EE203001 Linear Algebra

Solutions to Homework #13 Spring Semester, 2003

2. (a) $\begin{vmatrix} 2x & 2y & 2z \\ 3/2 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 2\begin{vmatrix} x & y & z \\ 3/2 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} x & y & z \\ 3 & 0 & 2 \\ 1 & 1 & 1 \end{vmatrix} = 1.$

(b) $\begin{vmatrix} x & y & z \\ 3x+3 & 3y & 3z+2 \\ x+1 & y+1 & z+1 \end{vmatrix} = \begin{vmatrix} x & y & z \\ 3 & 0 & 2 \\ x+1 & y+1 & z+1 \end{vmatrix} = \begin{vmatrix} x & y & z \\ 3 & 0 & 2 \\ 1 & 1 & 1 \end{vmatrix} = 1.$

(c) $\begin{vmatrix} x-1 & y-1 & z-1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} x & y & z \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} x & y & z \\ 3 & 0 & 2 \\ 1 & 1 & 1 \end{vmatrix} = 1.$

3.(b) i. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{bmatrix}$. We have

$$\begin{aligned}
\det A &= \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^3-a^3 & c^3-a^3 \end{bmatrix} \\
&= \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & (c^3-a^3)-(c-a)(b^2+ab+a^2) \end{bmatrix} \\
&= 1\cdot(b-a)[(c-a)(c^2+ac+a^2)-(c-a)(b^2+ab+a^2)] \\
&= (b-a)(c-a)(c^2+ac+a^2-b^2-ab-a^2) \\
&= (b-a)(c-a)(c^2-b^2+ac-ab) \\
&= (b-a)(c-a)[(c+b)(c-b)+a(c-b)] \\
&= (b-a)(c-a)(c-b)(a+b+c).
\end{aligned}$$

ii. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{bmatrix}$. We have

$$\begin{aligned}
\det A &= \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & b^2-a^2 & c^2-a^2 \\ 0 & b^3-a^3 & c^3-a^3 \end{bmatrix} \\
&= \frac{1}{b+a} \cdot \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & (b-a)(b+a) & c^2-a^2 \\ 0 & (b-a)(b^2+ab+a^2)(b+a) & (c^3-a^3)(b+a) \end{bmatrix} \quad \text{if } a+b \neq 0 \\
&= \frac{1}{b+a} \cdot \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & (b-a)(b+a) & c^2-a^2 \\ 0 & 0 & (c^3-a^3)(b+a)-(c^2-a^2)(b^2+ab+a^2) \end{bmatrix} \\
&= \frac{1}{b+a}(b-a)(b+a) \cdot \\
&\quad [(c-a)(c^2+ac+a^2)(b+a)-(c-a)(c+a)(b^2+ab+a^2)] \\
&= (b-a)(c-a)[(c^2+ac+a^2)(b+a)-(c+a)(b^2+ab+a^2)] \\
&= (b-a)(c-a) \cdot \\
&\quad (bc^2+abc+a^2b+ac^2+a^2c+a^3-b^2c-abc-a^2c-ab^2-a^2b-a^3) \\
&= (b-a)(c-a)(bc^2-b^2c+ac^2-ab^2) \\
&= (b-a)(c-a)[(bc)(c-b)+a(c+b)(c-b)] \\
&= (b-a)(c-a)(c-b)(bc+ac+ab). \\
\text{If } a+b &= 0, \\
\det A &= \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & (b-a)(b+a) & c^2-a^2 \\ 0 & b^3-a^3 & c^3-a^3 \end{bmatrix} \\
&= \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & (b-a)(b+a) & c^2-a^2 \\ 0 & (b-a)(b^2+2ab+a^2-ab) & (c^3-a^3)(b+a) \end{bmatrix} \\
&= \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & c^2-a^2 \\ 0 & (b-a)(-ab) & (c^3-a^3)(b+a) \end{bmatrix} \\
&= -\det \begin{bmatrix} 1 & 1 & 1 \\ 0 & (b-a)(-ab) & (c^3-a^3)(b+a) \\ 0 & 0 & c^2-a^2 \end{bmatrix} \\
&= -1(b-a)(-ab)(c^2-a^2) \\
&= (b-a)(c-a)(c+a)(ab) = (b-a)(c-a)(c-b)(ab) \\
&= (b-a)(c-a)(c+a)(bc+ac+ab).
\end{aligned}$$

