EE203001 Linear Algebra
Solutions for Quiz #6 Spring Semester, 2003

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1. (a) $x_1 = (0,0,1,1)$, $x_2 = (0,1,1,0)$, $x_3 = (0,-1,0,1)$, $x_4 = (1,0,0,1)$. We find an orthogonal basis $\{y_1', y_2', y_3', y_4'\}$ first. Let $y_1' = (0,0,1,1)$ and then we can start by the Gram-Schmidt process.
$y_2' = x_2 - \frac{(x_2,y_1')}{(y_1',y_1')}y_1' = (0,1,1,0) - \frac{1}{2}(0,0,1,1) = \frac{1}{2}(0,2,1,-1)$.
$y_3' = x_3 - \frac{(x_3,y_2')}{(y_2',y_2')}y_2' - \frac{(x_3,y_1')}{(y_1',y_1')}y_1' = (0,-1,0,1) - \frac{1}{2}(0,2,1,-1) - \frac{1}{2}(0,0,1,1) = (0,0,0,0)$.
$y_4' = x_4 - \frac{(x_4,y_3')}{(y_3',y_3')}y_3' - \frac{(x_4,y_2')}{(y_2',y_2')}y_2' - \frac{(x_4,y_1')}{(y_1',y_1')}y_1' = (1,0,0,1) + \frac{1}{6}(0,2,1,-1) - \frac{1}{2}(0,0,1,1) = \frac{1}{3}(3,1,-1,1)$.
We normalize y_1', y_2' and y_3' next. Thus we get an orthonormal basis:
$y_1 = \frac{1}{\sqrt{2}}(0,0,1,1)$, $y_2 = \frac{1}{\sqrt{6}}(0,2,1,-1)$, $y_3 = \frac{1}{2\sqrt{3}}(3,1,-1,1)$.

(b) $x_1 = (1,2,-2,1)$, $x_2 = (1,0,2,1)$, $x_3 = (1,1,0,1)$. We find an orthogonal basis $\{y_1', y_2', y_3'\}$ first. Let $y_1' = (1,2,-2,1)$ and then we can start by the Gram-Schmidt process.
$y_2' = x_2 - \frac{(x_2,y_1')}{(y_1',y_1')}y_1' = (1,0,2,1) + \frac{1}{5}(1,2,-2,1) = \frac{1}{5}(3,1,4,3)$.
$y_3' = x_3 - \frac{(x_3,y_2')}{(y_2',y_2')}y_2' - \frac{(x_3,y_1')}{(y_1',y_1')}y_1' = (1,1,0,1) - \frac{1}{5}(3,1,4,3) - \frac{2}{5}(1,2,-2,1) = (0,0,0,0)$.
We normalize y_1', y_2' and y_3' next. Thus we get an orthonormal basis:
$y_1 = \frac{1}{\sqrt{10}}(1,2,-2,1)$, $y_2 = \frac{1}{\sqrt{35}}(3,1,4,3)$.

2. Since x_1, $x_2 \cdots$, x_i are independent, $0 \le i \le k$, $dim(L(x_1, x_2, \cdots, x_i)) = i$.
Since y_1, $y_2 \cdots$, y_j are orthogonal, $0 \le j \le k$, $dim(L(y_1, y_2, \cdots, y_j)) = j$.
$(\Rightarrow)$
If $y_1, y_2 \cdots, y_k$ are nonzero, then $dim(L(y_1, y_2, \cdots, y_k)) = k$. Since $L(x_1, x_2, \cdots, x_k) = L(y_1, y_2, \cdots, y_k)$, $dim(L(x_1, x_2, \cdots, x_k)) = k$. Thus $x_1, x_2, \cdots, x_k$ are independent.
$(\Leftarrow)$
If $x_1, x_2 \cdots, x_k$ are independent, then $dim(L(x_1, x_2, \cdots, x_k)) = k$. Since $L(x_1, x_2, \cdots, x_k) = L(y_1, y_2, \cdots, y_k)$, $dim(L(y_1, y_2, \cdots, y_k)) = k$. Thus $y_1, y_2, \cdots, y_k$ are nonzero.

3. In the linear space of all real polynomials, let $\{z_0, z_1, z_2\}$ be the orthogonal basis for the subspace $L(\{x_0, x_1, x_2\})$, obtained by the Gram-Schmidt process. Then

$$z_0(t) = x_0(t) = 1.$$

Since

$$(z_0, z_0) = \int_0^1 1\,dt = t|_0^1 = 1 \quad \text{and} \quad (x_1, z_0) = \int_0^1 t\,dt = \left.\frac{t^2}{2}\right|_0^1 = \frac{1}{2},$$

we find that

$$z_1(t) = x_1(t) - \frac{(x_1, z_0)}{(z_0, z_0)}z_0(t) = x_1(t) - \frac{1}{2}z_0(t) = t - \frac{1}{2}.$$

Next, we use the relations

$$(x_2, z_0) = \int_0^1 t^2\,dt = \left.\frac{t^3}{3}\right|_0^1 = \frac{1}{3},$$

$$(x_2, z_1) = \int_0^1 t^2(t - \frac{1}{2})\,dt = \int_0^1 (t^3 - \frac{t^2}{2})\,dt = (\frac{t^4}{4} - \frac{t^3}{6})\Big|_0^1 = \frac{1}{12},$$

$$(z_1, z_1) = \int_0^1 (t - \frac{1}{2})^2\,dt = \int_0^1 t^2 - t + \frac{1}{4}\,dt = (\frac{t^3}{3} - \frac{t^2}{2} + \frac{t}{4})\Big|_0^1 = \frac{1}{3} - \frac{1}{2} + \frac{1}{4} = \frac{1}{12},$$

to obtain

$$\begin{aligned}
z_2(t) &= x_2(t) - \frac{(x_2, z_0)}{(z_0, z_0)} z_0(t) - \frac{(x_2, z_1)}{(z_1, z_1)} z_1(t) \\
&= x_2(t) - \frac{1}{3} \cdot 1 \cdot z_0(t) - \frac{1}{12} \cdot 12 \cdot z_1(t) \\
&= t^2 - \frac{1}{3} - (t - \frac{1}{2}) \\
&= t^2 - t + \frac{1}{6}.
\end{aligned}$$

Since

$$\begin{aligned}
(z_2, z_2) &= \int_0^1 (t^2 - t + \frac{1}{6})^2\,dt \\
&= \int_0^1 (t^4 - 2t^3 + \frac{4}{3}t^2 - \frac{1}{3}t + \frac{1}{36})\,dt \\
&= (\frac{t^5}{5} - \frac{t^4}{2} + \frac{4}{9}t^3 - \frac{t^2}{6} + \frac{t}{36})\Big|_0^1 \\
&= \frac{1}{5} - \frac{1}{2} + \frac{4}{9} - \frac{1}{6} + \frac{1}{36} \\
&= \frac{1}{180},
\end{aligned}$$

we normalize z_0, z_1 and z_2 to obtain an orthonormal basis $\{y_0, y_1, y_2\}$, where

$$\begin{aligned}
y_0(t) &= \frac{z_0(t)}{||z_0||} = 1, \\
y_1(t) &= \frac{z_1(t)}{||z_1||} = \sqrt{12}(t - \frac{1}{2}) = \sqrt{3}(2t - 1), \\
y_2(t) &= \frac{z_2(t)}{||z_2||} = \sqrt{180}(t^2 - t + \frac{1}{6}) = 6\sqrt{5}(t^2 - t + \frac{1}{6}) = \sqrt{5}(6t^2 - 6t + 1).
\end{aligned}$$