EE203001 Linear Algebra
Solutions for Homework #6 Spring Semester, 2003

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2. (a) $x_1 = (1,1,0,0)$, $x_2 = (0,1,1,0)$, $x_3 = (0,0,1,1)$, $x_4 = (1,0,0,1)$. We find an orthogonal basis $\{y_1', y_2', y_3', y_4'\}$ first. Let $y_1' = (1,1,0,0)$ and then we can start by the Gram-Schmidt process.
$y_2' = x_2 - \frac{(x_2,y_1')}{(y_1',y_1')}y_1' = (0,1,1,0) - \frac{1}{2}(1,1,0,0) = \frac{1}{2}(-1,1,2,0)$.
$y_3' = x_3 - \frac{(x_3,y_2')}{(y_2',y_2')}y_2' - \frac{(x_3,y_1')}{(y_1',y_1')}y_1' = (0,0,1,1) - \frac{1}{3}(-1,1,2,0) - 0 = \frac{1}{3}(1,-1,1,3)$.
$y_4' = x_4 - \frac{(x_4,y_3')}{(y_3',y_3')}y_3' - \frac{(x_4,y_2')}{(y_2',y_2')}y_2' - \frac{(x_4,y_1')}{(y_1',y_1')}y_1' = (1,0,0,1) - \frac{1}{2}(1,1,0,0) + \frac{1}{6}(-1,1,2,0) - \frac{1}{3}(1,-1,1,3) = (0,0,0,0)$.
We normalize y_1', y_2' and y_3' next. Thus we get an orthonormal basis:
$y_1 = \frac{1}{\sqrt{2}}(1,1,0,0)$, $y_2 = \frac{1}{\sqrt{6}}(-1,1,2,0)$, $y_3 = \frac{1}{2\sqrt{3}}(1,-1,1,3)$.

(b) $x_1 = (1,1,0,1)$, $x_2 = (1,0,2,1)$, $x_3 = (1,2,-2,1)$. We find an orthogonal basis $\{y_1', y_2', y_3'\}$ first. Let $y_1' = (1,1,0,0)$ and then we can start by the Gram-Schmidt process.
$y_2' = x_2 - \frac{(x_2,y_1')}{(y_1',y_1')}y_1' = (1,0,2,1) - \frac{2}{3}(1,1,0,1) = \frac{1}{3}(1,-2,6,1)$.
$y_3' = x_3 - \frac{(x_3,y_2')}{(y_2',y_2')}y_2' - \frac{(x_3,y_1')}{(y_1',y_1')}y_1' = (1,2,-2,1) + \frac{2}{3}(1,-2,6,1) - \frac{4}{3}(1,1,0,1) = (0,0,0,0)$.
We normalize y_1', y_2' and y_3' next. Thus we get an orthonormal basis:
$y_1 = \frac{1}{\sqrt{3}}(1,1,0,1)$, $y_2 = \frac{1}{\sqrt{42}}(1,-2,6,1)$.

3. First, we want to check if functions $y_0, y_1, y_2, ...$ are orthogonal to each other and $||y_n|| = 1$ for all n.

(a) $||y_0|| = (\int_0^\pi \frac{1}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{\pi}} dt)^{1/2} = (\frac{1}{\pi} \cdot t|_0^\pi)^{1/2} = 1^{1/2} = 1$.

And for all $n \neq 0$,

$$
\begin{aligned}
||y_n|| &= (\int_0^\pi \sqrt{\frac{2}{\pi}} \cos nt \cdot \sqrt{\frac{2}{\pi}} \cos nt \ dt)^{1/2} \\
&= (\frac{2}{\pi} \int_0^\pi \cos^2 nt \ dt)^{1/2} \\
&= (\frac{2}{\pi} \int_0^\pi \frac{1 + \cos 2nt}{2} \ dt)^{1/2} \\
&= (\frac{1}{\pi} \int_0^\pi \cos 2nt \ dt + \frac{1}{\pi} \int_0^\pi 1 \ dt)^{1/2} \\
&= (0 + \pi \frac{1}{\pi})^{1/2} \\
&= 1.
\end{aligned}
$$

(b) $(y_0, y_n) = \int_0^\pi \sqrt{\frac{1}{\pi}} \cdot \sqrt{\frac{2}{\pi}} \cos nt \ dt = \frac{\sqrt{2}}{\pi} \int_0^\pi \cos nt = 0$, for all $n \neq 0$.

