EE203001 Linear Algebra
Solutions for Homework #3 Spring Semester, 2003

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7. Since $\mathbb{R}^3$ is already a space, we need only to check the closure axioms for a subset S of $\mathbb{R}^3$ to be a subspace of $\mathbb{R}^3$. Let $S = \{(x, y, z) \text{ in } \mathbb{R}^3 \mid x^2 - y^2 = 0\}$ and $v_1 = (x_1, y_1, z_1)$, and $v_2 = (x_2, y_2, z_2)$ in S.
Since $v_1 + v_2 = (x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$, and

$$\begin{aligned}(x_1+x_2)^2 - (y_1+y_2)^2 &= (x_1^2 + 2x_1x_2 + x_2^2) - (y_1^2 + 2y_1y_2 + y_2^2)\\ &= (x_1^2 - y_1^2) + (x_2^2 - y_2^2) + 2(x_1x_2 - y_1y_2)\\ &= 0 + 0 + 2(x_1x_2 - y_1y_2)\\ &= 2(x_1x_2 - y_1y_2).\end{aligned}$$

But if $x_1 = 1$, $y_1 = 1$ and $x_2 = 1$, $y_2 = -1$, we have $2(x_1x_2 - y_1y_2) = 4 \neq 0$. Thus the closure axiom for addition on S does not hold and S is not a subspace of $\mathbb{R}^3$.

10. Since $\mathbb{R}^3$ is already a space, we need only to check the closure axioms for a subset S of $\mathbb{R}^3$ to be a subspace of $\mathbb{R}^3$. Let $S = \{(x, y, z) \text{ in } \mathbb{R}^3 \mid x + y + z = 0 \text{ and } x - y - z = 0\}$. Let $v = (x, y, z)$, $v_1 = (x_1, y_1, z_1)$,and $v_2 = (x_2, y_2, z_2)$ be elements in S and a be a real number.

i. Since $v_1 + v_2 = (x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$,
$(x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2) = (x_1 + y_1 + z_1) + (x_2 + y_2 + z_2) = 0 + 0 = 0$, and
$(x_1 + x_2) - (y_1 + y_2) - (z_1 + z_2) = (x_1 - y_1 - z_1) + (x_2 - y_2 - z_2) = 0 + 0 = 0$.
So $v_1 + v_2$ is in S.

ii. Since $av = a(x, y, z) = (ax, ay, az)$, $(ax) + (ay) + (az) = a(x + y + z) = a0 = 0$ and $(ax) - (ay) - (az) = a(x - y - z) = a0 = 0$. So av is in S.

Hence, S is a subspace of $\mathbb{R}^3$.

13. Let P_n denote the linear space of all real polynomials of degree $\leq n$, where n is fixed.
Let S denote the set of all polynomials f in P_n satisfying $f(0) + f(1) = 0$.
Let f, f_1, f_2 be elements in S and a be a real number.

i. Since

$$\begin{aligned}(f_1+f_2)(0) + (f_1+f_2)(1) &= f_1(0) + f_2(0) + f_1(1) + f_2(1)\\ &= (f_1(0) + f_1(1)) + (f_2(0) + f_2(1))\\ &= 0 + 0 = 0,\end{aligned}$$

$f_1 + f_2$ is in S.

ii. Since $(af)(0) + (af)(1) = a(f(0) + f(1)) = a0 = 0$, af is in S.

Hence, S is a subspace of $\mathbb{R}^3$.

20 Let f_1, f_2 be two polynomials in S. Since $f_1(0) = f_2(0) = 0$ and $f_1'(0) = f_2'(0) = 0$, we have $(f_1 + f_2)(0) = f_1(0) + f_2(0) = 0$ and $(f_1 + f_2)'(0) = f_1'(0) + f_2'(0) = 0$. Thus $f_1 + f_2$ is in S. Also since, $af(0) = 0$ and $af'(0) = 0$, af is in S. Thus the closure axioms hold and S is a subspace of P_n.

24 (a) i. If $a \neq 0$ and $b \neq 0$, consider an arbitrary linear relation of 1, e^{ax}, e^{bx} as $c_1 \cdot 1 + c_2 e^{ax} + c_3 e^{bx} = 0 \cdots (1)$. Taking derivitive on both sides, we get $ac_2 e^{ax} + bc_3 e^{bx} = 0$. If $b - a < 0$, divided by e^{ax} on both sides, we get $ac_2 + bc_3 e^{(b-a)x} = 0 \cdots (2)$. Now, if we let x$\rightarrow +\infty$ in (2), $e^{(b-a)x}$ tends to zero and we find $c_2 = 0$ since $a \neq 0$. Then returning to (2), we must have $c_3 = 0$ since $b \neq 0$. Similarly if $a - b < 0$, the result is the same. And returning to (1), we must have $c_1 = 0$. Thus c_1, c_2, c_3 are all zero and the set is independent.

ii. If $a = 0$, $b \neq 0$, $e^{ax} = e^0 = 1$. Since $1 - e^{ax} = 0$, then 1 and e^{ax} are linearly dependent and the set is a linearly dependent set. Similarly if $b = 0$, $a \neq 0$, the set is a linearly dependent set.

