EE203001 Linear Algebra
Solutions for Homework #8   Spring Semester, 2003

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6. $T(x,y) = (e^x, e^y)$. For all (x', y') in R^2, then

$$T((x,y)+(x',y')) = T(x+x', y+y') = (e^{x+x'}, e^{y+y'})$$

$$T(x,y)+T(x',y') = (e^x,e^y)+(e^{x'},e^{y'}) = (e^x+e^{x'}, e^y+e^{y'})$$

Since $(e^{x+x'}, e^{y+y'}) \neq (e^x+e^{x'}, e^y+e^{y'})$ in general, $T((x,y)+(x',y')) \neq T(x,y)+T(x',y')$
Thus T is nonlinear.

10. $T(x,y) = (2x-y, x+y)$. For all (x',y') in R^2 and all scalars a and b, then

$$\begin{aligned} T(a(x,y)+b(x',y')) &= T(ax+bx', ay+by') \\ &= (2(ax+bx')-(ay+by'), (ax+bx')+(ay+by')) \\ &= (a(2x-y)+b(2x'-y'), a(x+y)+b(x'+y')) \\ &= a(2x-y, x+y)+b(2x'-y', x'+y') \end{aligned}$$

and

$$aT(x,y)+bT(x',y') = a(2x-y,x+y)+b(2x'-y',x'+y')$$

Since $T(a(x,y)+b(x',y')) = aT(x,y)+bT(x',y')$, T is linear.
To find the null space, it's equivalent to finding $T(x,y) = O$.
$\Rightarrow T(x,y) = (2x-y, x+y) = O$
$\Rightarrow x = 0$ and $y = 0$.
$\Rightarrow N(T) = \{O\}$ and $T(R^2) = \{(x,y) : (x,y) \in R^2\}$
$\Rightarrow$ Its nullity $= 0$, and rank $= 2$.

12. Let $ax+by = 0, a, b \in \mathbb{R}$ and $a^2+b^2 \neq 0$ be a fixed line in $\mathbb{R}^2$ through the origin. We know $\mathbf{u} = (\frac{-b}{\sqrt{a^2+b^2}}, \frac{a}{\sqrt{a^2+b^2}})$ is a point on line $ax+by=0$ and $\mathbf{v} = (\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}})$ is a unit vector orthogonal to $(\frac{-b}{\sqrt{a^2+b^2}}, \frac{a}{\sqrt{a^2+b^2}})$. Thus the set $B = \{(\frac{-b}{\sqrt{a^2+b^2}}, \frac{a}{\sqrt{a^2+b^2}}), (\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}})\}$ is an orthonormal basis for $\mathbb{R}^2$. To find the reflection $T(x,y)$ of (x,y) with respect to the line $ax+by=0$, we consider $(x,y) = (\frac{-bx+ay}{\sqrt{a^2+b^2}})\mathbf{u} + (\frac{ax+by}{\sqrt{a^2+b^2}})\mathbf{v}$. Thus $T(x,y) = (\frac{-bx+ay}{\sqrt{a^2+b^2}})\mathbf{u} - (\frac{ax+by}{\sqrt{a^2+b^2}})\mathbf{v} = (\frac{(-b^2+a^2)x-2aby}{\sqrt{a^2+b^2}}, \frac{(a^2-b^2)y-2abx}{\sqrt{a^2+b^2}})$.
Hence the transformation T is linear. Since for null space $T(x,y) = O$, $(\frac{(-b^2+a^2)x-2aby}{\sqrt{a^2+b^2}}, \frac{(a^2-b^2)y-2abx}{\sqrt{a^2+b^2}}) = O$, $(0,0)$ is the only solution of (x,y). $N(T) = O$ and nullity is 0. Range is all $\mathbb{R}^2$ and rank $= 2$.

15. We can take a counterexample to prove T is not linear. Let $(1,0)$ and $(1,\pi)$ be points in $\mathbb{R}^2$. For $T(r,\theta) = (r, 2\theta)$, we have $T((1,0)+(1,\frac{\pi}{2})) = T(\sqrt{2}, \frac{\pi}{4}) = (\sqrt{2}, \frac{\pi}{2})$, and $T(1,0)+T(1,\frac{\pi}{2}) = (1,0)+(1,\pi) = 0$. Since $T((1,0)+(1,\frac{\pi}{2})) \neq T(1,0)+T(1,\frac{\pi}{2})$, T is not linear.

