EE203001 Linear Algebra
Solutions to Homework #5 Spring Semester, 2003

Meng-Hua Chang, Chao-Chung Chang, Chen-Wei Hsu, Wen-Yao Chen

3. (a) If $\|x+y\| = \|x-y\|$, then

$$\begin{aligned}
& (x+y, x+y) = (x-y, x-y) \\
\Rightarrow\ & (x,x) + (x,y) + (y,x) + (y,y) = (x,x) - (x,y) - (y,x) + (y,y), \\
\Rightarrow\ & (x,x) + 2(x,y) + (y,y) = (x,x) - 2(x,y) + (y,y), \qquad \text{(By Axiom 1)} \\
\Rightarrow\ & 4(x,y) = 0, \\
\Rightarrow\ & (x,y) = 0.
\end{aligned}$$

(b) If $(x,y) = 0$, then

$$\begin{aligned}
\|x+y\| &= (x+y, x+y)^{1/2} \\
&= \{(x,x) + 2(x,y) + (y,y)\}^{1/2} \\
&= \{(x,x) + (y,y)\}^{1/2} \\
&= \{(x,x) - 2(x,y) + (y,y)\}^{1/2} \\
&= (x-y, x-y)^{1/2} \\
&= \|x-y\|.
\end{aligned}$$

8. (b)

$$\begin{aligned}
\|x+y\|^2 - \|x-y\|^2 &= (x+y, x+y) - (x-y, x-y) \\
&= (x,x) + (x,y) + (y,x) + (y,y) - \{(x,x) - (x,y) - (y,x) + (y,y)\} \\
&= 2(x,y) + 2(y,x).
\end{aligned}$$

(c)

$$\begin{aligned}
\|x+y\|^2 + \|x-y\|^2 &= (x+y, x+y) + (x-y, x-y) \\
&= (x,x) + (x,y) + (y,x) + (y,y) + (x,x) - (x,y) - (y,x) + (y,y) \\
&= 2(x,x) + 2(y,y) \\
&= 2\|x\|^2 + 2\|y\|^2.
\end{aligned}$$

11. (a) Given $(f,g) = \int_1^e (\log(x)) f(x) g(x) dx$. If $f(x) = \sqrt{x}$, then

$$\begin{aligned}
\|f\| &= (f,f)^{1/2} \\
&= \left(\int_1^e (\log x) f(x) f(x) dx\right)^{1/2} \\
&= \left(\int_1^e (\log x) x dx\right)^{1/2}.
\end{aligned}$$

Let $u = \log x$, then $du = \frac{1}{x}dx$. Let $v = \frac{x^2}{2}$, then $dv = xdx$. Thus

$$\begin{aligned}
(\int (\log x) x dx)^{1/2} &= (\int u dv)^{1/2} \\
&= (uv - \int v du)^{1/2} \\
&= (\log x \frac{x^2}{2} - \int \frac{x^2}{2}\frac{1}{x} dx)^{1/2}.
\end{aligned}$$

So

$$\begin{aligned}
(\int_1^e (\log x) x dx)^{1/2} &= (\log x \frac{x^2}{2}|_1^e - \int_1^e \frac{x^2}{2}\frac{1}{x} dx)^{1/2} \\
&= (\frac{e^2}{2} - \frac{x^2}{4}|_1^e)^{1/2} \\
&= (\frac{e^2}{2} - \frac{e^2-1}{4})^{1/2} \\
&= (\frac{e^2+1}{4})^{1/2} \\
&= \frac{1}{2}\sqrt{e^2+1}.
\end{aligned}$$

(b) First we evaluate $\int_1^e \log x dx$. Let $u = x^2 \log x$, then $du = (2x\log x + x)dx$. Let $v = -x^{-1}$, then $dv = \frac{1}{x^2}dx$. Thus

$$\begin{aligned}
\int \log x dx &= \int (\frac{1}{x^2}) x^2 \log x dx \\
&= \int u dv \\
&= uv - \int v du \\
&= -x \log x + \int \frac{1}{x}(2x\log x + x) dx.
\end{aligned}$$