(c) i. If $a \neq 0$, consider an arbitrary linear relation $c_1 \cdot 1 + c_2 e^{ax} + c_3 x e^{ax} = 0 \cdots (1)$. Taking dirivative on both sides,we get $ac_2 e^{ax} + c_3 e^{ax} + ac_3 x e^{ax} = 0 \cdots (2)$. Taking derivative again, we get $a^2 c_2 e^{ax} + 2ac_3 e^{ax} + a^2 c_3 x e^{ax} = 0$. Divided by a and comparing with (2), we conclude that $c_3 = 0$. Then returning to (2), we must have c_2=0 since $a \neq 0$. And returning to (1), we must have $c_1 = 0$. Thus the set is independent.

ii. If $a = 0$, $e^{ax} = e^0 = 1$ and $xe^{ax} = x$. Then 1 and e^{ax} are linearly dependent since $1 - e^{ax} = 0$ and the set is a linearly dependent set.

(e) Because $0.5e^x + 0.5e^{-x} + (-1)\text{cosh } x = 0.5e^x + 0.5e^{-x} + (-1)(\frac{e^x + e^{-x}}{2}) = 0$, this subset is depedent.

(g) Consider an arbitrary linear relation of $\cos x^2$ and $\sin x^2$, $c_0 \cos^2 x + c_1 \sin^2 x = 0$. By taking $x = 0$, we have $c_0 \cdot 1 + c_1 \cdot 0 = 0 \;\Rightarrow\; c_0 = 0$. By taking $x = \frac{\pi}{2}$, we have $c_0 \cdot 0 + c_1 \cdot 1 = 0 \;\Rightarrow\; c_1 = 0$. Thus, this subset is independent.

(i)

$$\begin{aligned} c_0 \sin x + c_1 \sin 2x = 0 \;&\Rightarrow\; c_0 \sin x + c_1(2 \sin x \cos x) = 0 \\ &\Rightarrow\; \sin x(c_0 + 2c_1 \cos x) = 0 \\ &\Rightarrow\; 1(c_0 + 0) = 0 \text{ when } x = \frac{\pi}{2} \\ &\Rightarrow\; c_0 = 0 \\ &\Rightarrow\; c_1 = 0. \end{aligned}$$

Thus, this subset is independent.

25. (a). If $x \in S$, then $x = 1x \in L(S) = \{\sum_{i=1}^{k} c_i x_i | c_i$ is a scalar and $x_i \in S$, for $i = 1, ..., k\}$. Thus $S \subset L(S)$.

(b). Let $x = \sum_{i=1}^{k} c_i x_i$ be an element in $L(S)$, where $x_1, \ldots, x_k$ are in S and $c_1, \ldots, c_k$ are scalars. Since $x_1, x_2, ..., x_k \in S \subseteq T$ and T is a subspace of V, x must be in T due to the closure axioms of T. So $L(S) \subseteq T$.

(c). Let S be a subset of V.

i. $(\Rightarrow)$
If S is a subspace, then S satisfies the closure axioms (by Theorem 3.4). Hence $L(S) = \{\sum_{i=1}^{k} c_i x_i | c_i$ is a scalar and $x_i \in S,$ for $i = 1, ..., k\} \subset S$. We have seen that $S \subset L(S)$ in 25(a), thus $L(S) = S$.

ii. $(\Leftarrow)$
Suppose we have that $L(S) = S$. By the definition of $L(S)$, for any scalar a and $x, y \in S$, we have ax, $x + y \in L(S)$. Since $L(S) = S$, $x + y$ and $ax \in S$. We have shown that S satisfies the closure axioms, that is, S is a subspace of V.

(d). If $S \subset T$, then

$$\begin{aligned} L(S) &= \{\sum_{i=1}^{k} c_i x_i | c_i \text{ is a scalar}, x_i \in S, \text{ for } i = 1, ..., k\} \\ &\subset \{\sum_{i=1}^{k} c_i x_i | c_i \text{ is a scalar}, x_i \in T, \text{ for } i = 1, ..., k\} \text{ (since } S \subset T) \\ &= L(T). \end{aligned}$$

(e). (i). Let $x, y \in S \cap T$, then $x, y \in S$ and $x, y \in T$. Hence $x + y \in S$ and $x + y \in T$. Thus $x + y \in S \cap T$.

(ii). If $x \in S \cap T$, then $x \in S$ and $x \in T$. Let a be any scalar, then $ax \in S$ and $ax \in T$, and hence $ax \in S \cap T$.

By (i) and (ii), the closure axioms hold and $S \cap T$ is a subspace of V.

(f). Since $S \cap T \subset S$ and $S \cap T \subset T$, we have $L(S \cap T) \subset L(S)$ and $L(S \cap T) \subset L(T)$ by (d). Thus $L(S \cap T) \subset L(S) \cap L(T)$.

(g). Let $V = R^2$. Let $S = \{(1, 0)\}$ and $T = \{(-1, 0)\}$, then $S \cap T = \emptyset$ and hence $L(S \cap T) = \{(0, 0)\}$. But $L(S) = L(T) = \{(x, 0) | x \in R\}$.