And for $n, m \neq 0$,

$$\begin{aligned}
(y_n, y_m) &= \int_0^\pi \sqrt{\frac{2}{\pi}} \cos nt \cdot \sqrt{\frac{2}{\pi}} \cos mt \; dt \\
&= \frac{2}{\pi} \int_0^\pi \cos nt \cdot \cos mt \; dt \\
&= \frac{1}{\pi} \int_0^\pi (\cos(n+m)t + \cos(n-m)t) dt \\
&= \frac{1}{\pi} \int_0^\pi \cos(n-m)t \; dt \\
& \quad \text{(Since } n \text{ and } m \text{ are positive, first term is zero.)} \\
&= 0 \text{ , when } n \neq m.
\end{aligned}$$

Thus, functions $y_o, y_1, y_2, ...$ form an orthonormal set. Since $y_0 = \frac{1}{\sqrt{\pi}} x_0$ and $y_n = \sqrt{\frac{2}{\pi}} x_n$ for all $n \neq 0$, linear combination of $x_0, x_1, x_2, ...$ and linear combination of $y_o, y_1, y_2, ...$ are the same. Hence $y_o, y_1, y_2, ...$ form an orthonormal set and span the same subspace spanned by $x_0, x_1, x_2, ...$

4. In the linear space of all real polynomials, let $\{z_0, z_1, z_2\}$ be the orthogonal basis for the subspace $L(\{x_0, x_1, x_2\})$, obtained by the Gram-Schmidt process. Then

$$z_0(t) = x_0(t) = 1.$$

Since

$$(z_0, z_0) = \int_0^1 1 \, dt = t|_0^1 = 1 \quad \text{and} \quad (x_1, z_0) = \int_0^1 t \, dt = \left.\frac{t^2}{2}\right|_0^1 = \frac{1}{2},$$

we find that

$$z_1(t) = x_1(t) - \frac{(x_1, z_0)}{(z_0, z_0)} z_0(t) = x_1(t) - \frac{1}{2} z_0(t) = t - \frac{1}{2}.$$

Next, we use the relations

$$(x_2, z_0) = \int_0^1 t^2 \, dt = \left.\frac{t^3}{3}\right|_0^1 = \frac{1}{3},$$

$$(x_2, z_1) = \int_0^1 t^2 (t - \frac{1}{2}) \, dt = \int_0^1 (t^3 - \frac{t^2}{2}) \, dt = \left.(\frac{t^4}{4} - \frac{t^3}{6})\right|_0^1 = \frac{1}{12},$$

$$(z_1, z_1) = \int_0^1 (t - \frac{1}{2})^2 \, dt = \int_0^1 t^2 - t + \frac{1}{4} \, dt = \left.(\frac{t^3}{3} - \frac{t^2}{2} + \frac{t}{4})\right|_0^1 = \frac{1}{3} - \frac{1}{2} + \frac{1}{4} = \frac{1}{12},$$

to obtain

$$\begin{aligned}
z_2(t) &= x_2(t) - \frac{(x_2, z_0)}{(z_0, z_0)} z_0(t) - \frac{(x_2, z_1)}{(z_1, z_1)} z_1(t) \\
&= x_2(t) - \frac{1}{3} \cdot 1 \cdot z_0(t) - \frac{1}{12} \cdot 12 \cdot z_1(t) \\
&= t^2 - \frac{1}{3} - (t - \frac{1}{2}) \\
&= t^2 - t + \frac{1}{6}.
\end{aligned}$$

Since

$$
\begin{aligned}
(z_2, z_2) &= \int_0^1 (t^2 - t + \frac{1}{6})^2 \, dt \\
&= \int_0^1 (t^4 - 2t^3 + \frac{4}{3}t^2 - \frac{1}{3}t + \frac{1}{36}) \, dt \\
&= (\frac{t^5}{5} - \frac{t^4}{2} + \frac{4}{9}t^3 - \frac{t^2}{6} + \frac{t}{36})\Big|_0^1 \\
&= \frac{1}{5} - \frac{1}{2} + \frac{4}{9} - \frac{1}{6} + \frac{1}{36} \\
&= \frac{1}{180},
\end{aligned}
$$

we normalize z_0, z_1 and z_2 to obtain an orthonormal basis $\{y_0, y_1, y_2\}$, where

$$
\begin{aligned}
y_0(t) &= \frac{z_0(t)}{||z_0||} = 1, \\
y_1(t) &= \frac{z_1(t)}{||z_1||} = \sqrt{12}(t - \frac{1}{2}) = \sqrt{3}(2t - 1), \\
y_2(t) &= \frac{z_2(t)}{||z_2||} = \sqrt{180}(t^2 - t + \frac{1}{6}) = 6\sqrt{5}(t^2 - t + \frac{1}{6}) = \sqrt{5}(6t^2 - 6t + 1).
\end{aligned}
$$

