# EE203001 Linear Algebra

# Solutions for Homework #2 Spring Semester, 2003

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**10.** Let $V$ denote the nonempty set of all rational functions $f/g$, with the degree of $f \leq$ the degree of $g$ (including $f = 0$). Let $x, y, z$ and $w$ be four arbitrary elements in $V$ and $x = \frac{f_1}{g_1}, y = \frac{f_2}{g_2}, z = \frac{f_3}{g_3}, w = \frac{f}{g}$ where $f, f_1, f_2, f_3, g, g_1, g_2, g_3$ are all polynomials . Let $a, b$ be two scalars. For simplicity, we define $deg(h)$ as the degree of a polynomial $h$. Now we show that $V$ is a linear space indeed.

(a) $x+y = \frac{f_1}{g_1} + \frac{f_2}{g_2} = \frac{f_1g_2+f_2g_1}{g_1g_2}$. Note that $(g_1g_2)$ and $(f_1g_2+f_2g_1)$ are also polynomials, and hence we denote them as $g'$, $f'$ ,respectively. So $x+y = \frac{f'}{g'}$. Besides, $deg(g') = deg(g_1) + deg(g_2)$ and $deg(f') \leq max(deg(f_1) + deg(g_2), deg(f_2) + deg(g_1))$. But $deg(f_1) + deg(g_2) \leq deg(g_1) + deg(g_2)$ $deg(f_2) + deg(g_1) \leq deg(g_2) + deg(g_1) = deg(g_1) + deg(g_2)$, we have

$$\begin{aligned} deg(f') &\leq max(deg(f_1) + deg(g_2), deg(f_2) + deg(g_1)) \\ &\leq max(deg(g_1) + deg(g_2), deg(g_1) + deg(g_2)) \\ &= deg(g_1) + deg(g_2) \\ &= deg(g') \end{aligned}$$

Thus, Axiom 1 holds.

(b) We find $aw = a(\frac{f}{g}) = \frac{af}{g}$, But $af$ is still a polynomial, so $aw$ is a rational function. Besides, $deg(af) \leq deg(f) \leq deg(g)$ and hence Axiom 2 holds.

(c)

$$\begin{aligned} x + y &= \frac{f_1}{g_1} + \frac{f_2}{g_2} \\ &= \frac{f_1g_2 + f_2g_1}{g_1g_2} \\ &= \frac{f_2g_1 + f_1g_2}{g_1g_2} \\ &= \frac{f_2}{g_2} + \frac{f_1}{g_1} \\ &= y + x \end{aligned}$$

So Axiom 3 holds.

(d)

$$\begin{aligned}(x+y)+z &= (\frac{f_1}{g_1}+\frac{f_2}{g_2})+\frac{f_3}{g_3}\\ &= \frac{f_1g_2+f_2g_1}{g_1g_2}+\frac{f_3}{g_3}\\ &= \frac{f_1g_2g_3+f_2g_1g_3+f_3g_1g_2}{g_1g_2g_3}\\ &= \frac{f_1}{g_1}+\frac{f_2g_3+f_3g_2}{g_2g_3}\\ &= \frac{f_1}{g_1}+(\frac{f_2}{g_2}+\frac{f_3}{g_3})\\ &= x+(y+z)\end{aligned}$$

So Axiom 4 holds.

(e) We find the zero rational function 0 is in $V$ and is a zero element of $V$ since $w+0=\frac{f}{g}+0=\frac{f+0g}{g}=\frac{f}{g}=w$. Hence Axiom 5 holds.

(f) The rational function $(-1)w=\frac{-f}{g}$ with $deg(-f)=deg(f)\leq deg(g)$ is clearly in $V$. Since $w+(-1)w=\frac{f}{g}+(-1)\frac{f}{g}=\frac{f}{g}+\frac{-f}{g}=\frac{f-f}{g}=\frac{0}{g}=0$, Axiom 6 holds.

(g) $a(bw)=a(b\frac{f}{g})=a\frac{bf}{g}=\frac{abf}{g}=(ab)\frac{f}{g}=(ab)w$. Hence Axiom 7 holds.

(h) $a(x+y)=a(\frac{f_1}{g_1}+\frac{f_2}{g_2})=a\frac{f_1g_2+f_2g_1}{g_1g_2}=a\frac{f_1g_2}{g_1g_2}+a\frac{f_2g_1}{g_1g_2}=a\frac{f_1}{g_1}+a\frac{f_2}{g_2}=ax+ay$. Hence Axiom 8 holds.