4. Let C_1 be obtained from A by multiplying the first row of A by c, C_2 be obtained from C_1 by multiplying the second row of C_1 by c, ..., C_n be obtained from C_{n-1} by multiplying the n-th row of C_{n-1} by c. Thus $C_n = cA$ and $\det(cA) = \det C_n = c \det C_{n-1} = c(c \det C_{n-2}) = \cdots = c^{n-1} \det C_1 = c^n \det A$.

5. (b). Add $-a$ times row 1 to row 2, add $-a^2$ times to row 3, and add $-a^3$ times to row 4,

$$D = \begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & b-a & c-a & d-a \\ 0 & b^2-a^2 & c^2-a^2 & d^2-a^2 \\ 0 & b^3-a^3 & c^3-a^3 & d^3-a^3 \end{vmatrix}.$$

Add $-(b+a)$ times row 2 to row 3 and add $-(b^2+ab+a^2)$ times row 2 to row 4,

$$\begin{aligned} D &= \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & b-a & c-a & d-a \\ 0 & 0 & c^2-a^2-(c-a)(b+a) & d^2-a^2-(d-a)(b+a) \\ 0 & 0 & c^3-a^3-(c-a)(b^2+ab+a^2) & d^3-a^3-(d-a)(b^2+ab+a^2) \end{vmatrix} \\ &= \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & b-a & c-a & d-a \\ 0 & 0 & (c-a)(c-b) & (d-a)(d-b) \\ 0 & 0 & (c-a)(c-b)(a+b+c) & (d-a)(d-b)(a+b+d) \end{vmatrix}. \end{aligned}$$

Add $-(a+b+c)$ times row 3 to row 4,

$$\begin{aligned} D &= \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & b-a & c-a & d-a \\ 0 & 0 & (c-a)(c-b) & (d-a)(d-b) \\ 0 & 0 & 0 & (d-a)(d-b)(d-c) \end{vmatrix} \\ &= (b-a)(c-a)(c-b)(d-a)(d-b)(d-c). \end{aligned}$$

(e). Add (-1) times row 1 to the other rows

$$D = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -2 & -2 & -2 \\ 0 & 0 & -2 & -2 & 0 & 0 \\ 0 & -2 & -2 & 0 & -2 & 0 \\ 0 & -2 & 0 & -2 & 0 & 0 \\ 0 & -2 & -2 & 0 & 0 & -2 \end{vmatrix}.$$

Swap row 2 and row 4

$$D = -\begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 0 & -2 & 0 \\ 0 & 0 & -2 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 & -2 & -2 \\ 0 & -2 & 0 & -2 & 0 & 0 \\ 0 & -2 & -2 & 0 & 0 & -2 \end{vmatrix}.$$

Add (-1) times row 2 to row 5 and row 6

$$D = -\begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 0 & -2 & 0 \\ 0 & 0 & -2 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 & -2 & -2 \\ 0 & 0 & 2 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 \end{vmatrix}.$$

Add 1 times row 3 to row 5,

$$D = -\begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 0 & -2 & 0 \\ 0 & 0 & -2 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 & -2 & -2 \\ 0 & 0 & 0 & -4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 \end{vmatrix}.$$

Add -2 times row 4 to row 5,

$$D = -\begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 0 & -2 & 0 \\ 0 & 0 & -2 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 & -2 & -2 \\ 0 & 0 & 0 & 0 & 6 & 4 \\ 0 & 0 & 0 & 0 & 2 & -2 \end{vmatrix}.$$

