EE203001 Linear Algebra
Solutions to Homework #5 Spring Semester, 2003

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2. (a) Given $(f,g) = \int_1^e (\log(x)) f(x) g(x) dx$. If $f(x) = \sqrt{x}$, then

$$\begin{aligned}\|f\| &= (f,f)^{1/2}\\ &= (\int_1^e (\log x) f(x) f(x) dx)^{1/2}\\ &= (\int_1^e (\log x) x dx)^{1/2}.\end{aligned}$$

Let $u = \log x$, then $du = \frac{1}{x}dx$. Let $v = \frac{x^2}{2}$, then $dv = xdx$. Thus

$$\begin{aligned}(\int (\log x) x dx)^{1/2} &= (\int u dv)^{1/2}\\ &= (uv - \int v du)^{1/2}\\ &= (\log x \frac{x^2}{2} - \int \frac{x^2}{2}\frac{1}{x} dx)^{1/2}.\end{aligned}$$

So

$$\begin{aligned}(\int_1^e (\log x) x dx)^{1/2} &= (\log x \frac{x^2}{2}|_1^e - \int_1^e \frac{x^2}{2}\frac{1}{x} dx)^{1/2}\\ &= (\frac{e^2}{2} - \frac{x^2}{4}|_1^e)^{1/2}\\ &= (\frac{e^2}{2} - \frac{e^2-1}{4})^{1/2}\\ &= (\frac{e^2+1}{4})^{1/2}\\ &= \frac{1}{2}\sqrt{e^2+1}.\end{aligned}$$

Hence, $\|f\|^2 = \frac{1}{4}(e^2+1)$.

(b) Let $u = x^2 \log x$, then $du = (2x\log x + x)dx$. Let $v = -x^{-1}$, then $dv = \frac{1}{x^2}dx$. Thus

$$\begin{aligned}\int \log x dx &= \int (\frac{1}{x^2}) x^2 \log x dx\\ &= \int u dv\\ &= uv - \int v du\\ &= -x\log x + \int \frac{1}{x}(2x\log x + x) dx.\end{aligned}$$

So

$$
\begin{aligned}
\int_1^e \log x dx &= -x\log x|_1^e + \int_1^e \frac{1}{x}(2x\log x + x)dx \\
&= -e + \int_1^e 2\log x dx + \int_1^e dx \\
&= -e + \int_1^e 2\log x dx + e - 1 \\
&= \int_1^e 2\log x dx - 1.
\end{aligned}
$$

Thus, $\int_1^e \log x dx = 1$

(c) Now we want to find a linear polynomial $g(x) = a + bx$ nonzero and orthogonal to $f(x) = 1$, i.e., $(f, g) = 0$. Since

$$
\begin{aligned}
(f, g) &= \int_1^e \log x(a + bx)dx \\
&= a\int_1^e \log x dx + b\int_1^e x\log x dx \\
&= a + b(\frac{e^2+1}{4}) \qquad \text{(by (a))},
\end{aligned}
$$

we have $(f, g) = 0$ when $a = -b(\frac{e^2+1}{4})$. So $g(x) = b(x - \frac{e^2+1}{4}), b$ is an arbitrary real number.