EE203001 Linear Algebra
Solutions to Homework #4 Spring Semester, 2003

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23. Let P be the linear space of all real polynomials.

(a). Let $T = \{1, t^2, t^4\}$ and $S = \{a + bt^2 + ct^4 | a, b, c \in \mathbb{R}\}$. S is a subspace of P and $T \subset S$, so $L(T) \subset S$. On the other hand, by the definition of $L(T)$, $S \subset L(T)$. Thus $L(T) = S$.

If $a + bt^2 + ct^4 = 0$, let $t = 0$, we have $a = 0$. Then divide $bt^2 + ct^4 = 0$ by t^2, and let $t = 0$, we have $b = 0$. Repeating this process we find that $c = 0$. Thus T is independent, hence $\dim L(T) = 3$.

(Another way to see that T is independent is by the definition of polynomials, $a + bt^2 + ct^4 = 0$ if and only if $a = b = c = 0$.)

(b). Let $T = \{t, t^3, t^5\}$ and $S = \{at + bt^3 + ct^5 | a, b, c \in \mathbb{R}\}$. S is a subspace of P and $T \subset S$, so $L(T) \subset S$. On the other hand, by the definition of $L(T)$, $S \subset L(T)$. Thus $L(T) = S$. If $at + bt^3 + ct^5 = 0$, divide $at + bt^3 + ct^5 = 0$ by t, then put $t = 0$, we have $a = 0$. Then divide $bt^3 + ct^5 = 0$ by t^3, and let $t = 0$, we have $b = 0$. Repeating this process we find that $c = 0$. Thus T is independent, hence $\dim L(T) = 3$.

(Another way to see that T is independent is by the definition of polynomials, $a + bt^3 + ct^5 = 0$ if and only if $a = b = c = 0$.)

(c). Let $T = \{t, t^2\}$ and $S = \{at + bt^2 | a, b \in \mathbb{R}\}$. S is a subspace of P and $T \subset S$, so $L(T) \subset S$. On the other hand, by the definition of $L(T)$, $S \subset L(T)$. Thus $L(T) = S$. If $at + bt^2 = 0$, divide $at + bt^2 = 0$ by t, then let $t = 0$, we have $a = 0$. Then let $t = 1$, we have $b = 0$. Thus T is independent, hence $\dim L(T) = 2$.

(Another way to see that T is independent is by the definition of polynomials, $at + bt^2 = 0$ if and only if $a = b = 0$.)

(d). Let $T = \{1 + t, (1 + t)^2\}$ and $S = \{a(1 + t) + b(1 + t)^2 | a, b \in \mathbb{R}\}$. S is a subspace of P and $T \subset S$, so $L(T) \subset S$. On the other hand, by the definition of $L(T)$, $S \subset L(T)$. Thus $L(T) = S$. If $a(1+t)+b(1+t)^2 = 0$, divide $a(1+t)+b(1+t)^2 = 0$ by $1 + t$ then let $t = -1$, we have $a = 0$. Then let $t = 0$, we have $b = 0$. Thus T is independent, hence $\dim L(T) = 2$.

26. We use the notation $|T|$ to denote the number of elements in a set T.

(a). Let T be a basis for S, then T is a set of independent elements of T. Applying Theorem 3.7(a), we have that T is a subset of some basis T' for V. Then $|T| \leq |T'|$, that is, $\dim S \leq \dim V$ and S is finite dimensional.

(b). ($\Rightarrow$) Let T be a basis for S, if $\dim S = \dim V = n$, then $|T| = n$. By Theorem 3.7(b), T is a basis for V, Thus $S = L(T) = V$.

($\Leftarrow$) If $S = V$, it is clear that $\dim S = \dim V$.

(c). If T is a basis for S, then T consists of independent elements in $S \subset V$. By Theorem 3.7(a), T is a part of a basis for V.

(d). Let $V=\mathbb{R}^2$, and $T=\{(1,0),(0,1)\}$ be a basis for V. Let $S=L(\{(1,1)\})$ be the subspace spanned by $\{(1,1)\}$. Then T does not contain $(1,1)$.