(i) $(a+b)w=(a+b)\frac{f}{g}=\frac{(a+b)f}{g}=\frac{af+bf}{g}=\frac{af}{g}+\frac{bf}{g}=a\frac{f}{g}+b\frac{f}{g}=aw+bw$. Hence Axiom 9 holds.

(j) $1w=1\frac{f}{g}=\frac{1f}{g}=\frac{f}{g}=w$. Hence Axiom 10 holds.

**17.** Let $V$=$\{f : |f(x)|\leq M_f$ for all $x$, $M_f$ depends on $f\}$. Let $f, f_1, f_2, f_3$ be four arbitrary elements in $V$ and $|f|\leq M_f$, $|f_1|\leq M_{f_1}$, $|f_2|\leq M_{f_2}$, $|f_3|\leq M_{f_3}$ with $M_f, M_{f_1}, M_{f_2}, M_{f_3}\geq 0$ . Let $a, b$ be two scalars. For clarity, we let the domain of this function be $X$. Now we show that $V$ is a linear space indeed.

(a) If we denote the sum of $f_1$ and $f_2$ as $f'$ and the sum of $M_{f_1}$ and $M_{f_2}$ as $M_{f'}$, then

$$\begin{aligned}|f'| &= |f_1+f_2|\\ &\leq |f_1|+|f_2|\\ &\leq M_{f_1}+M_{f_2}\\ &= M_{f'}\end{aligned}$$

Thus, $f'$ is in $V$ and Axiom 1 holds.

(b) If we denote the product of $f$ and $a$ as $f$" and the product of $M_f$ and $|a|$ as $M_{f"}$, then $|f"|=|af|=|a||f|\leq|a|M_f=M_{f"}$.Hence Axiom 2 holds.

(c) $(f_1+f_2)(x)=f_1(x)+f_2(x)=f_2(x)+f_1(x)=(f_2+f_1)(x)$. Hence Axiom 3 holds.

(d)

$$\begin{aligned}((f_1+f_2)+f_3)(x) &= (f_1+f_2)(x)+f_3(x)\\ &= f_1(x)+f_2(x)+f_3(x)\\ &= f_1(x)+(f_2(x)+f_3(x))\\ &= f_1(x)+(f_2+f_3)(x)\\ &= (f_1+(f_2+f_3))(x)\end{aligned}$$

Hence Axiom 4 holds.

(e) The zero function 0 is in V since $|0(x)| \leq 0 \ \forall x \in X$. Also, $f(x)+0 = f(x)$, where 0 is the zero function. Hence Axiom 5 holds.

(f) The function $(-1)f$ is clearly in $V$ by (b). Since $(f+(-1)f)(x) = f(x) + (-1)f(x) = 0$. Hence Axiom 6 holds.

(g) $a(bf)(x) = a(bf(x)) = abf(x) = (ab)f(x)$. Hence Axiom 7 holds.

(h) $a(f_1+f_2)(x) = a(f_1(x)+f_2(x)) = af_1(x)+af_2(x) = (af_1+af_2)(x)$. Hence Axiom 8 holds.

(i) $((a+b)f)(x) = (a+b)f(x) = af(x)+bf(x) = (af+bf)(x)$. Hence Axiom 9 holds.

(j) $(1f)(x) = 1f(x) = f(x)$. Hence Axiom 10 holds.

**20.** Let $V = \{a\sin x + b\cos x; a, b \in R\}$.

(1) For $a_1, b_1, a_2$ and $b_2 \in R$,

$$(a_1\sin x + b_1\cos x) + (a_2\sin x + b_2\cos x) = (a_1+a_2)\sin x + (b_1+b_2)\cos x,$$

for all $x \in R$. Thus $(a_1+a_2)\sin x + (b_1+b_2)\cos x \in V$.

(2) For every $r \in R$ and $a\sin x + b\cos x \in V$, we have

$$r(a\sin x + b\cos x) = ra\sin x + rb\cos x,$$

for all $x \in R$. Thus $r(a\sin x + b\cos x) \in V$.

