \cdot 2x \, dx \\
& \quad (u = x^2, dv = \sin \pi x \, dx \Rightarrow du = 2xdx, v = \frac{-\cos \pi x}{\pi}) \\
&= -x^2\frac{\cos \pi x}{\pi} + \frac{2}{\pi}\int \cos \pi x \cdot x dx \\
&= -x^2\frac{\cos \pi x}{\pi} + \frac{2}{\pi}(x\frac{\sin \pi x}{\pi} + \frac{\cos \pi x}{\pi^2}) + C,
\end{aligned}
$$

Then,

$$
\begin{aligned}
(f, 1) &= \int_{-1}^{1} \sin \pi x \, dx = \frac{-\cos \pi x}{\pi}|_{-1}^{1} = \frac{-\cos \pi}{\pi} + \frac{\cos -\pi}{\pi} = 0, \\
(f, x) &= \int_{-1}^{1} x \sin \pi x \, dx = (-x\frac{\cos \pi x}{\pi} + \frac{\sin \pi x}{\pi^2})\Big|_{-1}^{1} = \frac{1}{\pi}(-\cos \pi - \cos(-\pi)) = \frac{2}{\pi}, \\
(f, x^2) &= \int_{-1}^{1} x^2 \sin \pi x \, dx = (-x^2\frac{\cos \pi x}{\pi} + \frac{2}{\pi}(x\frac{\sin \pi x}{\pi} + \frac{\cos \pi x}{\pi^2}))\Big|_{-1}^{1} = 0.
\end{aligned}
$$

Thus

$$(f, \phi_0) = \sqrt{\frac{1}{2}}(f, 1) = \sqrt{\frac{1}{2}} \cdot 0 = 0,$$

$$(f, \phi_1) = \sqrt{\frac{3}{2}}(f, x) = \sqrt{\frac{3}{2}}\frac{2}{\pi},$$

$$(f, \phi_2) = \frac{1}{2}\sqrt{\frac{5}{2}}(f, 3x^2 - 1) = \frac{1}{2}\sqrt{\frac{5}{2}} \cdot (3(f, x^2) - (f, 1)) = 0,$$

Hence the quadratic polynomial nearest to $f(x) = \sin \pi x$ is

$$(f, \phi_0)\phi_0 + (f, \phi_1)\phi_1 + (f, \phi_2)\phi_2 = \sqrt{\frac{3}{2}}\frac{2}{\pi}\sqrt{\frac{3}{2}}x = \frac{3}{\pi}x.$$