Swap row 5 and row 6,

$$D = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 0 & -2 & 0 \\ 0 & 0 & -2 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 & -2 & -2 \\ 0 & 0 & 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 0 & 6 & 4 \end{vmatrix}.$$

Add -3 times row 5 to row 6

$$D = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 0 & -2 & 0 \\ 0 & 0 & -2 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 & -2 & -2 \\ 0 & 0 & 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 & 10 \end{vmatrix} = 1(-2)(-2)(-2)(2)(10) = -160.$$

6. (a) i. Since U and V are $n \times n$ upper triangular matrices, we have $u_{ij} = v_{ij} = 0$ $\forall n \geq i > j \geq 1$. Let $W = U + V$. Then $\forall n \geq i > j \geq 1$, $w_{ij} = u_{ij} + v_{ij} = 0$. Thus $U + V$ is an upper triangular matrix.

   ii. Let $D = UV$ and d_{ij}, u_{ij}, v_{ij} $(i, j = 1, 2, \ldots, n)$ be the entries of D, U, V. Since U and V are upper triangular matrices, $u_{ij} = v_{ij} = 0$ when $i > j$. Consider $d_{ij} = \sum_{k=1}^{n} u_{ik}v_{kj}$ with $i > j$. If $i > k$, then $u_{ik} = 0$. If $k > j$, then $v_{kj} = 0$. Since $i > j$, each k, $1 \leq k \leq n$, is either $i > k$ or $k > j$. Thus $u_{ik}v_{kj} = 0$ and then $d_{ij} = 0$. Thus UV is an upper triangular matrix.

   (b) Since U and V are upper triangular matrices, $\det U = \Pi_{i=1}^{n} u_{ii}$ and $\det V = \Pi_{i=1}^{n} v_{ii}$. Let $D = UV$. Then from (a), D is an upper triangular matrix. So $\det D = \Pi_{i=1}^{n} d_{ii}$, and $d_{ii} = \sum_{k=1}^{n} u_{ik}v_{ki} = u_{ii}v_{ii}$, Thus $\det D = \Pi_{i=1}^{n} u_{ii}v_{ii} =(\det U)(\det V)$.

(c) First $\left[\begin{array}{cccc|cccc} u_{11} & u_{12} & \cdots & u_{1n} & 1 & 0 & \cdots & 0 \\ 0 & u_{22} & \cdots & u_{2n} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} & 0 & 0 & \cdots & 1 \end{array}\right]$ with each u_{ii} nonzero since det$U \neq 0$.

We can apply a sequence of elementary row operations to get $\left[\begin{array}{cccc|cccc} u_{11} & 0 & \cdots & 0 & 1 & u'_{12} & \cdots & u'_{1n} \\ 0 & u_{22} & \cdots & 0 & 0 & 1 & \cdots & u'_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} & 0 & 0 & \cdots & 1 \end{array}\right]$. Then divide first row by u_{11}, second row by u_{22}, ... , nth row by u_{nn}, we have $\left[\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & \frac{1}{u_{11}} & u_{12}^{-1} & \cdots & u_{1n}^{-1} \\ 0 & 1 & \cdots & 0 & 0 & \frac{1}{u_{22}} & \cdots & u_{2n}^{-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & \frac{1}{u_{nn}} \end{array}\right]$.

Then $U^{-1} = \begin{bmatrix} \frac{1}{u_{11}} & u_{12}^{-1} & \cdots & u_{1n}^{-1} \\ 0 & \frac{1}{u_{22}} & \cdots & u_{2n}^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{u_{nn}} \end{bmatrix}$, which is an upper triangular matrix, too.

Thus det(U^{-1})=$\Pi_{i=1}^{n}(1/u_{ii}) = 1/detU$.