20. $T : \mathbb{R}^3 \to \mathbb{R}^3$, $T(x, y, z) = (x + 1, y + 1, z - 1)$.

T is not linear, since $T(2(0, 0, 0)) = T(0, 0, 0) = (0 + 1, 0 + 1, 0 - 1) = (1, 1, -1)$ but $2T((0, 0, 0)) = 2T(0, 0, 0) = 2(0 + 1, 0 + 1, 0 - 1) = (2, 2, -2)$.

24. Assume $\dim N(T) = k < \infty$ and $\dim T(V) = r < \infty$. Let $\{e_1, e_2, \ldots, e_k\}$ be a basis for $N(T)$. Since V is infinite-dimensional, there exist infinitely many $e_{k+1}, e_{k+2}, \ldots, e_{k+n}, \ldots$ in V such that $e_1, e_2, \ldots, e_k, e_{k+1}, \ldots, e_{k+n}, \ldots$ are linearly independent. We choose $n > \dim T(V) = r$, then the n vectors $T(e_{k+1}), T(e_{k+2}), \ldots, T(e_{k+n})$ are linearly dependent. Thus, there are $a_1, a_2, \ldots, a_n$ not all zeros, such that $a_1T(e_{k+1}) + a_2T(e_{k+2}) + \cdots + a_nT(e_{k+n}) = 0$. Since

$$\begin{aligned} 0 &= a_1T(e_{k+1}) + a_2T(e_{k+2}) + \cdots + a_nT(e_{k+n}) \\ &= T(a_1e_{k+1} + a_2e_{k+2} + \cdots + a_ne_{k+n}), \quad \text{(linearity of } T\text{)} \end{aligned}$$

$a_1e_{k+1} + a_2e_{k+2} + \cdots + a_ne_{k+n}$ is in $N(T)$. That is $a_1e_{k+1} + a_2e_{k+2} + \cdots + a_ne_{k+n}$ is a linear combination of $e_1, e_2, \ldots, e_k$, a contradiction to that $e_1, e_2, \ldots, e_k, \ldots, e_{k+n}$ are linearly independent. Hence at least one of $T(V)$ or $N(T)$ is infinite-dimensional.

25. Let $p(x) = \sum_{i=0}^{n} p_ix^i$, $r(x) = \sum_{i=0}^{n} r_ix^i$ be two real polynomials of degree $\leq n$, and $a, b \in \mathbb{R}$. Since

$$\begin{aligned} T(ap(x) + br(x)) &= T(a\sum_{i=0}^{n} p_ix^i + b\sum_{i=0}^{n} r_ix^i) \\ &= T(\sum_{i=0}^{n}(ap_i + br_i)x^i) \\ &= \sum_{i=0}^{n}(ap_i + br_i)(x + 1)^i \\ &= a\sum_{i=0}^{n} p_i(x + 1)^i + b\sum_{i=0}^{n} r_i(x + 1)^i \\ &= aT(p) + bT(r), \end{aligned}$$

T is a linear transformation.

If $T(p(x)) = 0$, then $\sum_{i=0}^{n} p_i(x + 1)^i = 0$. We know that $\{1, 1 + x, \ldots, (1 + x)^n\}$ is a basis for V (Section 3.6), hence $p_i = 0$, for $1 \leq i \leq n$. Thus $N(T) = \{O\}$ and $\dim N(T) = 0$.

The dimension of V is $n+1$ which is finite. Thus by rank-nullity theorem, $\dim N(T) + \dim T(V) = \dim V$, we have $\dim T(V) = \dim V = n + 1$. But $T(V) \subset V$, we have $T(V) = V$.