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1. (a) $(x,y)=\sum_{i=1}^{n} x_i|y_i|$.

i. Axiom 1 fails to hold.
Let $n=1$, $x=1$ and $y=-1$.
$(x,y)=1\cdot|-1|=1$, but $(y,x)=(-1)\cdot 1=-1$.
Thus $(x,y)\neq(y,x)$.

ii. Axiom 2 fails to hold.
Let $n=1$, $x=1$, $y=1$ and $z=-1$.
$(x,y+z)=1\cdot|1+(-1)|=1\cdot 0=0$, but $(x,y)+(x,z)=1\cdot|1|+1\cdot|-1|=2$.
Thus $(x,y+z)\neq(x,y)+(x,z)$.

iii. Axiom 3 fails to hold.
Let $n=2$, $x=(1,-1)$. Then $x\neq O$.
$(x,x)=\sum_{n=1}^{2} x_n=1\cdot|1|+(-1)\cdot|-1|=1-1=0$.

Hence, (x,y) is not an inner product for $\mathbb{R}^n$.

(b) $(x,y)=|\sum_{i=1}^{n} x_i y_i|$

i. Axiom 2 fails to hold.
Let $n=1$, $x=1$, $y=1$ and $z=-1$.
$(x,y+z)=|1\cdot(1+(-1))|=|1\cdot 0|=0$, but $(x,y)+(x,z)=|1\cdot 1|+|1\cdot(-1)|=2$.
Thus $(x,y+z)\neq(x,y)+(x,z)$.

ii. Axiom 3 fails to hold.
Let $n=1$, $x=1$, $y=1$ and $c=-1$.
$c(x,y)=(-1)|1\cdot 1|=-1$, but $(cx,y)=|(-1)\cdot 1\cdot 1|=1$.
Thus $c(x,y)\neq(cx,y)$.

Hence, (x,y) is not an inner product for $\mathbb{R}^n$.

(c) $(x,y)=\sum_{i=1}^{n} x_i \sum_{j=1}^{n} y_j$
Let $n=2$, $x=(1,-1)$. Then $x\neq O$.
But $(x,x)=(1+(-1))(1+(-1))=0$.
$\Rightarrow$ Axiom4 fails hold.

(d) $(x,y)=(\sum_{i=1}^{n} x_i^2 y_i^2)^{\frac{1}{2}}$.

i. Let $x=(1,1), y=(3,4), z=(4,3)$ be elements in $\mathbb{R}^2$. We have
$(x,y+z)=(1^2\cdot(3+4)^2+1^2\cdot(4+3)^2)^{\frac{1}{2}}=(49+49)^{\frac{1}{2}}=7\sqrt{2}$,
$(x,y)=(1^2\cdot 3^2+1^2\cdot 4^2)^{\frac{1}{2}}=(9+16)^{\frac{1}{2}}=5$, and
$(x,z)=(1^2\cdot 4^2+1^2\cdot 3^2)^{\frac{1}{2}}=(16+9)^{\frac{1}{2}}=5$.
Thus $(x,y+z)=7\sqrt{2}\neq 10=5+5=(x,y)+(x,z)$. Hence Axiom for distributivity fails to hold.

ii. Let $x=(1,1), y=(3,4)$ be elements in $\mathbb{R}^2$ and $c=-2$ be a scalar. We have
$c(x,y)=-2(1^2\cdot 3^2+1^2\cdot 4^2)^{\frac{1}{2}}=-2(9+16)^{\frac{1}{2}}=-10$, and
$(cx,y)=((-2)^2\cdot 3^2+(-2)^2\cdot 4^2)^{\frac{1}{2}}=(4\cdot 9+4\cdot 16)^{\frac{1}{2}}=100^{\frac{1}{2}}=10$.
Thus $c(x,y)\neq(cx,y)$. Hence Axiom for associativity fails to hold.

Hence, (x,y) is not an inner product for $\mathbb{R}^n$.