(3) $(a_1\sin x + b_1\cos x) + (a_2\sin x + b_2\cos x) = (a_1+a_2)\sin x + (b_1+b_2)\cos x = (a_2+a_1)\sin x + (b_2+b_1)\cos x = (a_2\sin x + b_2\cos x) + (a_1\sin x + b_1\cos x)$, for all $x \in R$. Thus Axiom 3 holds.

(4) For $a_1\sin x + b_1\cos x$, $a_2\sin x + b_2\cos x$ and $a_3\sin x + b_3\cos x$ in $V$,

$$\begin{aligned}&[(a_1\sin x + b_1\cos x) + (a_2\sin x + b_2\cos x)] + (a_3\sin x + b_3\cos x)\\ =\ & (a_1+a_2)\sin x + (b_1+b_2)\cos x + (a_3\sin x + b_3\cos x)\\ =\ & (a_1+a_2+a_3)\sin x + (b_1+b_2+b_3)\cos x, \quad \text{for all} x \in R\end{aligned}$$

and

$$\begin{aligned}&(a_1\sin x + b_1\cos x) + [(a_2\sin x + b_2\cos x) + (a_3\sin x + b_3\cos x)]\\ =\ & (a_1\sin x + b_1\cos x) + (a_2+a_3)\sin x + (b_2+b_3)\cos x\\ =\ & (a_1+a_2+a_3)\sin x + (b_1+b_2+b_3)\cos x, \quad \text{for all } x \in R\end{aligned}$$

The associative law for addition holds.

(5) Let $O$ be the zero function, i.e. $O(x) = 0$ for all $x \in R$, then $O(x) = 0 \cdot \sin x + 0 \cdot \cos x \in V$. For any $a \sin x + b \cos x$ in $V$, $(a \sin x + b \cos x) + O(x) = a \sin x + b \cos x + 0 = a \sin x + b \cos x$, $\forall x \in R$. Thus the zero function is a zero element in $V$.

(6) For every $a \sin x + b \cos x$ in $V$, $(-1)(a \sin x + b \cos x)$ is also in $V$ by (2) and $(a \sin x + b \cos x) + (-1)(a \sin x + b \cos x) = (a - a) \sin x + (b - b) \cos x = O(x)$ for all $x \in R$. Thus Axiom 6 holds.

(7) For $r, s \in R$ and $a \sin x + b \cos x \in V$, $r(s(a \sin x + b \cos x)) = r(sa \sin x + sb \cos x) = rsa \sin x + rsb \cos x = (rs)a \sin x + (rs)b \cos x = (rs)(a \sin x + b \cos x)$. The associative law for multiplication by numbers holds.

(8) For $(a_1 \sin x + b_1 \cos x)$ and $(a_2 \sin x + b_2 \cos x) \in V$, and $r \in R$,

$$\begin{aligned} & r[(a_1 \sin x + b_1 \cos x) + (a_2 \sin x + b_2 \cos x)] \\ = \; & r[(a_1 + a_2) \sin x + (b_1 + b_2) \cos x] \\ = \; & r(a_1 + a_2) \sin x + r(b_1 + b_2) \cos x \\ = \; & (ra_1 + ra_2) \sin x + (rb_1 + rb_2) \cos x \\ = \; & (ra_1 \sin x + rb_1 \cos x) + (ra_2 \sin x + rb_2 \cos x) \\ = \; & r(a_1 \sin x + b_1 \cos x) + r(a_2 \sin x + b_2 \cos x), \; \forall x \in R. \end{aligned}$$

Thus the Axiom 8 holds.

(9) For $r$ and $s \in R$, $a \sin x + b \cos x \in V$, $(r + s)(a \sin x + b \cos x) = (r + s)a \sin x + (r + s)b \cos x = r(a \sin x + b \cos x) + s(a \sin x + b \cos x)$ $\forall x \in R$, thus Axiom 9 holds.

(10) For every $a \sin x + b \cos x \in V$, $1(a \sin x + b \cos x) = a \sin x + b \cos x$ $\forall x \in R$.

**24.** (d) If $a = 0$, there is nothing to prove. If $a \neq 0$, multiplying $a^{-1}$ to both sides of $ax = O$, we have $a^{-1}ax = a^{-1}O$. Using Theorem 3.3 (b), we have $x = a^{-1}ax = a^{-1}O = O$.

(e) Because $a \neq 0$, we can multiply $a^{-1}$ to both sides of $ax = ay$ to get $a^{-1}ax = a^{-1}ay$. Thus $x = y$.