27. We find

$$\begin{aligned} T(ax + by) &= (ax + by)'' + A(ax + by)' + B(ax + by) \\ &= ax'' + by'' + Aax' + Aby' + Bax + Bby \\ &= a(x'' + Ax' + Bx) + b(y'' + Ay' + By) \\ &= aT(x) + bT(y), \end{aligned}$$

so T is linear. To derive its null space, we need $T(y)=y''+Ay'+By=0$. Let $y_h=e^{\lambda x}$, then $\lambda^2 e^{\lambda x}+A\lambda e^{\lambda x}+Be^{\lambda x}=0.\Rightarrow e^{\lambda x}(\lambda^2+A\lambda+B)=0.\Rightarrow \lambda^2+A\lambda+B=0$. Then the solution is $\lambda=\frac{-A\pm\sqrt{A^2-4B}}{2}$.

(a) If $A^2-4B=0$, $T(e^{-Ax/2})=0$ and $T(xe^{-Ax/2})=0$. So the null space $N(T)=L\{e^{-Ax/2},xe^{-Ax/2}\}$ with nullity 2, and the range $T(V)=\{y''(t)+Ay'(t)+By(t): y(t)\in V\}$ with rank infinity since dim $V=\infty$ and by Exercise 24.

(b) If $A^2-4B>0$, $T(e^{\frac{-A+\sqrt{A^2-4B}}{2}x})=0$ and $T(e^{\frac{-A-\sqrt{A^2-4B}}{2}x})=0$. So the null space $N(T)=L\{e^{\frac{-A+\sqrt{A^2-4B}}{2}x},e^{\frac{-A-\sqrt{A^2-4B}}{2}x}\}$ with nullity 2, and the range $T(V)=\{y''(t)+Ay'(t)+By(t): y(t)\in V\}$ with rank infinity since dim $V=\infty$ and by Exercise 24.

(c) If $A^2-4B<0$,$\lambda=\frac{-A\pm j\sqrt{4B-A^2}}{2}$. So $T(e^{\frac{-A+j\sqrt{4B-A^2}}{2}x})=0$ and $T(e^{\frac{-A-j\sqrt{4B-A^2}}{2}x})=0$. This means if $y=c_1e^{\frac{-A+j\sqrt{4B-A^2}}{2}x}+c_2e^{\frac{-A-j\sqrt{4B-A^2}}{2}x}$ for arbitrary c_1 and c_2, then $T(y)=0$. But in this example, y must be real, so we take c_1 as $a+\frac{b}{j}$ and c_2 as $a-\frac{b}{j}$ where a,b are two arbitrary real numbers. Then

$$\begin{aligned}
y &= (a+\frac{b}{j})e^{\frac{-A+j\sqrt{4B-A^2}}{2}x}+(a-\frac{b}{j})e^{\frac{-A-j\sqrt{4B-A^2}}{2}x}\\
&= ae^{-A/2}(e^{\frac{j\sqrt{4B-A^2}}{2}}+e^{\frac{-j\sqrt{4B-A^2}}{2}})+be^{-A/2}(e^{\frac{j\sqrt{4B-A^2}}{2}}-e^{\frac{-j\sqrt{4B-A^2}}{2}})/j\\
&= 2ae^{-A/2}\cos\frac{\sqrt{4B-A^2}}{2}+2be^{-A/2}\sin\frac{\sqrt{4B-A^2}}{2}
\end{aligned}$$

Hence $N(T)=L\{\cos\frac{\sqrt{4B-A^2}}{2},\sin\frac{\sqrt{4B-A^2}}{2}\}$ with nullity 2 and the range $T(V)=\{y''(t)+Ay'(t)+By(t): y(t)\in V\}$ with rank infinity since dim $V=\infty$ and by Exercese 24.

