EE203001 Linear Algebra
Solutions for Homework #9 Spring Semester, 2003

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2. Let $V = \{0, 1, 2\}$. There are six functions $T_1, T_2, T_3, T_4, T_5, T_6 : V \to V$ for which $T(V) = V$. Hence

	0	1	2
$T_1(\bullet)$	0	1	2
$T_2(\bullet)$	0	2	1
$T_3(\bullet)$	1	0	2
$T_4(\bullet)$	1	2	0
$T_5(\bullet)$	2	0	1
$T_6(\bullet)$	2	1	0

and each of $T_1, T_2, T_3, T_4, T_5, T_6$ is a one-to-one function on V.
The composition of each pair of functions is listed below.

$T_{row}T_{col}$	T_1	T_2	T_3	T_4	T_5	T_6
T_1	T_1	T_2	T_3	T_4	T_5	T_6
T_2	T_2	T_1	T_5	T_6	T_3	T_4
T_3	T_3	T_4	T_1	T_2	T_6	T_5
T_4	T_4	T_3	T_6	T_5	T_1	T_2
T_5	T_5	T_6	T_2	T_1	T_4	T_3
T_6	T_6	T_5	T_4	T_3	T_2	T_1

Since $T_1(v) = v$ for all element $v \in V$, T_1 is the identity mapping I. By the table above, we know that $T_1T_1 = T_1 = I$, $T_2T_2 = T_1 = I$, $T_3T_3 = T_1 = I$, $T_4T_5 = T_1 = I$, $T_5T_4 = T_1 = I$, and $T_6T_6 = T_1 = I$. Hence $T_1^{-1} = T_1$, $T_2^{-1} = T_2$, $T_3^{-1} = T_3$, $T_4^{-1} = T_5$, $T_5^{-1} = T_4$, and $T_6^{-1} = T_6$.

14. No, because $T(x_1, y_1, z_1) = T(x_1, y_1, z_2) = (x_1, y_1, 0)$ for arbitrary real number x_1, y_1, z_1 and z_2, even though $z_1 \neq z_2$.

19. $T(x, y, z) = (x, x + y, x + y + z)$.
Let (x_1, y_1, z_1), (x_2, y_2, z_2) be in R^3 such that $T(x_1, y_1, z_1) = T(x_2, y_2, z_2)$, *i.e.*, $(x_1, x_1 + y_1, x_1 + y_1 + z_1) = (x_2, x_2 + y_2, x_2 + y_2 + z_2)$. Thus we have $x_1 = x_2$, $y_1 = y_2$, $z_1 = z_2$ which says that T is one-to-one.

22. We prove it by induction.

(a) When $n = 0$, we have $(ST)^0 = I = I \cdot I = S^0T^0$.

(b) Assume for $n = k$, $(ST)^k = S^kT^k$.

(c) when $n = k+1$, $(ST)^{k+1} = (ST)^k(ST) = S^kT^kST = S^kT^{k-1}(TS)T = S^kT^{k-1}(ST)T = S^kT^{k-1}ST^2 = S^kT^{k-2}ST^3 = \ldots = S^kTST^k = S^{k+1}T^{k+1}$.

Thus, the proof is completed.

25. Let V be a linear space. Let S and T denote functions with domain V and values in V. If S and T commute, then

$$\begin{aligned}(S+T)^2 &= (S+T)(S+T)\\ &= SS+ST+TS+TT\\ &= S^2+ST+ST+T^2\\ &= S^2+2ST+T^2, \text{ and}\end{aligned}$$

$$\begin{aligned}(S+T)^3 &= (S+T)(S+T)^2\\ &= (S+T)(S^2+2ST+T^2)\\ &= S^3+2SST+ST^2+TS^2+2TST+T^3\\ &= S^3+2S^2T+ST^2+S^2T+2ST^2+T^3\\ &= S^3+3S^2T+3ST^2+T^3.\end{aligned}$$

If $ST \neq TS$, then

$$\begin{aligned}(S+T)^2 &= (S+T)(S+T)\\ &= SS+ST+TS+TT\\ &= S^2+ST+TS+T^2, \text{ and}\end{aligned}$$

$$\begin{aligned}(S+T)^3 &= (S+T)(S+T)^2\\ &= (S+T)(SS+ST+TS+TT)\\ &= SSS+SST+STS+STT+TSS+TST+TTS+TTT\\ &= S^3+S^2T+STS+ST^2+TS^2+TST+T^2S+T^3.\end{aligned}$$

26. We prove it by induction.

(a) When $n=1$, we have $ST-TS=I=1I=1T^0=1T^{1-1}$ as given.

