EE203001 Linear Algebra
Solutions to Quiz #4 Spring Semester, 2003

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1. Let $T = \{1+t, (1+t)^2, (1+t)^3\}$ and $S = \{a(1+t)+b(1+t)^2+c(1+t)^3 | a, b, c \in \mathbb{R}\}$. S is a subspace of the linear space of all real polynomials and $T \subset S$, so $L(T) \subset S$, by Exercise 25(b). On the other hand, by the definition of $L(T)$, $S \subset L(T)$. Thus $L(T) = S$.

   If $a(1+t)+b(1+t)^2+c(1+t)^3 = 0$, divide $a(1+t)+b(1+t)^2+c(1+t)^3 = 0$ by $1+t$, then put $t = -1$, we have $a = 0$. Applying the same process, we find $b = c = 0$. Thus T is independent, hence $\dim L(T) = 3$.

2. (a). (i). For f, $g \in P_1$,

$$(f, g) = f(0)g(0) + f(1)g(1) = g(0)f(0) + g(1)f(1) = (g, f),$$

thus the symmetry axiom holds.

(ii). For f, g and $h \in P_1$,

$$\begin{aligned}(f, g+h) &= f(0)(g+h)(0) + f(1)(g+h)(1)\\ &= f(0)(g(0)+h(0)) + f(1)(g(1)+h(1))\\ &= f(0)g(0) + f(0)h(0) + f(1)g(1) + f(1)h(1)\\ &= f(0)g(0) + f(1)g(1) + f(0)h(0) + f(1)h(1)\\ &= (f, g) + (f, h),\end{aligned}$$

thus the linearity axiom holds.

(iii). For f, $g \in P_1$, and $c \in \mathbb{R}$,

$$\begin{aligned}c(f, g) &= c[f(0)g(0) + f(1)h(1)]\\ &= cf(0)g(0) + cf(1)g(1)\\ &= [cf(0)]g(0) + [cf(1)]g(1)\\ &= (cf, g),\end{aligned}$$

thus the homogeneity axiom holds.

(iv). If f is a non-zero polynomial of degree 1 in P_1, then f has at most one real root. Thus at least one of $f(0)$ and $f(1)$ is non-zero. Thus we have

$$(f, f) = [f(0)f(0) + f(1)f(1)] = f(0)^2 + f(1)^2 > 0.$$

Thus the positivity axiom holds.

(b). If $f(t) = t$ and $g(t) = at + b$, then

$$(f, g) = f(0)g(0) + f(1)g(1) = 0 \cdot b + 1 \cdot (a+b) = a + b.$$

(c). By (b), $f^{\perp} = \{at + b | a + b = 0; a, b \in \mathbb{R}\}$.

(i). Given a_1t+b_1 and a_2t+b_2 in $f^\perp$, then

$$(a_1t+b_1)+(a_2t+b_2)=(a_1+a_2)t+(b_1+b_2).$$

Since

$$(a_1+a_2)+(b_1+b_2)=(a_1+b_1)+(a_2+b_2)=0+0=0,$$

$(a_1t+b_1)+(a_2t+b_2)\in f^\perp$.

(ii). If $c\in\mathbb{R}$ and $at+b\in f^\perp$, then $c(at+b)=(ca)t+cb$ and $ca+cb=c(a+b)=c\cdot 0=0$. Thus $c(at+b)\in f^\perp$.

By (i) and (ii), $f^\perp$ satisfies the closure axioms, thus $f^\perp$ is a subspace of P_1. Since the linear polynomial $at+b$ such that $a+b=0$ can be written as $at+(-a)=a(t-1)$, we have $f^\perp=\{a(t-1)|a\in\mathbb{R}\}$. It is clear that $\{t-1\}$ is a basis for $f^\perp$, hence $\dim f^\perp=1$.

3. (a). Firstly, we simplify this product:

$$\begin{aligned}(x,y) &= \sum_{i=1}^{n}(x_i+y_i)^2-\sum_{i=1}^{n}x_i^2-\sum_{i=1}^{n}y_i^2\\ &= \sum_{i=1}^{n}[(y_i+x_i)^2-x_i^2-y_i^2]\\ &= \sum_{i=1}^{n}2x_iy_i.\end{aligned}$$

Then, for all choices of x, y, z in $\mathbb{R}^n$ and all real scalars c:

(i).

$$(x,y)=\sum_{i=1}^{n}2x_iy_i=\sum_{i=1}^{n}2y_ix_i=(y,x).$$

(ii).

$$\begin{aligned}(x,y+z) &= \sum_{i=1}^{n}2x_i(y_i+z_i)\\ &= \sum_{i=1}^{n}(2x_iy_i)+(2x_iz_i)\\ &= \sum_{i=1}^{n}2x_iy_i+\sum_{i=1}^{n}2x_iz_i\\ &= (x,y)+(x,z).\end{aligned}$$

(iii).

$$c(x,y)=c\sum_{i=1}^{n}2x_iy_i=\sum_{i=1}^{n}2(cx_i)y_i=(cx,y).$$

(iv). If $x\neq O$, then

$$(x,x)=\sum_{i=1}^{n}2x_ix_i=\sum_{i=1}^{n}2x_i^2>0.$$

By (i), (ii), (iii) and (iv), (x, y) is an inner product.

(b). $(x, y) = \sum_{i=1}^{n}(x_i + y_i)^2 - \sum_{i=1}^{n} x_i y_i = \sum_{i=1}^{n}(x_i^2 + y_i^2 + x_i y_i)$.

(i). Let $x = (1, 1, \ldots, 1)$, $y = z = O = (0, 0, \ldots, 0)$ be elements in $\mathbb{R}^n$. $(x, y+z) = (x, O + O) = (x, O) = \sum_{i=1}^{n}(1^2 + 0^2 + 1 \cdot 0) = n$, but $(x, y) + (x, z) = (x, O) + (x, O) = 2\sum_{i=1}^{n} 1 = 2n$. Thus the linearity axiom fails to hold.

(ii). If $x = (1, 1, \ldots, 1)$ and $c = 2$, then $c(x, x) = c\sum_{i=1}^{n}(1^2 + 1^2 + 1 \cdot 1) = 2 \cdot 3n$. But $(cx, x) = \sum_{i=1}^{n}(2^2 + 1^2 + 2 \cdot 1) = 7n$, the homogeneity axiom fails to hold.