28. Since

$$\begin{aligned}
T(\alpha f_1+\beta f_2) &= \int_a^b(\alpha f_1(t)+\beta f_2(t))\sin(x-t)dt\\
&= \alpha\int_a^b f_1(t)\sin(x-t)dt+\beta\int_a^b f_2(t)\sin(x-t)dt\\
&= \alpha T(f_1)(x)+\beta T(f_2)(x),
\end{aligned}$$

T is linear. Note that

$$\begin{aligned}
T(f) &= \int_a^b f(t)\sin(x-t)dt\\
&= \sin x\int_a^b f(t)\cos t dt-\cos x\int_a^b f(t)\sin t dt
\end{aligned}$$

Thus $T(f) \in L(\sin x, \cos x)$. Consider the equation $\int_a^b \cos(t+k)\sin t dt = 0$. We have

$$\begin{aligned}
& \int_a^b (\cos t \cos k - \sin t \sin k)\sin t dt = 0 \\
\Rightarrow\ & \cos k \int_a^b \cos t \sin t dt - \sin k \int_a^b \sin^2 t dt = 0 \\
\Rightarrow\ & \cos k \int_a^b \frac{1}{2}\sin 2t dt - \sin k \int_a^b \frac{1-\cos 2t}{2} dt = 0 \\
\Rightarrow\ & \frac{1}{2}\cos k \frac{\cos 2t}{-2}|_a^b - \frac{1}{2}\sin k (t - \frac{\sin 2t}{2})|_a^b = 0 \\
\Rightarrow\ & \frac{-1}{4}\cos k(\cos 2b - \cos 2a) - \frac{1}{2}\sin k(b - a - \frac{\sin 2b - \sin 2a}{2}) = 0 \\
\Rightarrow\ & -\cos k(\cos 2b - \cos 2a) = 2\sin k(b - a - \frac{\sin 2b - \sin 2a}{2}) \\
\Rightarrow\ & \tan k = \frac{\sin k}{\cos k} = \frac{\cos 2a - \cos 2b}{2(b-a) - (\sin 2b - \sin 2a)} \quad \text{if } 2(b-a) \neq (\sin 2b - \sin 2a)
\end{aligned}$$

We have to examine when $2b - \sin 2b = 2a - \sin 2a$. Let $f(b) = 2b - \sin 2b$, we have $f'(b) = 2 - 2\cos 2b \geq 0$ with equality iff $b = n\pi$ where n is an integer. Thus, $f(b)$ is a nondecreasing function and $f'(b) = 0$ when $b = n\pi$ where n is an integer. Therefore, $2b - \sin 2b = 2a - \sin 2a$ only when $a = b$.

Next, we consider the equation $\int_a^b \cos(t+k)\cos t dt = 0$. Similarly, we have $\tan k = \frac{2(b-a)+\sin 2b - \sin 2a}{\cos 2a - \cos 2b}$ if $\cos 2b \neq \cos 2a$. Note that $\cos 2b = \cos 2a$ when $b = \pm a + n\pi$ where n is an integer. We next show that we cannot have $\int_a^b \cos(t+k)\sin t dt = 0$ and $\int_a^b \cos(t+k)\cos t dt = 0$ simultaneously for any k when $b \neq \pm a + n\pi$ where n is an integer. Otherwise, we could have

$$\begin{aligned}
\Rightarrow\ & \frac{\cos 2a - \cos 2b}{2(b-a) - (\sin 2b - \sin 2a)} = \frac{2(b-a) + \sin 2b - \sin 2a}{\cos 2a - \cos 2b} \\
\Rightarrow\ & (\cos 2a - \cos 2b)^2 = 4(b-a)^2 - (\sin 2b - \sin 2a)^2 \\
\Rightarrow\ & \cos^2 2a - 2\cos 2a \cos 2b + \cos^2 2b = 4(b-a)^2 - \sin^2 2b + 2\sin 2b \sin 2a - \sin^2 2a \\
\Rightarrow\ & 4(b-a)^2 = -2\cos(2a - 2b) \\
\Rightarrow\ & 2(b-a)^2 = -\cos(2(b-a))
\end{aligned}$$

Let $x = b - a$, then we have $2x^2 + \cos 2x = 0$. Define $f(x) = 2x^2 + \cos 2x$. Then $f'(x) = 4x - 2\sin 2x$ and $f''(x) = 4(1 - \cos 2x)$. Note that $f''(x) \geq 0$ for all x and therefore, $f(x)$ is a convex function. So the local minimum of $f(x)$ is the global minimum of it. Since $f'(x) = 0 \Leftrightarrow x = 0$, the global minimum happens when $x = 0$. But $f(0) = 1$, so $2(b-a)^2 \neq -\cos(2(b-a))$, a contradiction, when $b \neq \pm a + n\pi$ where n is an integer.