(b) Assume for $n=k$, $ST^k-T^kS=kT^{k-1}$.

(c) When $n=k+1$, $ST^{k+1}-T^{k+1}S=ST^kT-T^{k+1}S=(kT^{k-1}+T^kS)T-T^{k+1}S=kT^k+T^kST-T^{k+1}S=kT^k+T^k(ST-TS)=kT^k+T^k=(k+1)T^k$.

Thus, the proof is completed.

28. Given $p(x)=c_0+c_1x+\cdots+c_nx^n \in V$, according to the definitions of R, S, and T,

$$R(p(x))=c_0, \quad S(p(x))=c_1+c_2x+\cdots+c_nx^{n-1}, \quad T(p(x))=c_0x+c_1x^2+\cdots+c_nx^{n+1},$$

we have

$$ST=I, \quad TS=I-R,$$

since

$$ST(p(x)) = S(c_0x + c_1x^2 + \cdots + c_nx^{n+1}) = c_0 + c_1x + \cdots + c_nx^n = p(x)$$

and

$$TS(p(x)) = T(c_1 + c_2x + \cdots + c_nx^{n-1}) = c_1x + c_2x^2 \cdots + c_nx^n = p(x) - c_0.$$

(a). Let $p(x) = 2 + 3x - x^2 + x^3$.

$$\begin{aligned}
R(p(x)) &= 2, \\
S(p(x)) &= 3 - x + x^2, \\
T(p(x)) &= 2x + 3x^2 - x^3 + x^4, \\
ST(p(x)) &= p(x), \text{ since } ST = I, \\
TS(p(x)) &= 3x - x^2 + x^3, \text{ since } TS = I - R, \\
(TS)^2(p(x)) &= 3x - x^2 + x^3, \text{ since } (TS)^2 = (TS)(TS) = T(ST)S = TIS = TS, \\
T^2S^2(p(x)) &= T^2S(3 - x + x^2) = T^2(-1 + x) = T(-x + x^2) = -x^2 + x^3, \\
S^2T^2(p(x)) &= p(x), \text{ since } S^2T^2 = (SS)(TT) = S(ST)T = SIT = ST = I, \\
TRS(p(x)) &= TR(3 - x + x^2) = T(3) = 3x, \\
RST(p(x)) &= R(p(x)) = 2, \text{ since } RST = R(ST) = RI = R.
\end{aligned}$$

(b). Given $p(x) = c_0 + c_1x + \cdots + c_nx^n, q(x) = d_0 + d_1x + \cdots + d_mx^m \in V$, and $a, b \in \mathbb{R}$. Without loss of generality, we assume $n \leq m$. Since

$$\begin{aligned}
R(ap(x) + b(q(x)) &= R(ac_0 + bd_0 + (ac_1 + bd_1)x + \cdots + (ac_n + bd_n)x^n + \cdots + bd_mx^m) \\
&= ac_0 + bd_0 \\
&= aR(p(x)) + bR(q(x)), \\
S(ap(x) + b(q(x)) &= S(ac_0 + bd_0 + (ac_1 + bd_1)x + \cdots (ac_n + bd_n)x^n + \cdots + bd_mx^m) \\
&= (ac_1 + bd_1) + \cdots + (ac_n + bd_n)x^{n-1} + \cdots + bd_mx^{m-1}) \\
&= a(c_1 + \cdots + c_nx^{n-1}) + b(d_1 + \cdots + bd_nx^{n-1} + \cdots + d_mx^{m-1}) \\
&= aS(p(x)) + bS((q(x)), \\
T(ap(x) + b(q(x)) &= T(ac_0 + bd_0 + (ac_1 + bd_1)x + \cdots (ac_n + bd_n)x^n + \cdots + bd_mx^m) \\
&= (ac_1 + bd_1)x + \cdots + (ac_n + bd_n)x^{n+1} + \cdots + bd_mx^{m+1}) \\
&= a(c_1x + \cdots + c_nx^{n+1}) + b(d_1x + \cdots + bd_nx^{n+1} + \cdots + d_mx^{m+1}) \\
&= aS(p(x)) + bS(q(x)),
\end{aligned}$$

