EE203001 Linear Algebra
Solutions for Quiz-Solution #11 Spring Semester, 2003

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1. (a) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$. A and B are orthogonal. Then $A+B = \begin{bmatrix} 1+cos\theta & -sin\theta \\ sin\theta & 1+cos\theta \end{bmatrix}$.

$\Rightarrow (A+B)(A+B)^t = \begin{bmatrix} 1+cos\theta & -sin\theta \\ sin\theta & 1+cos\theta \end{bmatrix} \begin{bmatrix} 1+cos\theta & sin\theta \\ -sin\theta & 1+cos\theta \end{bmatrix} = \begin{bmatrix} 2+2cos\theta & 0 \\ 0 & 2+2cos\theta \end{bmatrix}$. Since $(A+B)(A+B)^t \neq I$, $A+B$ is not orthogonal.

(b) First, we want to know what dose $(AB)^t$ look like. Let $C = AB$. Then $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$. Then $c^t_{ij} = c_{ji} = \sum_{k=1}^{n} a_{jk}b_{ki} = \sum_{k=1}^{n} b^t_{ik}a^t_{kj}$. Thus we get $(AB)^t = B^tA^t$. Since A and B are orthogonal,
$\Rightarrow (AB)(AB)^t = (AB)(B^tA^t) = A(BB^t)A^t = AIA^t = AA^t = I$.
Thus AB is orthogonal.

(c) $\Rightarrow (AB)(AB)^t = I$.(Since AB is orthogonal.)
$\Rightarrow ABB^tA^t = I$.
$\Rightarrow A^tABB^tA^t = A^t$.
$\Rightarrow BB^tA^t = A^t$.(Since A is orthogonal.)
$\Rightarrow BB^t = I$.
Thus B is orthogonal.

2. We have to find a nonsigular matrix P such that $AP = P\begin{bmatrix} 6 & 0 \\ 0 & -1 \end{bmatrix}$. Let $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then we have $\begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 6 & 0 \\ 0 & -1 \end{bmatrix}$. Thus,

$$a + 2c = 6a, b + 2d = -b, 5a + 4c = 6c, 5b + 4d = -d.$$

$\Rightarrow 5a = 2c, b = -d$. Choose $a = 2, b = 1, c = 5$, and $d = -1$, then $ad - bc = -2 - 5 = -7 \neq 0$. Thus $P = \begin{bmatrix} 2 & 1 \\ 5 & -1 \end{bmatrix}$ is nonsigular and hence it is a solution.

3. We prove it by induction.

(a) When $n = 1$, $(A + I)^1 = A + I = I + A = I + (2^1 - 1)A$.

(b) For $n = k$, we assume $(A + I)^k = I + (2^k - 1)A$.

(c) When $n = k + 1$, we have

$$\begin{aligned} (A+I)^{k+1} &= (A+I)(A+I)^k \\ &= (A+I)(I+(2^k-1)A) \\ &= A + I + (2^k-1)A^2 + 2^kA - A \\ &= I + (2^k-1)A + 2^kA \\ &= I + (2^{k+1}-1)A \end{aligned}$$