Let us consider two different cases:

(a) $b \neq \pm a + n\pi$ where n is an integer.

Let k_1 and k_2 satisfy the following equations

$$\int_a^b \cos(t+k_1)\sin t dt = 0 \text{ and } \int_a^b \cos(t+k_2)\cos t dt = 0$$

respectively. Then by the previous result, we have $\int_a^b \cos(t+k_1)\cos t dt = C_1 \neq 0$ and $\int_a^b \cos(t+k_2)\sin t dt = C_2 \neq 0$ respectively. Thus, $T(\frac{\cos(t+k_1)}{C_1}) = \sin x$ with $k_1 = \tan^{-1}(\frac{\cos 2a - \cos 2b}{2(b-a)-(\sin 2b - \sin 2a)})$ and $T(\frac{\cos(t+k_2)}{-C_2}) = \cos x$ with $k_2 = \tan^{-1}(\frac{2(b-a)+\sin 2b - \sin 2a}{\cos 2a - \cos 2b})$. Hence $L(\cos x, \sin x) \subseteq T(V)$. We conclude that $T(V) = L(\cos x, \sin x)$ and $\{\cos x, \sin x\}$ is a basis of $T(V)$ since it is a linearly independent set. Therefore, the rank=2. The null space $N(T) = \{f \in V | \int_a^b f(x)\cos x dx = \int_a^b f(x)\sin x dx = 0\}$. Since $L\{\cos 2x, \sin 2x, \cos 3x, \sin 3x, \ldots\} \subseteq N(T)$, and the nullity is infinity.

(b) $b = \pm a + n\pi$ but $b \neq a$ where n is an integer. In this case, we have

$$\begin{aligned}
\int_a^b \cos t \cos t dt &= \int_a^b \cos^2 t dt \\
&= \int_a^b \left(\frac{1+\cos 2t}{2}\right) dt \\
&= \left(\frac{t}{2} + \frac{\sin 2t}{4}\right)\Big|_a^b \\
&= \frac{b-a}{2} + \frac{\sin 2b - \sin 2a}{4} \\
&= \frac{1}{4}(2b - 2a + \sin 2b - \sin 2a) \\
&= C_1 \neq 0 \quad \text{(The reason is similar to that of } 2b - 2a - \sin 2b + \sin 2a)
\end{aligned}$$

and

$$\begin{aligned}
\int_a^b \cos t \sin t dt &= \frac{1}{2}\int_a^b \sin 2t dt \\
&= -\frac{1}{4}\cos 2t\Big|_a^b \\
&= -\frac{1}{4}(\cos 2b - \cos 2a) \\
&= 0.
\end{aligned}$$

So

$$T\left(\frac{\cos t}{C_1}\right) = \sin x.$$

In a similar way, we have

$$\int_a^b \sin t \cos t dt = 0$$

and

$$\int_a^b \sin t \sin t dt = \frac{1}{4}(2b - 2a - \sin 2b + \sin 2a) = C_2 \neq 0.$$

Hence

$$T\left(\frac{\sin t}{C_2}\right) = \cos x.$$

Therefore, $L(\cos x, \sin x) \subseteq T(V)$. We conclude that $T(V) = L(\cos x, \sin x)$ and $\{\cos x, \sin x\}$ is a basis of $T(f)$ since it is a linearly independent set. Therefore, the rank=2. Since $L\{\cos 2x, \sin 2x, \cos 3x, \sin 3x, \ldots\} \subseteq N(T)$, the nullity is infinity.