(e) $(x, y) = \sum_{i=1}^n (x_i + y_i)^2 - \sum_{i=1}^n x_i^2 - \sum_{i=1}^n y_i^2$.

i. Since $(x, y) = \sum_{i=1}^n (x_i+y_i)^2 - \sum_{i=1}^n x_i^2 - \sum_{i=1}^n y_i^2 = \sum_{i=1}^n (x_i+y_i)^2 - \sum_{i=1}^n y_i^2 - \sum_{i=1}^n x_i^2 = (y, x)$, Axiom 1 holds.

ii. Since

$$\begin{aligned}
(x, y) &= \sum_{i=1}^n (x_i + y_i)^2 - \sum_{i=1}^n x_i^2 - \sum_{i=1}^n y_i^2 \\
&= \sum_{i=1}^n (x_i^2 + 2x_iy_i + y_i^2) - \sum_{i=1}^n x_i^2 - \sum_{i=1}^n y_i^2 \\
&= \sum_{i=1}^n (x_i^2 + 2x_iy_i + y_i^2 - x_i^2 - y_i^2) \\
&= \sum_{i=1}^n (2x_iy_i),
\end{aligned}$$

we have

$$\begin{aligned}
(x, y + z) &= \sum_{i=1}^n [2x_i(y_i + z_i)] \\
&= \sum_{i=1}^n [(2x_iy_i) + 2(x_iz_i)] \\
&= \sum_{i=1}^n (2x_iy_i) + \sum_{i=1}^n (2x_iz_i) \\
&= (x, y) + (x, z).
\end{aligned}$$

Thus Axiom 2 holds.

iii. Since $(x, y) = \sum_{i=1}^n (2x_iy_i)$ (by ii.), we have $(cx, y) = \sum_{i=1}^n (2cx_iy_i) = c[\sum_{i=1}^n (2x_iy_i)] = c(x, y)$. Thus Axiom 3 holds.

iv. First, we note that $(0, 0, ...0)$ is the zero element O in $\mathbb{R}^n$ since $x+(0, 0, ..., 0) = x$ for all x. Then

$$\begin{aligned}
(x, x) &= \sum_{i=1}^n (x_i + x_i)^2 - \sum_{i=1}^n x_i^2 - \sum_{i=1}^n x_i^2 \\
&= \sum_{i=1}^n (4x_i^2 - x_i^2 - x_i^2) \\
&= \sum_{i=1}^n 2x_i^2 > 0 \text{ , if } x \neq O.
\end{aligned}$$

Thus Axiom 4 holds.

Hence, (x, y) is an inner product for $\mathbb{R}^n$.

2. Let V be a real linear space and have a real inner product with the axiom 4'. Suppose there are two non-zero elements x and y in V such that $(x, x) > 0$ and $(y, y) < 0$.

Let $z = ax + by$, where a and b are real scalars and neither of them is zero. Then $(cx, cx) = c^2(x, x) > 0$ for all real $c \neq 0$ since $(x, x) > 0$. Hence y is independent of x and $z = ax + by \neq 0$ for all $a \neq 0$ and $b \neq 0$. Let $b = 1$. Then we have

$$\begin{aligned}
(z, z) &= (ax + y, ax + y) \\
&= (ax, ax) + (ax, y) + (y, ax) + (y, y) \text{ (by Axioms 1 and 2)} \\
&= a^2(x, x) + a(x, y) + a(y, x) + (y, y) \text{ (by Axioms 1 and 3, and } a \text{ is a real number)} \\
&= a^2(x, x) + 2a(x, y) + (y, y) \text{ (by Axiom 1).}
\end{aligned}$$

Since $(y, y) < 0$ and $(x, x) > 0$, we have $4(x, y)^2 - 4(y, y)(x, x) > 0$. Then we can choose a such that $a = \frac{-(x,y) \pm \sqrt{(x,y)^2 - (y,y)(x,x)}}{(x,x)}$ such that $(z, z) = 0$. But in this case z is not O, a contradiction to the axiom 4'. Then we must have either $(x, x) > 0$ for all $x \neq O$ or $(x, x) < 0$ for all $x \neq O$.

