EE203001 Linear Algebra
Solutions to Homework #7   Spring Semester, 2003

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6. Let $g(x) = K$, where K is a constant. To find the $g(x)$ nearest to f is equal to find the minimum value of $\|g-f\|^2$.
$\|g-f\|^2 = (K-\frac{1}{x}, K-\frac{1}{x}) = \int_1^3 (K^2 - 2K\frac{1}{x} + \frac{1}{x^2})dx = (K^2x - 2K\log x - \frac{1}{x})|_1^3 = 2K^2 - (2\log 3)K + \frac{2}{3} = 2(K-\frac{\log 3}{2})^2 + \frac{2}{3} - \frac{1}{2}\log^2 3$.
Since $(K-\frac{\log 3}{2})^2 \geq 0$, the minimum value of $\|g-f\|^2$ occurs at $(K-\frac{\log 3}{2})^2 = 0$. That is $K = \frac{1}{2}\log 3$. Thus $g(x) = \frac{1}{2}\log 3$ and $\|g-f\|^2 = \frac{2}{3} - \frac{1}{2}\log^2 3$.

7. Finding a constant polynomial $g(x) = c$ nearest to $f(x) = e^x$ is equivalent to finding an element g in the subspace $L(\{1\}) = \{a; a \in \mathbb{R}\}$ of $C(0,2)$, such that g is nearest to f.

Normalizing the constant function 1, we obtain $e_1(x) = \frac{\sqrt{2}}{2}$. By the Approximation Theorem, the projection $g(x)$ of f on $L(\{e_1\})$ is nearest to f. Since

$$(f, e_1) = \int_0^2 \frac{\sqrt{2}}{2} e^x\, dx = \frac{\sqrt{2}}{2} e^x|_0^2 = \frac{\sqrt{2}}{2}e^2 - \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}(e^2-1),$$

we have

$$g(x) = (f, e_1)e_1 = \frac{\sqrt{2}}{2}(e^2-1)\cdot\frac{\sqrt{2}}{2} = \frac{1}{2}(e^2-1).$$

Since $(f-g)\perp g$, we have

$$\begin{aligned}||f-g||^2 &= ||f||^2 - ||g||^2, \quad \text{(by Pythagorean Theorem)}\\ &= ||f||^2 - ||(f,e_1)e_1||^2 = ||f||^2 - (f,e_1)^2\\ &= \int_0^2 e^{2x}\, dx - \frac{1}{2}(e^2-1)^2\\ &= \frac{e^4-1}{2} - \frac{1}{2}(e^4 - 2e^2 + 1) = e^2 - 1.\end{aligned}$$

8. Finding a linear polynomial $g(x) = ax+b$ nearest to $f(x) = e^x$ is equivalent to finding an element g in the subspace $L(\{1,x\}) = \{ax+b; a,b \in \mathbb{R}\}$, such that g is nearest to f.

Because $(1,x) = \int_{-1}^1 1\cdot x\, dx = \frac{x^2}{2}|_{-1}^1 = 0$, $\{1,x\}$ is an orthogonal basis for $L(\{1,x\})$. We normalize $\{1,x\}$ to obtain an orthonormal basis $\{e_1, e_2\}$, where

$$e_1(x) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \quad \text{and} \quad e_2(x) = \frac{\sqrt{6}x}{2},$$

since

$$||1|| = (\int_{-1}^1 1\, dx)^{1/2} = \sqrt{2},$$

$$||x|| = (\int_{-1}^1 x^2\, dx)^{1/2} = (\frac{x^3}{3}|_{-1}^1)^{1/2} = \sqrt{\frac{2}{3}}.$$

By the Approximation Theorem, $g(x) = \sum_{i=1}^{2} (f, e_i)e_i$ is nearest to f.

Since

$$(f, e_1) = \frac{\sqrt{2}}{2}(f, 1) = \frac{\sqrt{2}}{2}\int_{-1}^{1} e^x\, dx = \frac{\sqrt{2}}{2}(e - e^{-1})$$

and

$$\begin{aligned}(f, e_2) &= \frac{\sqrt{6}}{2}(f, x) = \frac{\sqrt{6}}{2}\int_{-1}^{1} xe^x\, dx \\ &= \frac{\sqrt{6}}{2}(xe^x|_{-1}^{1} - \int_{-1}^{1} e^x\, dx), \quad (u = x, dv = e^x dx \;\Rightarrow\; v = e^x, du = dx) \\ &= \frac{\sqrt{6}}{2}(x-1)e^x|_{-1}^{1} \\ &= \frac{\sqrt{6}}{2}(-2e^{-1}) = -\sqrt{6}(e^{-1}),\end{aligned}$$

we have

$$\begin{aligned}g(x) &= \sum_{i=1}^{2}(f, e_i)e_i \\ &= \frac{\sqrt{2}}{2}(e - e^{-1})\frac{\sqrt{2}}{2} - \sqrt{6}(e^{-1})\frac{\sqrt{6}x}{2} \\ &= \frac{e - e^{-1}}{2} - 3e^{-1}x.\end{aligned}$$

