EE203001 Linear Algebra
Solutions for Homework #11 Spring Semester, 2003

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1. Let $A = [a_{ij}]$ be an $n \times n$ matrix having a row of zeros or a column of zeros. Suppose A is nonsingular with inverse $B = [b_{ij}]$ such that $AB = BA = I_{n\times n}$.

i. If the i-th row of A is a row of zeros, ($a_{ij} = 0$ for $j = 1, \ldots, n$), then the i-th row of AB must be a row of zeros, since the ik-entry of AB is given by $\sum_{j=1}^{n} a_{ij}b_{jk} = \sum_{j=1}^{n} 0 \cdot b_{jk} = 0$, for $k = 1, \ldots, n$.

ii. If the j-th column of A is a column of zeros, ($a_{ij} = 0$ for $i = 1, \ldots, n$), then the j-th column of BA must be a column of zeros, since the kj-entry of BA is given by $\sum_{i=1}^{n} b_{ki}a_{ij} = \sum_{j=1}^{n} b_{ki} \cdot 0 = 0$, for $k = 1, \ldots, n$.

Both cases contradict to $AB = BA = I$.

3. We have to find a nonsigular matrix P such that $AP = P\begin{bmatrix} 6 & 0 \\ 0 & -1 \end{bmatrix}$. Let $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then we have $\begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 6 & 0 \\ 0 & -1 \end{bmatrix}$. Thus,

$$a + 2c = 6a, b + 2d = -b, 5a + 4c = 6c, 5b + 4d = -d.$$

$\Rightarrow 5a = 2c, b = -d$. Choose $a = 2, b = 1, c = 5$, and $d = -1$, then $ad - bc = -2 - 5 = -7 \neq 0$. Thus $P = \begin{bmatrix} 2 & 1 \\ 5 & -1 \end{bmatrix}$ is nonsigular and hence it is a solution.

5. We prove it by induction.

(a) When $n = 1$, $(A + I)^1 = A + I = I + A = I + (2^1 - 1)A$.

(b) For $n = k$, we assume $(A + I)^k = I + (2^k - 1)A$.

(c) When $n = k + 1$, we have

$$\begin{aligned} (A + I)^{k+1} &= (A + I)(A + I)^k \\ &= (A + I)(I + (2^k - 1)A) \\ &= A + I + (2^k - 1)A^2 + (2^k - 1)A \\ &= I + 2(2^k - 1)A + A \\ &= I + (2^{k+1} - 1)A \end{aligned}$$

6. By Definition, *Lorentz transformation* $L(v) = \frac{c}{\sqrt{c^2-v^2}}\begin{bmatrix} 1 & -v \\ -vc^{-2} & 1 \end{bmatrix}$ and $L(u) = \frac{c}{\sqrt{c^2-u^2}}\begin{bmatrix} 1 & -u \\ -uc^{-2} & 1 \end{bmatrix}$. We have

$$
\begin{aligned}
L(v)L(u) &= \frac{c}{\sqrt{c^2-v^2}}\begin{bmatrix} 1 & -v \\ -vc^{-2} & 1 \end{bmatrix}\frac{c}{\sqrt{c^2-u^2}}\begin{bmatrix} 1 & -u \\ -uc^{-2} & 1 \end{bmatrix} \\
&= \frac{c}{\sqrt{c^2-v^2}}\cdot\frac{c}{\sqrt{c^2-u^2}}\begin{bmatrix} 1 & -v \\ -vc^{-2} & 1 \end{bmatrix}\begin{bmatrix} 1 & -u \\ -uc^{-2} & 1 \end{bmatrix} \\
&= \frac{c^2}{\sqrt{c^2-v^2}\sqrt{c^2-u^2}}\begin{bmatrix} 1+uvc^{-2} & -v-u \\ -vc^{-2}-uc^{-2} & 1+uvc^{-2} \end{bmatrix} \\
&= \frac{c^2\cdot(1+uvc^{-2})}{\sqrt{c^2-v^2}\sqrt{c^2-u^2}}\begin{bmatrix} 1 & \frac{-v-u}{1+uvc^{-2}} \\ \frac{-vc^{-2}-uc^{-2}}{1+uvc^{-2}} & 1 \end{bmatrix} \\
&= \frac{c^2+uv}{\sqrt{c^2-v^2}\sqrt{c^2-u^2}}\begin{bmatrix} 1 & -\frac{(v+u)c^2}{c^2+uv} \\ -\frac{v+u}{c^2+uv} & 1 \end{bmatrix} \\
&= \frac{c^2+uv}{\sqrt{c^2-v^2}\sqrt{c^2-u^2}}\begin{bmatrix} 1 & -w \\ -wc^{-2} & 1 \end{bmatrix}.
\end{aligned}
$$