7. By Theorem 5.5, we have det $A = 2\cdot 2\cdot 2\cdot 2 = 16$. Then we apply Gauss-Jordan process

for $[A|I]$.

$$\left[\begin{array}{cccc|cccc} 2 & 3 & 4 & 5 & 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 4 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 2 & 3 & 4 & 5 & 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 4 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1/2 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|cccc} 2 & 3 & 4 & 0 & 1 & 0 & 0 & -5/2 \\ 0 & 2 & 3 & 0 & 0 & 1 & 0 & -4/2 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & -3/2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1/2 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|cccc} 2 & 3 & 4 & 0 & 1 & 0 & 0 & -5/2 \\ 0 & 2 & 3 & 0 & 0 & 1 & 0 & -4/2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1/2 & -3/4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1/2 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|cccc} 2 & 3 & 0 & 0 & 1 & 0 & -4/2 & 1/2 \\ 0 & 2 & 0 & 0 & 0 & 1 & -3/2 & 1/4 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1/2 & -3/4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1/2 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|cccc} 2 & 3 & 0 & 0 & 1 & 0 & -4/2 & 1/2 \\ 0 & 1 & 0 & 0 & 0 & 1/2 & -3/4 & 1/8 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1/2 & -3/4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1/2 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|cccc} 2 & 0 & 0 & 0 & 1 & -3/2 & 1/4 & 1/8 \\ 0 & 1 & 0 & 0 & 0 & 1/2 & -3/4 & 1/8 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1/2 & -3/4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1/2 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/2 & -3/4 & 1/8 & 1/16 \\ 0 & 1 & 0 & 0 & 0 & 1/2 & -3/4 & 1/8 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1/2 & -3/4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1/2 \end{array}\right].$$

Hence $A^{-1} = \begin{bmatrix} 1/2 & -3/4 & 1/8 & 1/16 \\ 0 & 1/2 & -3/4 & 1/8 \\ 0 & 0 & 1/2 & -3/4 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$, and we know $\det(A^{-1}) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{16}$.

8. We follow the steps of the proof of Theorem 5.5.

Let $A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n_2} & \cdots & a_{nn} \end{bmatrix}$ be a lower triangular matrix.

If $a_{11} = 0$, then row 1 is a zero vector, thus $\det A = 0 = a_{11} \cdots a_{nn}$, by Theorem 5.1.

If $a_{11} \neq 0$, multiplying row 1 by $-a_{i1}/a_{11}$ then add it to row i, for $i = 2, \ldots, n$, we obtain a matrix A' whose entries below a_{11} are all zeros, the other entries are unchanged, and $\det A = \det A'$.

If $a'_{22} = a_{22} = 0$, then row 2 is a zero vector, hence $\det A = \det A' = 0 = a_{11} \cdots a_{nn}$.

If $a'_{22} = a_{22} \neq 0$, multiplying row 2 of A' by $-a'_{i2}/a'_{22}$ then add it to row i, for $i = 3, \ldots, n$, we obtain an matrix A'' whose entries below a'_{11}, a'_{22} are zeros, and $\det A = \det A' = \det A''$.

We continue this process. If no diagonal entry is zero, we eventually arrive at a diagonal matrix whose determinant is the product of the diagonal entries. And if some diagonal entry $a_{ii} = 0$, the process will produce a matrix with a row of zeros, in which case $\det A = a_{11}a_{22} \cdots a_{nn}$ holds trivially.