Finally,

$$\begin{aligned}||f - g|| &= ||f||^2 - ||g||^2 = \int_{-1}^{1} e^{2x}\, dx - [(f, e_1)^2 + (f, e_2)^2] \\ &= \frac{e^{2x}}{2}\bigg|_{-1}^{1} - [\frac{1}{2}(e - e^{-1})^2 + 6(e^{-1})^2] \\ &= \frac{e^2}{2} - \frac{e^{-2}}{2} - \frac{1}{2}e^2 + 1 - \frac{1}{2}e^{-2} - 6e^{-2} \\ &= 1 - 7e^{-2}.\end{aligned}$$

9. As in Example 1 of Section 3.12, $\{u_0(x), u_1(x), u_2(x)\}$ is an orthogonal set. We normalize these basis vectors $u_0(x) = 1, u_1(x) = \cos x$ and $u_2(x) = \sin x$ as $\|u_0\| = \sqrt{\int_0^{2\pi} dx} = \sqrt{2\pi}$.

$\|u_1\| = \sqrt{\int_0^{2\pi} \cos^2 x dx} = \sqrt{\int_0^{2\pi} \frac{1+\sin 2x}{2} dx} = \sqrt{\frac{2\pi}{2}} = \sqrt{\pi}$.

$\|u_2\| = \sqrt{\int_0^{2\pi} \sin^2 x dx} = \sqrt{\int_0^{2\pi} \frac{1-\cos 2x}{2} dx} = \sqrt{\frac{2\pi}{2}} = \sqrt{\pi}$.

So the orthonormal basis are $u_0'(x) = \frac{1}{\sqrt{2\pi}}, u_1'(x) = \frac{\cos x}{\sqrt{\pi}}$ and $u_2'(x) = \frac{\sin x}{\sqrt{\pi}}$. Now we compute $(f, u_0'), (f, u_1')$ and (f, u_2') where $f(x) = x$.

$(f, u_0') = \int_0^{2\pi} \frac{x}{\sqrt{2\pi}} dx = \frac{x^2}{2\sqrt{2\pi}}|_0^{2\pi} = \frac{4\pi^2}{2\sqrt{2\pi}} = \sqrt{2}\pi^{3/2}$.

$(f, u_1') = \frac{1}{\sqrt{\pi}}\int_0^{2\pi} x\cos x dx = \frac{1}{\sqrt{\pi}}(x \sin x|_0^{2\pi} - \int_0^{2\pi} \sin x dx) = \frac{1}{\sqrt{\pi}}(0 - 0) = 0$.

$(f, u_2') = \frac{1}{\sqrt{\pi}}\int_0^{2\pi} x\sin x dx = \frac{1}{\sqrt{\pi}}(-x \cos x|_0^{2\pi} + \int_0^{2\pi} \cos x dx) = \frac{-2\pi+0}{\sqrt{\pi}} = -2\sqrt{\pi}$.

Hence the trigonometric polynomial nearest to f is $\frac{\sqrt{2}\pi^{3/2}}{\sqrt{2\pi}} - 2\sqrt{\pi}\frac{\sin x}{\sqrt{\pi}} = \pi - 2\sin x$.

10. In the linear space V of Exercise 5, we know that the set of all linear polynomials is a subspace of V spanned by orthonormal basis $\{y_0, y_1\}$, where $y_0 = 1$, $y_1 = t - 1$. By THEOREM 3.16., $p(x)$, the projection of $f(x) = e^{-x}$ on the subspace of all linear polynomials, is nearest to f. By the definition of inner product $(f, g) = \int_0^\infty e^{-t} f(t) g(t)\, dt$, we have

$$\begin{aligned}
(f, y_0) &= \int_0^\infty e^{-t} e^{-t} \cdot 1\, dt \\
&= \int_0^\infty e^{-2t}\, dt \\
&= \frac{-1}{2} e^{-2t} |_0^\infty \\
&= \frac{1}{2}, \text{ and}
\end{aligned}$$

$$\begin{aligned}
(f, y_1) &= \int_0^\infty e^{-t} e^{-t} \cdot (t-1)\, dt \\
&= \int_0^\infty e^{-2t} \cdot t\, dt - \int_0^\infty e^{-2t} \cdot 1\, dt \\
&= \frac{-1}{2} e^{-2t} \cdot t|_0^\infty + \frac{1}{2} \int_0^\infty e^{-2t}\, dt - \int_0^\infty e^{-2t}\, dt \\
&= 0 + \frac{1}{2} \cdot \frac{1}{2} - \frac{1}{2} \\
&= -\frac{1}{4}.
\end{aligned}$$

Hence $p(x) = (f, y_0) y_0 + (f, y_1) y_1 = \frac{1}{2} \cdot 1 - \frac{1}{4}(x - 1) = -\frac{1}{4} x + \frac{3}{4}$.