We know that a of the $L(w)$ must be

$$
\begin{aligned}
\frac{c}{\sqrt{c^2-w^2}} &= \frac{c}{\sqrt{c^2-\frac{(u+v)^2c^4}{(c^2+uv)^2}}} \\
&= \frac{c}{c\sqrt{\frac{(c^2+uv)^2-(u+v)^2c^2}{(c^2+uv)^2}}} \\
&= \frac{c^2+uv}{\sqrt{c^4+2uvc^2+u^2v^2-u^2c^2-v^2c^2-2uvc^2}} \\
&= \frac{c^2+uv}{\sqrt{c^4-v^2c^2-u^2c^2+u^2v^2}} \\
&= \frac{c^2+uv}{\sqrt{(c^2-v^2)(c^2-u^2)}} \\
&= \frac{c^2+uv}{\sqrt{c^2-v^2}\sqrt{c^2-u^2}}.
\end{aligned}
$$

Hence $L(v)L(u) = L(w)$.

9. (a) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. A and B are orthogonal. Then $A+B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, which is obviously non-orthogonal.

(b) First, we want to know what dose $(AB)^t$ look like. Let $C = AB$. Then $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$. Then $c^t_{ij} = c_{ji} = \sum_{k=1}^{n} a_{jk}b_{ki} = \sum_{k=1}^{n} b^t_{ik}a^t_{kj}$. Thus we get $(AB)^t = B^tA^t$. Since A and B are orthogonal,
$\Rightarrow (AB)(AB)^t = (AB)(B^tA^t) = A(BB^t)A^t = AIA^t = AA^t = I$.
Thus AB is orthogonal.

(c) $(AB)(AB)^t = I$ (since AB is orthogonal.)
$\Rightarrow ABB^tA^t = I$
$\Rightarrow A^tABB^tA^t = A^t$

$\Rightarrow BB^tA^t = A^t$ (since A is orthogonal)
$\Rightarrow BB^t = I$.
Thus B is orthogonal.

10. (a). Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, where $|a_{ij}| = 1$, for $i, j = 1, 2$, and $a_{11}a_{21} + a_{12}a_{22} = 0$.
Multiplying a_{12} to both sides of $a_{11}a_{21} + a_{12}a_{22} = 0$, we have $a_{11}a_{12}a_{21} + a_{22} = 0$.
Thus,
$a_{11} = 1, a_{12} = 1, a_{21} = 1 \Rightarrow a_{22} = -1$.
$a_{11} = 1, a_{12} = 1, a_{21} = -1 \Rightarrow a_{22} = 1$.
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$a_{11} = 1, a_{12} = -1, a_{21} = -1 \Rightarrow a_{22} = -1$.
$a_{11} = -1, a_{12} = 1, a_{21} = 1 \Rightarrow a_{22} = 1$.
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$a_{11} = -1, a_{12} = -1, a_{21} = -1 \Rightarrow a_{22} = 1$.

(b). Proof of Lemma 4.22.

$$\begin{aligned} (X+Y)\cdot(X+Z) &= X\cdot X + X\cdot Z + Y\cdot X + Y\cdot Z \\ &= X\cdot X, \text{ (since dot product of any two of } X, Y, Z \text{ is zero)} \\ &= ||X||^2. \end{aligned}$$

Proof of Lemma 4.23.

$$(x_i + y_i)(x_i + z_i) = x_i^2 + x_iz_i + y_ix_i + y_iz_i = 1 + x_iz_i + y_ix_i + y_iz_i = 1 + M, \quad (1)$$

where $M = (y_iz_i + x_iy_i + x_iz_i)$. Then

$$\begin{aligned} M^2 &= (x_iz_i + y_ix_i + y_iz_i)^2 \\ &= x_i^2z_i^2 + y_i^2x_i^2 + y_i^2z_i^2 + 2x_i^2y_iz_i + 2x_iy_iz_i^2 + 2x_iy_i^2z_i \\ &= 3 + 2y_iz_i + 2x_iy_i + 2x_iz_i, \\ &= 3 + 2M. \end{aligned}$$

$M^2 - 2M - 3 = 0$ implies $M = 3$ or $M = -1$. Substitute M to equation (1), we have $(x_i + y_i)(x_i + z_i) = 0$ or 4.
Proof of the Theorem.
Let X, Y and Z be three distinct row vectors of an $n \times n$ Hadamard matrix A, $n \geq 3$. By II of the definition of Hadamard matrices and the Lemma 4.22,

$$\begin{aligned} (X+Y)\cdot(X+Z) &= ||X||^2 \\ &= (\sqrt{n})^2 = n. \end{aligned}$$

By Lemma 4.23, $(X+Y)\cdot(X+Z) = \sum_{i=1}^{n}(x_i + y_i)(x_i + z_i) = 4m$ for some $m \leq n, m \in \mathbb{Z}^+$, thus n is a multiple of 4.