R, S, T are all linear transformations.

i-1.

$$p(x) \in N(R) \quad \Leftrightarrow \quad R(p(x)) = p(0) = c_0 = 0.$$

Thus $N(R)$ is the set of all real polynomials without constant term.

i-2. Since $R(p(x)) = p(0) = c_0$, $R(V) = \mathbb{R}$.

ii-1.

$$\begin{aligned} p(x)\in N(S) &\Leftrightarrow S(p(x)) = c_1 + c_2x + \cdots + c_nx^{n-1} = 0, \forall x, \\ &\Leftrightarrow c_1 = \cdots = c_n = 0. \end{aligned}$$

Thus $N(S)$ is the set of all real constant polynomials, i.e., $N(S) = \mathbb{R}$.

ii-2. Given any $p(x) = c_0 + c_1x + \cdots + c_nx^n$ in V, then $q(x) = xp(x) = c_0x + c_1x^2 + \cdots + c_nx^{n+1}$ is in V and satisfies $S(q(x)) = p(x)$. Thus $S(V) = V$.

iii-1.

$$\begin{aligned} p(x)\in N(T) &\Leftrightarrow T(p(x)) = c_0x + c_1x^2 + c_2x^3 + \cdots + c_nx^{n+1} = 0, \forall x. \\ &\Leftrightarrow c_0 = c_1 = \cdots = c_n = 0. \end{aligned}$$

Thus $N(T) = \{O\}$.

iii-2. It is clear that T maps a real polynomial to a polynomial without constant term. On the other hand, given any polynomial $p(x) = c_1x + \cdots + c_nx^n$ without constant term, we may find a $q(x) = c_1 + c_2x + \cdots + c_nx^{n-1} \in V$ such that $T(q(x)) = p(x)$. Thus $T(V) = V \setminus \mathbb{R}$.

(c). If $T(p(x)) = O$, then $c_k = 0$ for $k = 0, \ldots, n$. Thus $N(T) = \{O\}$, i.e. T is one-to-one.

(d). $(TS)^n = \overbrace{(TS)(TS)\cdots(TS)}^{n \text{ terms}} = T\,\overbrace{(ST)\cdots(ST)}^{n-1 \text{ terms}}\,S = (TI)S = TS = I - R.$

$S^nT^n = \overbrace{S\cdots S}^{n \text{ terms}}\ \overbrace{T\cdots T}^{n \text{ terms}} = \overbrace{S\cdots S}^{n-1 \text{ terms}}\ (ST)\ \overbrace{T\cdots T}^{n-1 \text{ terms}} = \overbrace{S\cdots S}^{n-1 \text{ terms}}\ I\ \overbrace{T\cdots T}^{n-1 \text{ terms}} = \cdots = I.$

29. (a). $p(x) = 2 + 3x - x^2 + 4x^3$.

$$\begin{aligned} D(p) &= 3 - 2x + 12x^2. \\ T(p) &= x(3 - 2x + 12x^2) = 3x - 2x^2 + 12x^3. \\ DT(p) &= D(3x - 2x^2 + 12x^3) = 3 - 4x + 36x^2. \\ TD(p) &= T(3 - 2x + 12x^2) = x(-2 + 24x) = -2x + 24x^2. \\ (DT - TD)(p) &= (3 - 4x + 36x^2) - (-2x + 24x^2) = 3 - 2x + 12x^2. \\ T^2D^2(p) &= T^2D(3 - 2x + 12x^2) = T^2(-2 + 24x) = T(x\cdot 24) = 24x. \\ D^2T^2(p) &= D^2T(3x - 2x^2 + 12x^3) = D^2(x(3 - 4x + 36x^2)) \\ &= D(3 - 8x + 72x^2) = -8 + 144x. \\ (T^2D^2 - D^2T^2)(p) &= 24x - (-8 + 144x) = 8 - 192x. \end{aligned}$$

(b). Given $p(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n$,

$$\begin{aligned} & T(p) = p, \\ \Leftrightarrow\ & c_1x + 2c_2x^2 + \cdots + nc_nx^n = c_0 + c_1x + \cdots + c_nx^n, \\ \Leftrightarrow\ & c_0 = 0, c_1 = c_1, c_2 = 2c_2, \ldots, c_n = nc_n, \\ \Leftrightarrow\ & p \text{ is a linear polynomial without constant term.} \end{aligned}$$