5. First we prove $\int_0^\infty e^{-t}t^n dt = n!$ by induction.

(i) When $n = 0$, $\int_0^\infty e^{-t}dt = -e^{-t}|_0^\infty = -1(0-1) = 1 = 0!$.

(ii) Assume for $n = k$, $\int_0^\infty e^{-t}t^k dt = k!$.

(iii) For $n = k+1$,

$$
\begin{aligned}
\int_0^\infty e^{-t}t^{k+1}dt &= -e^{-t}t^{k+1}|_0^\infty + \int_0^\infty e^{-t}(k+1)t^n dt \\
&= (k+1)k! \\
&= (k+1)!.
\end{aligned}
$$

Now we compute $y_0(t), y_1(t), y_2(t)$ and $y_3(t)$.

(a) $y_0(t) = x_0(t) = 1$.

(b) $y_1(t) = x_1(t) - \frac{(x_1(t),y_0(t))}{(y_0(t),y_0(t))}y_0(t)$. Since $(x_1(t), y_0(t)) = \int_0^\infty e^{-t}tdt = 1! = 1$ and $(y_0(t), y_0(t)) = \int_0^\infty e^{-t}dt = 0! = 1$, we have $y_1(t) = t - \frac{1}{1}(1) = t - 1$.

(c) $y_2(t) = x_2(t) - \frac{(x_2(t),y_0(t))}{(y_0(t),y_0(t))}y_0(t) - \frac{(x_2(t),y_1(t))}{(y_1(t),y_1(t))}y_1(t)$. Since
$(x_2(t), y_0(t)) = \int_0^\infty e^{-t}t^2dt = 2! = 2$ and
$(x_2(t), y_1(t)) = \int_0^\infty e^{-t}t^2(t-1)dt = \int_0^\infty e^{-t}t^3dt - \int_0^\infty e^{t}(t^2)dt = 3! - 2! = 6 - 2 = 4$
and
$(y_1(t), y_1(t)) = \int_0^\infty e^{-t}(t-1)^2dt = \int_0^\infty e^{-t}(t^2 - 2t + 1)dt = 2! - 2 + 1 = 1$,
we have $y_2(t) = t^2 - 2 - 4(t-1) = t^2 - 4t + 2$.

(d) $y_3(t) = x_3(t) - \frac{(x_3(t),y_0(t))}{(y_0(t),y_0(t))} y_0 - \frac{(x_3(t),y_1(t))}{(y_1(t),y_1(t))} y_1(t) - \frac{(x_3(t),y_2(t))}{(y_2(t),y_2(t))} y_2(t)$. Since
$(x_3(t), y_0(t)) = \int_0^\infty e^{-t} t^3 dt = 3! = 6$ and
$(x_3(t), y_1(t)) = \int_0^\infty e^{-t} t^3 (t-1) dt = \int_0^\infty e^{-t} t^4 dt - \int_0^\infty e^{-t} t^3 dt = 4! - 3! = 18$ and
$(x_3(t), y_2(t)) = \int_0^\infty e^{-t} t^3 (t^2 - 4t + 2) dt = \int_0^\infty e^{-t} t^5 dt - 4 \int_0^\infty e^{-t} t^4 dt + 2 \int_0^\infty e^{-t} t^3 dt =$
$5! - 4 \cdot 4! + 2 \cdot 3! = 120 - 96 + 12 = 36$ and
$(y_2(t), y_2(t)) = \int_0^\infty e^{-t} (t^2 - 4t + 2)^2 dt = \int_0^\infty e^{-t} (t^4 - 8t^3 + 20t^2 - 16t + 4) dt =$
$4! - 8 \cdot 3! + 20 \cdot 2! - 16 + 4 = 4$,
we have $y_3(t) = t^3 - 6 - 18(t-1) - \frac{36}{4}(t^2 - 4t + 2) = t^3 - 9t^2 + 18t - 6$.