9. (a)

$$\begin{aligned}
(ax, by) &= a(x, by) && \text{(by Axiom 3)} \\
&= a\overline{(by, x)} && \text{(by Axiom 1)} \\
&= a\overline{b}\overline{(y, x)} \\
&= a\overline{b}(x, y). && \text{(by Axiom 1).}
\end{aligned}$$

(b)

$$\begin{aligned}
(x, ay + bz) &= \overline{(x, ay)} + \overline{(x, bz)} && \text{(by Axiom 2)} \\
&= \overline{(ay, x)} + \overline{(bz, x)} && \text{(by Axiom 1)} \\
&= \overline{a}\overline{(y, x)} + \overline{b}\overline{(z, x)} \\
&= \overline{a}(x, y) + \overline{b}(x, z). && \text{(by Axiom 1)}
\end{aligned}$$

10 (a) i. $(f, g) = \sum_{k=0}^{n} f(\frac{k}{n})g(\frac{k}{n}) = \sum_{k=0}^{n} g(\frac{k}{n})f(\frac{k}{n}) = (g, f)$. So Axiom 1 holds.

ii.

$$\begin{aligned}
(f, g + h) &= \sum_{k=0}^{n} f(\frac{k}{n})(g + h)(\frac{k}{n}) \\
&= \sum_{k=0}^{n} f(\frac{k}{n})(g(\frac{k}{n}) + h(\frac{k}{n})) \\
&= \sum_{k=0}^{n} f(\frac{k}{n})g(\frac{k}{n}) + \sum_{k=0}^{n} f(\frac{k}{n})h(\frac{k}{n}) \\
&= (f, g) + (f, h).
\end{aligned}$$

So Axiom 2 holds.

iii.

$$\begin{aligned} c(f,g) &= c\sum_{k=0}^{n} f(\frac{k}{n})g(\frac{k}{n}) \\ &= \sum_{k=0}^{n}(cf(\frac{k}{n}))g(\frac{k}{n}) \\ &= \sum_{k=0}^{n}(cf)(\frac{k}{n})g(\frac{k}{n}) \\ &= (cf,g). \end{aligned}$$

So Axiom 3 holds.

iv. First, we note that the zero function $0(x)$ is a zero element in P_n because $f(x) + 0(x) = f(x)$ for any $f(x)$ in P_n. Now, for a non-zero polynomial $f(x)$ of degree $\leq n$ in P_n, there exist at most n real roots. Thus at least one of the $n + 1$ numbers $f(\frac{k}{n}), 0 \leq k \leq n$, is non-zero. Thus,

$$\begin{aligned} (f,f) &= \sum_{k=0}^{n} f(\frac{k}{n})f(\frac{k}{n}) \\ &= \sum_{k=0}^{n} f(\frac{k}{n})^2 \\ &> 0. \end{aligned}$$

So Axiom 4 holds.

(b) Given $f(t) = t$ and $g(t) = at + b$,

$$\begin{aligned} (f,g) &= \sum_{k=0}^{n} f(\frac{k}{n})g(\frac{k}{n}) \\ &= \sum_{k=0}^{n} \frac{k}{n}(a\frac{k}{n} + b) \\ &= \frac{a}{n^2}\sum_{k=0}^{n} k^2 + \frac{b}{n}\sum_{k=0}^{n} k \\ &= \frac{a}{n^2}\frac{n(n+1)(2n+1)}{6} + \frac{b}{n}\frac{n(n+1)}{2} \\ &= \frac{a(n+1)(2n+1)}{6n} + \frac{b(n+1)}{2}. \end{aligned}$$

(c) We find that $g(t) = at + b$ in (b) is a linear polynomial, so by (b) we have the coefficients a, b of $g(t)$ such that

$$\begin{aligned}
&\frac{a(n+1)(2n+1)}{6n} + \frac{b(n+1)}{2} = 0, \\
\Rightarrow\quad &\frac{a(2n+1)}{6n} + \frac{b}{2} = 0, \qquad \text{because } n \geq 1 \\
\Rightarrow\quad &\frac{a(2n+1)}{6n} = -\frac{b}{2}, \\
\Rightarrow\quad &b = \frac{-a(2n+1)}{3n}.
\end{aligned}$$

So $g(t) = a(t - \frac{2n+1}{3n})$, for arbitrary a.