10.

$$\begin{aligned}
F(x) &= \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} \\
&= f_1(x)\begin{vmatrix} g_2(x) & g_3(x) \\ h_2(x) & h_3(x) \end{vmatrix} - f_2(x)\begin{vmatrix} g_1(x) & g_3(x) \\ h_1(x) & h_3(x) \end{vmatrix} + f_3(x)\begin{vmatrix} g_1(x) & g_2(x) \\ h_1(x) & h_2(x) \end{vmatrix} \\
&= f_1(x)F_1(X) - f_2(x)F_2(x) + f_2(x)F_3(x),
\end{aligned}$$

where

$$F_1(x) = \begin{vmatrix} g_2(x) & g_3(x) \\ h_2(x) & h_3(x) \end{vmatrix}, \quad F_2(x) = \begin{vmatrix} g_1(x) & g_3(x) \\ h_1(x) & h_3(x) \end{vmatrix}, \quad F_3(x) = \begin{vmatrix} g_1(x) & g_2(x) \\ h_1(x) & h_2(x) \end{vmatrix}.$$

Then by (9),

$$\begin{aligned}
F'(x) &= f'_1(x)F_1(x) + f_1(x)F'_1(x) - f'_2(x)F_2(x) - f_2(x)F'_3(x) + f'_3(x)F_3(x) + f_3(x)F'_3(x) \\
&= f'_1(x)\begin{vmatrix} g_2(x) & g_3(x) \\ h_2(x) & h_3(x) \end{vmatrix} + f_1(x)\Bigg(\begin{vmatrix} g'_2(x) & g'_3(x) \\ h_2(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} g_2(x) & g_3(x) \\ h'_2(x) & h'_3(x) \end{vmatrix}\Bigg) \\
&\quad -f'_2(x)\begin{vmatrix} g_1(x) & g_3(x) \\ h_1(x) & h_3(x) \end{vmatrix} - f_2(x)\Bigg(\begin{vmatrix} g'_1(x) & g'_3(x) \\ h_1(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} g_1(x) & g_3(x) \\ h'_1(x) & h'_3(x) \end{vmatrix}\Bigg) \\
&\quad +f'_3(x)\begin{vmatrix} g_1(x) & g_2(x) \\ h_1(x) & h_2(x) \end{vmatrix} + f_3(x)\Bigg(\begin{vmatrix} g'_1(x) & g'_2(x) \\ h_1(x) & h_2(x) \end{vmatrix} + \begin{vmatrix} g_1(x) & g_2(x) \\ h'_1(x) & h'_2(x) \end{vmatrix}\Bigg).
\end{aligned}$$

Rearranging the right hand side of the above equation, we have

$$\begin{aligned}
F'(x) &= \Bigg(f'_1(x)\begin{vmatrix} g_2(x) & g_3(x) \\ h_2(x) & h_3(x) \end{vmatrix} - f'_2(x)\begin{vmatrix} g_1(x) & g_3(x) \\ h_1(x) & h_3(x) \end{vmatrix} + f'_3(x)\begin{vmatrix} g_1(x) & g_2(x) \\ h_1(x) & h_2(x) \end{vmatrix}\Bigg) \\
&\quad \Bigg(+ f_1(x)\begin{vmatrix} g'_2(x) & g'_3(x) \\ h_2(x) & h_3(x) \end{vmatrix} - f_2(x)\begin{vmatrix} g'_1(x) & g'_3(x) \\ h_1(x) & h_3(x) \end{vmatrix} + f_3(x)\begin{vmatrix} g'_1(x) & g'_2(x) \\ h_1(x) & h_2(x) \end{vmatrix}\Bigg) \\
&\quad \Bigg(+ f_1(x)\begin{vmatrix} g_2(x) & g_3(x) \\ h'_2(x) & h'_3(x) \end{vmatrix} - f_2(x)\begin{vmatrix} g_1(x) & g_3(x) \\ h'_1(x) & h'_3(x) \end{vmatrix} + f_3(x)\begin{vmatrix} g_1(x) & g_2(x) \\ h'_1(x) & h'_2(x) \end{vmatrix}\Bigg) \\
&= \begin{vmatrix} f'_1(x) & f'_2(x) & f'_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g'_1(x) & g'_2(x) & g'_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h'_1(x) & h'_2(x) & h'_3(x) \end{vmatrix}.
\end{aligned}$$

11. (a) $F(x) = \begin{vmatrix} f_1(x) & f_2(x) \\ f'_1(x) & f'_2(x) \end{vmatrix} = f_1(x)f'_2(x) - f'_1(x)f_2(x)$, then $F'(x) = f'_1(x)f'_2(x) + f_1(x)f''_2(x) - f''_1(x)f_2(x) - f'_1(x)f'_2(x) = f_1(x)f''_2(x) - f''_1(x)f_2(x) = \begin{vmatrix} f_1(x) & f_2(x) \\ f''_1(x) & f''_2(x) \end{vmatrix}$.