So

$$\begin{aligned}
\int_1^e \log x dx &= -x \log x|_1^e + \int_1^e \frac{1}{x}(2x\log x + x) dx \\
&= -e + \int_1^e 2\log x dx + \int_1^e dx \\
&= -e + \int_1^e 2\log x dx + e - 1 \\
&= \int_1^e 2\log x dx - 1.
\end{aligned}$$

Thus, $\int_1^e \log x dx = 1$. Now we want to find a linear polynomial $g(x) = a + bx$ nonzero and orthogonal to $f(x) = 1$, i.e., $(f, g) = 0$. Since

$$\begin{aligned}(f, g) &= \int_1^e \log x(a + bx)dx \\ &= a\int_1^e \log x dx + b\int_1^e x \log x dx \\ &= a + b(\frac{e^2+1}{4}) \qquad \text{(by (a))},\end{aligned}$$

we have $(f, g) = 0$ when $a = -b(\frac{e^2+1}{4})$. So $g(x) = b(x - \frac{e^2+1}{4}), b$ is an arbitrary real number.

12. $(f, g) = \int_{-1}^1 f(t)g(t)\, dt$.

Since $u_1(t) = 1$ and $u_2(t) = t$, we have the following results:

$$\begin{aligned}(u_1, u_1) &= \int_{-1}^1 1 \cdot 1\, dt = t\,|_{-1}^1 = 2, \\ (u_2, u_2) &= \int_{-1}^1 t \cdot t\, dt = \left.\frac{t^3}{3}\right|_{-1}^1 = \frac{2}{3}, \quad \text{and} \\ (u_1, u_2) &= \int_{-1}^1 1 \cdot t\, dt = \left.\frac{t^2}{2}\right|_{-1}^1 = \frac{1}{2} - \frac{1}{2} = 0.\end{aligned}$$

Then $\|u_1\| = (u_1, u_1)^{1/2} = \sqrt{2}$, $\|u_2\| = (u_2, u_2)^{1/2} = \sqrt{\frac{2}{3}}$, and that u_1 and u_2 are orthogonal.

By the fact that $u_3(t) = 1 + t = u_1(t) + u_2(t)$ and $(u_1, u_2) = 0$, we have

$$(u_1, u_3) = (u_1, u_1 + u_2) = (u_1, u_1) + (u_1, u_2) = (u_1, u_1) = \|u_1\|^2,$$
$$(u_2, u_3) = (u_2, u_1 + u_2) = (u_2, u_1) + (u_2, u_2) = (u_2, u_2) = \|u_2\|^2,$$

and

$$\begin{aligned}(u_3, u_3) &= (u_1 + u_2, u_1 + u_2) = (u_1, u_1) + (u_2, u_1) + (u_2, u_1) + (u_2, u_2) \\ &= 2 + 0 + 0 + \frac{2}{3} = \frac{8}{3}.\end{aligned}$$

The last equation implies

$$\|u_3\| = (u_3, u_3)^{1/2} = \sqrt{\frac{8}{3}}.$$

Let θ_{ij} be the angle between u_i and u_j, for $1 \le i, j \le 3$ and $i \ne j$. Then

$$\begin{aligned}\cos\theta_{12} &= \frac{(u_1, u_2)}{\|u_1\|\,\|u_2\|} = 0, \\ \cos\theta_{13} &= \frac{(u_1, u_3)}{\|u_1\|\,\|u_3\|} = \frac{\|u_1\|^2}{\|u_1\|\,\|u_3\|} = \frac{\|u_1\|}{\|u_3\|} = \frac{\sqrt{2}}{\sqrt{\frac{8}{3}}} = \frac{\sqrt{3}}{2}, \quad \text{and} \\ \cos\theta_{23} &= \frac{(u_2, u_3)}{\|u_2\|\,\|u_3\|} = \frac{\|u_2\|^2}{\|u_2\|\,\|u_3\|} = \frac{\|u_2\|}{\|u_3\|} = \frac{\sqrt{\frac{2}{3}}}{\sqrt{\frac{8}{3}}} = \frac{1}{2}.\end{aligned}$$