(f) Adding the negative element of $bx$, $-1(bx)$, to both sides of $ax = bx$, we have $ax - bx = bx - bx$. The left hand side of the equality is equal to $(a - b)x$, and the right hand side is equal to $O$, so we have $(a - b)x = O$. By (d), $a - b = 0$, hence $a = b$.

(g)

$$\begin{aligned} x + y + (-x) + (-y) &= x + (-x) + y + (-y), \text{by Axiom 3} \\ &= O + O, \text{by Axiom 6} \\ &= O, \end{aligned}$$

hence $-(x + y) = (-x) + (-y)$

(h) i. $x + x = 1x + 1x = (1 + 1)x = 2x$.

ii. $x + x + x = (x + x) + x = 2x + x = 2x + 1x = (2 + 1)x = 3x$.

iii. We prove the general case, $\sum_{i=1}^{n} x = nx$, by mathematical induction.
For $n = 1$, $1x = x$ is true.
We assume the equality holds for $n = k$, that is,

$$\sum_{i=1}^{k} x = kx.$$

For the case $n = k + 1$,

$$\sum_{i=1}^{k+1} x = (\sum_{i=1}^{k} x) + x = kx + x = (k+1)x.$$

Hence the equality holds.

**25.** (a) Let $V$ be the set of all functions $f$ integrable on [0,1] such that $\int_0^1 f(x)dx = 0$. Let $f, f_1, f_2$ and $f_3$ be elements in $V$ and $a$ and $b$ be real numbers.

(1) Since $\int_0^1 (f_1(x) + f_2(x))dx = \int_0^1 f_1(x)dx + \int_0^1 f_2(x)dx = 0 + 0 = 0$, $f_1 + f_2$ is in $V$.

(2) Since $\int_0^1 (af(x))dx = a(\int_0^1 f(x)dx) = a0 = 0$, $af$ is in $V$.

(3) Since $(f_1 + f_2)(x) = f_1(x) + f_2(x) = f_2(x) + f_1(x) = (f_2 + f_1)(x)$ for all $x \in [0, 1]$, $f_1 + f_2 = f_2 + f_1$.

(4) Since $(f_1(x)+f_2(x))+f_3(x) = f_1(x)+f_2(x)+f_3(x) = f_1(x)+(f_2(x)+f_3(x))$ for all $x \in [0, 1]$, $(f_1 + f_2) + f_3 = f_1 + (f_2 + f_3)$.

(5) Let $O(x) = 0$ for all $x$. Since $\int_0^1 O(x)dx = 0$, $O(x) \in V$. Since $f(x)+O(x) = f(x) + 0 = f(x)$ for all $x \in [0, 1]$, $f + O = f$.

(6) By (2), $(-1)f$ is in $V$. Since $f(x) + (-1)f(x) = 0$ for all $x \in [0, 1]$, $f + (-f) = O$.

(7) Since $a(bf(x)) = abf(x) = (ab)f(x)$ for all $x \in [0, 1]$, $a(bf) = (ab)f$.

(8) Since $a(f_1(x)+f_2(x)) = af_1(x)+af_2(x)$ for all $x \in [0, 1]$, $a(f_1+f_2) = af_1+af_2$.

(9) Since $(a + b)f(x) = af(x) + bf(x)$ for all $x \in [0, 1]$, $(a + b)f = af + bf$.

(10) Since $1f(x) = f(x)$ for all $x \in [0, 1]$, $1f = f$.

(b) Let $V$ denote the nonempty set of all functions $f$ integrable on [0,1] with $\int_0^1 f(x)dx \geq 0$. Let $a < 0$ be a real number. And choose an $f \in V$, e.g., $f(x) = x$, such that $\int_0^1 f(x)dx > 0$, then

$$\int_0^1 af(x)dx = a\int_0^1 f(x)dx < 0.$$

So, $af \notin V$ and Axiom 2 fails to hold. Then Axioms 6, 7, 8, 9, which relate to the scalar multiplication, fail to hold.

(c) Let $V = \{f(x); \lim_{x\to\infty} f(x) = 0\}$.

(1). If $f_1(x)$ and $f_2(x) \in V$, then $\lim_{x\to\infty}(f_1 + f_2)(x) = \lim_{x\to\infty} f_1(x) + \lim_{x\to\infty} f_2(x) = 0+0$, thus $f_1(x) + f_2(x) \in V$.