(c). Given $p(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n$,

$$\begin{aligned} DT(p) &= D(c_1x + 2c_2x^2 + \cdots + nc_nx^n) = c_1 + 2^2c_2x + \cdots + n^2c_nx^{n-1}, \\ 2D(p) &= 2(c_1 + 2c_2x + \cdots + nc_nx^{n-1}), \\ (DT - 2D)(p) &= DT(p) - 2D(p) \\ &= c_1 + 2^2c_2x + \cdots + n^2c_nx^{n-1} - 2(c_1 + 2c_2x + \cdots + nc_nx^{n-1}) \\ &= (1-2)c_1 + (2^2 - 2\cdot 2)c_2x + \cdots + (n^2 - 2n)c_nx^{n-1}. \end{aligned}$$

Thus $(DT - 2D)(p) = O$ if and only if $(k^2 - 2k)c_k = 0$, for $1 \le k \le n$. Thus, if $c_k \neq 0$, then $k = 0$ or $k = 2$. Hence $p(x) = a + bx^2$ for $a, b \in \mathbb{R}$.

(d). Given $p(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n$. Because,

$$\begin{aligned} DT(p) &= D(c_1x + 2c_2x^2 + \cdots + nc_nx^n) = c_1 + 2^2c_2x + \cdots + n^2c_nx^{n-1}, \\ TD(p) &= T(c_1 + 2c_2x + \cdots + nc_nx^{n-1}) = x(2c_2 + \cdots + n(n-1)c_nx^{n-2}) \\ &= 2c_2x + \cdots + n(n-1)c_nx^{n-1}, \end{aligned}$$

we have

$$\begin{aligned} (DT - TD)(p) &= c_1 + 2^2c_2x + \cdots + n^2c_nx^{n-1} - (2c_2x + \cdots + n(n-1)c_nx^{n-1}) \\ &= c_1 + 2c_2x + \cdots + nc_nx^{n-1} \\ &= D(p). \end{aligned}$$

Thus $DT - TD = D$, and hence $(DT - TD)^n = D^n$. Thus any $p \in V$ will satisfy $(DT - TD)^n(p) = D^n(p)$.

30. (a) Since $DT(p(x)) - TD(p(x)) = D(xp(x)) - T(p'(x)) = (p(x) + xp'(x)) - xp'(x) = p(x)$, we have $DT - TD = I$.

(b) We prove it by induction.

i. When $n = 2$, we have $DT^2(p(x)) - T^2D(p(x)) = DT(xp(x)) - T^2(p'(x)) = D(x^2p(x)) - T(xp'(x)) = (2xp(x) + x^2p'(x)) - (x^2p'(x)) = 2xp(x) = 2T(p(x))$.

ii. Assume for $n = k$, $DT^k - T^kD = kT^{k-1}$.

iii. When $n = k+1$, we have

$$\begin{aligned} & DT^{k+1}(p(x)) - T^{k+1}D(p(x)) \\ =\ & DT^k(xp(x)) - T^{k+1}(p'(x)) \\ =\ & (kT^{k-1}(xp(x)) + T^kD(xp(x))) - T^k(xp'(x)) \\ =\ & kT^k(p(x)) + T^k(p(x) + xp'(x)) - T^k(xp'(x)) \\ =\ & kT^k(p(x)) + T^k(p(x)) + T^k(xp'(x)) - T^k(xp'(x)) (\text{because } T^k \text{ is linear}) \\ =\ & (k+1)T^k(p(x)) \end{aligned}$$

Thus, the proof is completed.

31. Since $DT(p(x)) = Dq(x) = D\int_0^x p(t)dt = p(x)$, $DT = I_V$.
We redefine $p(x) = a(x) + b$, for $a(0) = 0$.
$\Rightarrow TD(p(x)) = TD(a(x) + b) = T(a'(x)) = \int_0^x a'(t)dt = a(x) \neq p(x)$, if $b \neq 0$.
Thus $TD \neq I_V$.

Let $TD(p(x)) = 0 \Rightarrow a(x) = 0$.
It's equal to say that $p(x)$ is a constant polynomial.
Thus the null space of TD is L(1).
And it's range is the space of all real polynomials excluding the constant polynomials.