(b) If $F(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ f_1'(x) & f_2'(x) & f_3'(x) \\ f_1''(x) & f_2''(x) & f_3''(x) \end{vmatrix}$, then $F'(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ f_1'(x) & f_2'(x) & f_3'(x) \\ f_1'''(x) & f_2'''(x) & f_3'''(x) \end{vmatrix}$.

Proof:

$$\begin{aligned} F(x) &= \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ f_1'(x) & f_2'(x) & f_3'(x) \\ f_1''(x) & f_2''(x) & f_3''(x) \end{vmatrix} \\ &= f_1(x)\begin{vmatrix} f_2'(x) & f_3'(x) \\ f_2''(x) & f_3''(x) \end{vmatrix} - f_2(x)\begin{vmatrix} f_1'(x) & f_3'(x) \\ f_1''(x) & f_3''(x) \end{vmatrix} + f_3(x)\begin{vmatrix} f_1'(x) & f_2'(x) \\ f_1''(x) & f_2''(x) \end{vmatrix}, \end{aligned}$$

then $F'(x) = f_1'(x)\begin{vmatrix} f_2'(x) & f_3'(x) \\ f_2''(x) & f_3''(x) \end{vmatrix} + f_1(x)\begin{vmatrix} f_2'(x) & f_3'(x) \\ f_2'''(x) & f_3'''(x) \end{vmatrix} - f_2'(x)\begin{vmatrix} f_1'(x) & f_3'(x) \\ f_1''(x) & f_3''(x) \end{vmatrix} - f_2(x)\begin{vmatrix} f_1'(x) & f_3'(x) \\ f_1'''(x) & f_3'''(x) \end{vmatrix} + f_3'(x)\begin{vmatrix} f_1'(x) & f_2'(x) \\ f_1''(x) & f_2''(x) \end{vmatrix} + f_3(x)\begin{vmatrix} f_1'(x) & f_2'(x) \\ f_1'''(x) & f_2'''(x) \end{vmatrix}$.

But $f_1'(x)\begin{vmatrix} f_2'(x) & f_3'(x) \\ f_2''(x) & f_3''(x) \end{vmatrix} - f_2'(x)\begin{vmatrix} f_1'(x) & f_3'(x) \\ f_1''(x) & f_3''(x) \end{vmatrix} + f_3'(x)\begin{vmatrix} f_1'(x) & f_2'(x) \\ f_1''(x) & f_2''(x) \end{vmatrix} = \begin{vmatrix} f_1'(x) & f_2'(x) & f_3'(x) \\ f_1'(x) & f_2'(x) & f_3'(x) \\ f_1''(x) & f_2''(x) & f_3''(x) \end{vmatrix} = 0$, so $F'(x) = f_1(x)\begin{vmatrix} f_2'(x) & f_3'(x) \\ f_2'''(x) & f_3'''(x) \end{vmatrix} - f_2(x)\begin{vmatrix} f_1'(x) & f_3'(x) \\ f_1'''(x) & f_3'''(x) \end{vmatrix} + f_3(x)\begin{vmatrix} f_1'(x) & f_2'(x) \\ f_1'''(x) & f_2'''(x) \end{vmatrix} = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ f_1'(x) & f_2'(x) & f_3'(x) \\ f_1'''(x) & f_2'''(x) & f_3'''(x) \end{vmatrix}$.