Thus

$$\theta_{12} = \cos^{-1} 0 = \frac{\pi}{2}, \ \theta_{13} = \cos^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{6}, \quad \text{and} \quad \theta_{23} = \cos^{-1}\frac{1}{2} = \frac{\pi}{3}.$$

14. Let P be the linear space of all real polynomials and O be the zero element of P, that is, $O(t) = 0$.

(a). $(f, g) = f(1)g(1)$.
Let $f(t) = t-1$, then $(f, f) = f(1)f(1) = 0{\cdot}0 = 0$. Since $f \neq O$, the nonnegativity property is violated.

(b). $(f, g) = |\int_0^1 f(t)g(t)\ dt|$.
Let $c < 0$ and $f(t) \neq O(t)$. Then $c(f, f) = c|\int_0^1 f(t)f(t)\ dt| < 0$, and $(cf, f) = |\int_0^1 cf(t)f(t)\ dt| > 0$. Thus $c(f, f) \neq (cf, f)$ and the linearity property is violated.

(c). $(f, g) = \int_0^1 f'(t)g'(t)\ dt$.
Let $f(t)$ be a nonzero polynomial of degree 0, say $f(t) = 1$. Then $(f, f) = \int_0^1 f'(t)g'(t)\ dt = \int_0^1 0 \cdot 0\ dt = 0$. Thus the nonnegativity property is violated.

(d). $(f, g) = (\int_0^1 f(t)\ dt)(\int_0^1 g(t)\ dt)$.
Let $f(t) = t - \frac{1}{2}$. Then $(f, f) = [\int_0^1 (t - \frac{1}{2})\ dt]^2 = (\frac{t^2}{2} - \frac{t}{2})|_0^1 = 0 - 0 = 0$. Thus the nonnegativity property is violated.

15. (a) Let f and g be two elements of set V. Thus $\int_0^\infty e^{-t}f(t)^2dt$ and $\int_0^\infty e^{-t}g(t)^2dt$ converge. Since

$$\begin{aligned}
\lim_{M\to\infty}(\int_0^M e^{-t}|f(t)g(t)|dt)^2 &= \lim_{M\to\infty}|\int_0^M e^{-t}|f(t)||g(t)|dt|^2 \\
&\leq \lim_{M\to\infty}(\int_0^M e^{-t}|f(t)||f(t)|dt \cdot \int_0^M e^{-t}|g(t)||g(t)|dt) \\
&\quad \text{(by Cauchy-Schwarz inequality for functions } |f| \text{ and } |g| \\
&\quad \text{over } [0, M] \text{ with inner product as in Example 4 in the} \\
&\quad \text{textbook with } w(t) = e^{-t}.\ ) \\
&= \lim_{M\to\infty}(\int_0^M e^{-t}f(t)^2dt \cdot \int_0^M e^{-t}g(t)^2dt),
\end{aligned}$$

$\lim_{M\to\infty} \int_0^M e^{-t}|f(t)g(t)|dt$ converges. Thus $(f, g) = \int_0^\infty e^{-t}f(t)g(t)dt$ converges absolutely.

(b) Since the set of all functions continuous on a given interval is a linear space and V is a subset of it, we only need to check the closure axioms.

i Let f and g be two elements of set V. For $f + g$,

$$\begin{aligned}
\int_0^\infty e^{-t}(f(t) + g(t))^2dt &= \int_0^\infty e^{-t}(f(t)^2 + g(t)^2 + 2f(t)g(t))dt \\
&= \int_0^\infty e^{-t}f(t)^2dt + \int_0^\infty e^{-t}g(t)^2dt + 2\int_0^\infty e^{-t}f(t)g(t)dt.
\end{aligned}$$