(2). $\lim_{x\to\infty}(af)(x) = a \lim_{x\to\infty} f(x) = a \cdot 0 = 0$, thus $af(x) \in V$.

(3). For $f_1(x)$ and $f_2(x) \in V$, $f_1(x) + f_2(x) = f_1(x) + f_2(x)$ for all $x$ in $R$.

(4). For $f_1(x)$, $f_2(x)$ and $f_3(x) \in V$, $(f_1(x)+f_2(x))+f_3(x) = f_1(x)+(f_2(x)+f_3(x))$ for all $x$ in $R$. Thus Axiom 4 holds.

(5). Define $O(x) = 0$, for all $x \in R$. Since $\lim_{x\to\infty} O(x) = 0$, $O(x)$ is in $V$. Since $f(x) + O(x) = f(x) + 0 = f(x)$ for any $f(x) \in V$, $O(x)$ is a zero element in $V$.

(6). For any $f(x) \in V$, we have $(-1)f(x)$ in $V$ by (2). Since $f(x) + (-1)f(x) = f(x) - f(x) = 0 = O(x)$, Axiom 6 holds.

(7). For every $f$ in $V$, and all real numbers $r$ and $s$, $r(sf(x)) = (rs)f(x)$ for all $x$ in $R$. Thus Axiom 7 holds.

(8). For all $f_1$ and $f_2$ in $V$, and $r \in R$, $r(f_1(x) + f_2(x)) = rf_1(x) + rf_2(x)$ for all $x$ in $R$, thus Axiom 8 holds.

(9). For all $f \in V$ and all real numbers $r$ and $s$, we have $(r+s)f(x) = rf(x)+sf(x)$ for all $x$ in $R$. Thus Axiom 9 holds.

(10). For all $f \in V$, $1f(x) = f(x)$ for all $x$ in $R$, thus Axiom 10 holds.

(d) Let $f_1$, $f_2$ and $f_3$ satisfy the linear second-order differential equation. In other words, $f_i'' + P(x)f_i' + Q(x)f_i = 0$, for $i = 1$, 2 and 3. Let $a$ and $b$ be real numbers.

(1) Axiom 1.
Let $g = f_1 + f_2$.
Then $g'' + P(x)g' + Q(x)g$
$= (f_1'' + f_2'') + (P(x)f_1' + P(x)f_2') + (Q(x)f_1 + Q(x)f_2)$
$= (f_1'' + P(x)f_1' + Q(x)f_1) + (f_2'' + P(x)f_2' + Q(x)f_2)$
$= 0 + 0$
$= 0.$

(2) Axiom 2.
Let $g = af$
Then $g'' + P(x)g' + Q(x)g$
$= af'' + aP(x)f' + aQ(x)f$
$= a(f'' + P(x)f' + Q(x)f)$
$= a0$
$= 0.$

(3) Axiom 3.
Since $(f_1 + f_2)(x) = f_1(x) + f_2(x) = f_2(x) + f_1(x) = (f_2 + f_1)(x)$,
$f_1 + f_2 = f_2 + f_1$.

(4) Axiom 4.
Since $(f_1(x)+f_2(x))+f_3(x) = f_1(x)+f_2(x)+f_3(x) = f_1(x)+(f_2(x)+f_3(x))$,
$(f_1 + f_2) + f_3 = f_1 + (f_2 + f_3)$.

(5) Axiom 5.
Let $O(x)$ be the zero function.Then $O''(x) = O'(x) = O(x) = 0$. Thus $O'' + P(x)O' + Q(x)O = 0 + 0 + 0 = 0$ and $O(x)$ is in $V$.
Since $f(x) + 0 = f(x)$, $f + O = f$.

(6) Axiom 6.
By(2), $(-1)f$ is in $V$. Since $f(x) + (-1)(f(x)) = 0$, $f + (-1)f = O$.

(7) Axiom 7.
Since $a(bf(x)) = abf(x) = (ab)f(x)$, $a(bf) = (ab)f$.

(8) Axiom 8.
Since $a(f_1(x) + f_2(x)) = af_1(x) + af_2(x)$, $a(f_1 + f_2) = af_1 + af_2$.

(9) Axiom 8.
Since $(a + b)f(x) = af(x) + bf(x)$, $(a + b)f = af + bf$.

(10) Axiom 10.
Since $1f(x) = f(x)$, $1f = f$.