30. (a) To prove S is a subspace of V, we only need to check closure axioms because S is a subset of V.

i. If f_1 and f_2 are two elements of S, then
$\int_{-\pi}^{\pi}(f_1(t) + f_2(t))dt = \int_{-\pi}^{\pi} f_1(t)dt + \int_{-\pi}^{\pi} f_2(t)dt = 0 + 0 = 0.$
$\int_{-\pi}^{\pi}(f_1(t) + f_2(t))\cos tdt = \int_{-\pi}^{\pi} f_1(t)\cos tdt + \int_{-\pi}^{\pi} f_2(t)\cos tdt = 0 + 0 = 0.$
$\int_{-\pi}^{\pi}(f_1(t) + f_2(t))\sin tdt = \int_{-\pi}^{\pi} f_1(t)\sin tdt + \int_{-\pi}^{\pi} f_2(t)\sin tdt = 0 + 0 = 0.$

ii. If f are a elements of S, then
$\int_{-\pi}^{\pi} cf(t)dt = c\int_{-\pi}^{\pi} f(t)dt = c \cdot 0 = 0.$
$\int_{-\pi}^{\pi} cf(t)\cos tdt = c\int_{-\pi}^{\pi} f(t)\cos tdt = c \cdot 0 = 0.$
$\int_{-\pi}^{\pi} cf(t)\sin tdt = c\int_{-\pi}^{\pi} f(t)\sin tdt = c \cdot 0 = 0.$

Thus, S is a subspace of V.

(b) i. If $f(x) = \cos nx$ where $n = 2, 3, \ldots$, then
$\int_{-\pi}^{\pi} f(t)dt = \int_{-\pi}^{\pi} \cos ntdt = \frac{\sin nt}{n}|_{-\pi}^{\pi} = \frac{\sin n\pi - \sin(-n\pi)}{n} = \frac{0+0}{n} = 0.$
$\int_{-\pi}^{\pi} f(t)\cos tdt = \int_{-\pi}^{\pi} \cos nt\cos tdt = \frac{1}{2}(\int_{-\pi}^{\pi} \cos(n+1)tdt + \int_{-\pi}^{\pi} \cos(n-1)tdt) = \frac{1}{2}(0 + 0) = 0.$
$\int_{-\pi}^{\pi} f(t)\sin tdt = \int_{-\pi}^{\pi} \cos nt\sin tdt = \frac{1}{2}(\int_{-\pi}^{\pi} \sin(n+1)tdt - \int_{-\pi}^{\pi} \sin(n-1)tdt) = \frac{1}{2}(0 + 0) = 0.$

ii. If $f(x) = \sin nx$ where $n = 2, 3, \ldots$, then
$\int_{-\pi}^{\pi} f(t)dt = \int_{-\pi}^{\pi} \sin ntdt = -\frac{\cos nt}{n}|_{-\pi}^{\pi} = -\frac{\cos n\pi - \cos(-n\pi)}{n} = -\frac{\cos n\pi - \cos n\pi}{n} = 0.$
$\int_{-\pi}^{\pi} f(t)\cos tdt = \int_{-\pi}^{\pi} \sin nt\cos tdt = \frac{1}{2}(\int_{-\pi}^{\pi} \sin(n+1)tdt + \int_{-\pi}^{\pi} \sin(n-1)tdt) = \frac{1}{2}(0 + 0) = 0.$
$\int_{-\pi}^{\pi} f(t)\sin tdt = \int_{-\pi}^{\pi} \sin nt\sin tdt = -\frac{1}{2}(\int_{-\pi}^{\pi} \cos(n+1)tdt - \int_{-\pi}^{\pi} \cos(n-1)tdt) = -\frac{1}{2}(0 - 0) = 0.$
so S contains the functions $f(x) = \cos nx$ and $f(x) = \sin nx$ for each $n = 2, 3, \ldots$.

(c) We note that $\int_{-\pi}^{\pi} \cos mx\sin nxdx = \frac{1}{2}(\int_{-\pi}^{\pi} \sin(m+n)x - \sin(m-n)x)dx = \frac{0-0}{2} = 0 \forall m, n$. Also $\int_{-\pi}^{\pi} \cos mx cosnx dx = \int_{-\pi}^{\pi} \sin mx\sin nxdx = 0 \forall m \neq n$.Hence $\cos mx$ and $\sin nx$ are linearly indepedent. It is clear that $W = L\{\cos 2x, \sin 2x, \cos 3x, \sin 3x, \ldots\} \subseteq S$ and dim $W = +\infty$. Thus dim $S = +\infty$.