Since $\int_0^\infty e^{-t}f(t)g(t)dt$ converges by(a), and $\int_0^\infty e^{-t}f(t)^2dt$ and $\int_0^\infty e^{-t}g(t)^2dt$ converge, $\int_0^\infty e^{-t}(f(t)+g(t))^2dt$ converges. Hence $f+g$ is an element of V and Axiom for closure under addition holds.

ii Let f be the element of set V such that $\int_0^\infty e^{-t}f(t)^2dt$ converges, and c be a real scalar. Since $\int_0^\infty e^{-t}(af(t))^2dt = \int_0^\infty e^{-t}a^2f(t)^2dt = a^2\int_0^\infty e^{-t}f(t)^2dt$ converges, af is an elements of V. Hence Axiom for closure under scalar multiplication holds.

Hence V is a linear space. Then we need to check if (f,g) is an inner product for V. Let x, y, and z be elements of V , and c be a real scalar.

i Since $(x,y) = \int_0^\infty e^{-t}x(t)y(t)dt = \int_0^\infty e^{-t}y(t)x(t)dt = (y,x)$, axiom for commutativity holds.

ii Since

$$\begin{aligned}(\alpha x+\beta y,z) &= \int_0^\infty e^{-t}(\alpha x+\beta y)(t)z(t)dt\\ &= \int_0^\infty e^{-t}(\alpha x(t)z(t)+\beta y(t)z(t))dt\\ &= \alpha\int_0^\infty e^{-t}x(t)z(t)dt+\beta\int_0^\infty e^{-t}y(t)z(t)dt\\ &= \alpha(x,z)+\beta(y,z),\end{aligned}$$

axiom for linearity holds.

iii We note that zero function $0(t)$ is the zero element O in V since $x(t)+0(t) = x(t)$ for all x. Then for all $x \neq O$, $(x,x) = \int_0^\infty e^{-t}x(t)x(t)dt = \int_0^\infty e^{-t}x(t)^2dt > 0$ since $\int_0^\infty e^{-t}x(t)^2dt$ converges. Hence axiom for positivity holds.

(c) We prove $(f,g) = \frac{n!}{2^{n+1}}$ for $f = e^{-t}$ and $g = t^n$, where $n = 0,1,2,...$ by induction. When $n = 0$,

$$\begin{aligned}(f,g) &= \int_0^\infty e^{-t}\cdot e^{-t}\cdot 1dt\\ &= \int_0^\infty e^{-2t}dt\\ &= \frac{-1}{2}e^{-2t}|_0^\infty\\ &= \frac{1}{2} = \frac{0!}{2^{(0+1)}}.\end{aligned}$$

Let $(f,g) = \frac{k!}{2^{k+1}}$ when $n = k$.
When $n = k+1$,

$$\begin{aligned}(f,g) &= \int_0^\infty e^{-t}\cdot e^{-t}\cdot t^{k+1}dt\\ &= \int_0^\infty t^{k+1}\cdot e^{-2t}dt\\ &= \frac{-1}{2}t^{k+1}e^{-2t}|_0^\infty + \frac{k+1}{2}\int_0^\infty e^{-2t}t^kdt\\ &= 0+\frac{k+1}{2}\cdot\frac{k!}{2^{k+1}} = \frac{(k+1)!}{2^{k+2}}.\end{aligned}$$

Hence $(f,g) = \frac{n!}{2^{n+1}}$ where $g(t) = t^n$ and $f(t) = e^{-t}$ by induction.

16. (a) $\sum_{n=1}^{\infty} x_n y_n$ converges absolutely $\Leftrightarrow \sum_{n=1}^{\infty} |x_n y_n|$ converges.
Consider two new sequences $x' = \{|x_n|\}$ and $y' = \{|y_n|\}$ both in V.
$\sum_{n=1}^{\infty} |x_n y_n| = (x', y')$ since $\sum_{n=1}^{\infty} |x_n y_n| = \sum_{n=1}^{\infty} |x_n||y_n|$.
Then using Cauchy-Schwarz inequality for the inner product space R^M with standard inner product,
$(\sum_{n=1}^{M} |x_n y_n|)^2 \leq (\sum_{n=1}^{M} |x_n|^2)(\sum_{n=1}^{M} |y_n|^2) \leq (\sum_{n=1}^{\infty} x_n^2)(\sum_{n=1}^{\infty} y_n^2)$, $\forall\ M$.
By taking $M \to \infty$, we have $(\sum_{n=1}^{\infty} |x_n y_n|)^2 \leq (\sum_{n=1}^{\infty} x_n^2)(\sum_{n=1}^{\infty} y_n^2) < \infty$.
Thus $(\sum_{n=1}^{\infty} |x_n y_n|)^2$ converges and $\sum_{n=1}^{\infty} x_n y_n$ converges absolutely.

(b) Since the set of all sequences of real numbers is a linear space and V is a subset of it, we only need to check the closure axioms.

i. Let $x = \{x_n\}$ and $y = \{y_n\}$ be two sequences in V. Consider $x+y = \{x_n+y_n\}$,

$$\begin{aligned} \sum_{n=1}^{M} (x_n + y_n)^2 &= \sum_{n=1}^{M} (x_n^2 + 2x_n y_n + y_n^2) \\ &= \sum_{n=1}^{M} x_n^2 + 2\sum_{n=1}^{M} x_n y_n + \sum_{n=1}^{M} y_n^2 \end{aligned}$$

From (a) we know that $\sum_{n=1}^{M} x_n y_n$ converges absolutely as M $\to \infty$ In addition, $\sum_{n=1}^{\infty} x_n^2$ and $\sum_{n=1}^{\infty} y_n^2$ converge. Thus $\sum_{n=1}^{\infty} (x_n + y_n)^2$ converges and $x + y$ is in V.

ii. Let $x = \{x_n\}$ in V, and $y = cx = \{cx_n\}$ where c is a real scalar.
Then $\sum_{n=1}^{\infty} (cx_n)^2 = c^2 \sum_{n=1}^{\infty} x_n^2$ converges.
Thus cx is in V.

Hence V is a linear space. Next, we test if V is a linear space with (x, y) as an inner procuet. Consider all choices of x, y, z in V and all real scalars c:

i. $(x,y) = \sum_{n=1}^{\infty} x_n y_n = \sum_{n=1}^{\infty} y_n x_n = (y,x)$.
ii. $(x, y+z) = \sum_{n=1}^{\infty} x_n(y_n+z_n) = \sum_{n=1}^{\infty}(x_n y_n+x_n z_n) = \sum_{n=1}^{\infty} x_n y_n+\sum_{n=1}^{\infty} x_n z_n = (x,y) + (x,z)$.
iii. $c(x,y) = c\sum_{n=1}^{\infty} x_n y_n = \sum_{n=1}^{\infty}(cx_n)y_n = (cx, y)$.
iv. Since $(x,x) = \sum_{n=1}^{\infty} x_n^2$, $(x,x) \geq 0$ and $(x,x) = 0\ iff\ x = O$.
Thus $(x,x) > 0$ if $x \neq O$.

Hence the four axioms all hold, V is a linear space with (x, y) as an inner product.

(c) $(x,y) = \sum_{n=1}^{\infty} \frac{1}{n}\frac{1}{n+1} = \sum_{n=1}^{\infty}(\frac{1}{n} - \frac{1}{n+1}) = (1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \cdots)$
$= (1 + (\frac{1}{2} - \frac{1}{2}) + (\frac{1}{3} - \frac{1}{3}) - \cdots) = 1$.

(d) Recall that $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$
Then $(x,y) = \sum_{n=1}^{\infty}(2^{-n})(\frac{1}{n!}) = \sum_{n=1}^{\infty} \frac{2^{-n}}{n!} = -1 + (1 + \frac{2^{-1}}{1!} + \frac{2^{-2}}{2!} + \cdots)$
$= -1 + e^{2^{-1}} = e^{\frac{1}{2}} - 1$.