(d) For any $f(x) \in V$, the image $g(x) = T(f)$ of f is

$$\begin{aligned} g(x) &= \int_{-\pi}^{\pi}\{1 + cos(x-t)\}f(t)dt \\ &= \int_{-\pi}^{\pi} f(t)dt + \cos x\int_{-\pi}^{\pi} \cos tf(t)dt + \sin x\int_{-\pi}^{\pi} \sin tf(t)dt \\ &\in L(1, \cos x, \sin x). \end{aligned}$$

Thus $T(V) \subseteq L(1, \cos x, \sin x)$.

Consider $f(t) = \frac{1}{2\pi}, \frac{\cos t}{\pi}, \frac{\sin t}{\pi}$, we have

$$\int_{-\pi}^{\pi} \frac{1}{2\pi} dt = 1, \int_{-\pi}^{\pi} \frac{1}{2\pi} \cos t dt = 0, \int_{-\pi}^{\pi} \frac{1}{2\pi} \sin t dt = 0,$$

$$\int_{-\pi}^{\pi} \frac{\cos t}{\pi} dt = 0, \int_{-\pi}^{\pi} \frac{\cos t}{\pi} \cos t dt = 1, \int_{-\pi}^{\pi} \frac{\cos t}{\pi} \sin t dt = 0,$$

$$\int_{-\pi}^{\pi} \frac{\sin t}{\pi} dt = 0, \int_{-\pi}^{\pi} \frac{\sin t}{\pi} \cos t dt = 0, \int_{-\pi}^{\pi} \frac{\sin t}{\pi} \sin t dt = 1.$$

Thus, we have $T(\frac{1}{2\pi}) = 1, T(\frac{\cos x}{\pi}) = \cos x, T(\frac{\sin x}{\pi}) = \sin x$, which implies that $L(1, \cos x, \sin x) \subseteq T(V)$. We conclude that $T(V) = L(1, \cos x, \sin x)$ and $\{1, \cos x, \sin x\}$ is a basis of $T(V)$ since it is a linearly independent set.

(e) $f(x) \in N(T)$ if and only if $\int_{-\pi}^{\pi} f(t)dt + \cos x \int_{-\pi}^{\pi} \cos t f(t)dt + \sin x \int_{-\pi}^{\pi} \sin t f(t)dt = 0$. Since $1, \cos x$ and $\sin x$ are linearly independent in V, we have $\int_{-\pi}^{\pi} f(t)dt = 0$, $\int_{-\pi}^{\pi} \cos t f(t)dt = 0$ and $\int_{-\pi}^{\pi} \sin t f(t)dt = 0$. Thus $N(T) = S$.

(f) If $T(f) = cf$, $c \neq 0, f \neq 0$, then cf is in $T(V)$. Hence $f = c_1 + c_2 \cos x + c_3 \sin x$. so

$$\begin{aligned} T(f) &= 2\pi c_1 T(\frac{1}{2\pi}) + \pi c_2 T(\frac{\cos x}{\pi}) + \pi c_3 T(\frac{\sin x}{\pi}) \\ &= 2\pi c_1 + \pi c_2 \cos x + \pi c_3 \sin x. \end{aligned}$$

Thus, $2\pi c_1 + \pi c_2 \cos x + \pi c_3 \sin x = c(c_1 + c_2 \cos x + c_3 \sin x)$. $\Rightarrow (2\pi - c)c_1 + (\pi - c)c_2 \cos x + (\pi - c)c_3 \sin x = 0$. $\Rightarrow (2\pi - c)c_1 = (\pi - c)c_2 = (\pi - c)c_3 = 0$.
If $c_1 \neq 0$, then $c = 2\pi, c_2 = c_3 = 0$, and $f(x) = c_1$ where $c_1 \neq 0$ but otherwise arbitrary.If one of c_2 and c_3 is non-zero, then $c = \pi, c_1 = 0$ and $f(x) = c_2 \cos x + c_3 \sin x$ where c_2, c_3 are not both 0 but otherwise arbitrary.