EE203001 Linear Algebra
Solutions to Homework #11 Spring Semester, 2003

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11. (a). Let $E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

($\Rightarrow$).
A commutes with every 2×2 matrix, it commutes with the four matrices, of course.

($\Leftarrow$).
Let $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ be any 2×2 matrix.

$$\begin{aligned}
AB &= A\Big(b_{11}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b_{21}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + b_{12}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + b_{22}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\Big) \\
&= b_{11}A\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b_{21}A\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + b_{12}A\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + b_{22}A\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\
&= b_{11}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}A + b_{21}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}A + b_{12}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}A + b_{22}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}A \\
&= \Big(b_{11}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b_{21}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + b_{12}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + b_{22}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\Big)A \\
&= BA
\end{aligned}$$

(b). $AE_1 = E_1A$

$\Leftrightarrow \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$\Leftrightarrow \begin{bmatrix} a_{11} & 0 \\ a_{21} & 0 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ 0 & 0 \end{bmatrix}$

$\Leftrightarrow a_{12} = a_{21} = 0.$

$AE_2 = E_2A$

$\Leftrightarrow \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$\Leftrightarrow \begin{bmatrix} a_{12} & 0 \\ a_{22} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ a_{11} & a_{12} \end{bmatrix}$

$\Leftrightarrow a_{12} = 0, a_{22} = a_{11}.$

$AE_3 = E_3A$

$\Leftrightarrow \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$\Leftrightarrow \begin{bmatrix} 0 & a_{11} \\ 0 & a_{21} \end{bmatrix} = \begin{bmatrix} a_{21} & a_{22} \\ 0 & 0 \end{bmatrix}$

$\Leftrightarrow a_{11} = a_{22}, a_{21} = 0.$

$AE_4 = E_4 A$
$\Leftrightarrow \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$
$\Leftrightarrow \begin{bmatrix} 0 & a_{12} \\ 0 & a_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ a_{21} & a_{22} \end{bmatrix}$
$\Leftrightarrow a_{12} = a_{21} = 0.$

So A commutes with every 2×2 matrix
$\Leftrightarrow a_{11} = a_{22}, a_{12} = a_{21} = 0.$
$\Leftrightarrow A = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}.$

14. (a). $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix},$

$A + B = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix},$

$(A+B)^2 = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}^2 = \begin{bmatrix} 3 & -6 \\ 6 & 15 \end{bmatrix},$

$A^2 = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 0 & 4 \end{bmatrix},$

$AB = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 2 & 4 \end{bmatrix},$

$B^2 = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix},$

$A^2 + 2AB + B^2 = \begin{bmatrix} 1 & -3 \\ 0 & 4 \end{bmatrix} + 2 \begin{bmatrix} 0 & -2 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & -7 \\ 7 & 16 \end{bmatrix},$

$\Rightarrow A^2 + 2AB + B^2 \neq (A+B)^2.$

$A - B = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$

$(A+B)(A-B) = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -4 & -1 \end{bmatrix},$

$A^2 - B^2 = \begin{bmatrix} 1 & -3 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix},$

$\Rightarrow (A+B)(A-B) \neq A^2 - B^2.$

(b). $(A+B)^2 = (A+B)(A+B) = A^2 + AB + BA + B^2$, $(A+B)(A-B) = A^2 - AB + BA - B^2$.

(c). If $AB = BA$, identities in (b) can be simplified as identities in (a).

15. (a). Since $(A-B)^2 = A^2 - AB - BA + B^2$ and $(A+B)^2 = A^2 + AB + BA + B^2$, if $(A+B)^2 = (A-B)^2$ then $AB = -BA$. Thus, $A^2B = (AA)B = A(AB) = A(-BA) = (-AB)A = (BA)A = BA^2$.

(b). Let $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, then $A^2 = I$ but $A \neq I$ and $A \neq I$.

9. The augmented matrix of the system $x+y+2z=2$, $2x-y+3z=2$, $5x-y+az=6$ is

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 2 & -1 & 3 & 2 \\ 5 & -1 & a & 6 \end{array}\right].$$

By Gauss-Jordan method, we have

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & -3 & -1 & -2 \\ 0 & -6 & (a-10) & -4 \end{array}\right],$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & 1/3 & 2/3 \\ 0 & -6 & (a-10) & -4 \end{array}\right],$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 5/3 & 4/3 \\ 0 & 1 & 1/3 & 2/3 \\ 0 & 0 & (a-8) & 0 \end{array}\right].$$

If $a \neq 8$, then z must be 0, and $x = 4/3$, $y = 2/3$.
If $a = 8$, then $(x,y,z) = (\frac{4}{3} - \frac{5}{3}z, \frac{2}{3} - \frac{1}{3}z, z) = (\frac{4}{3}, \frac{2}{3}, 0) + z(-\frac{5}{3}, -\frac{1}{3}, 1)$.

11. We multiply the two matrixs $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & -ab+ab \\ cd-cd & -bc+ad \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = (ad-bc)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = (ad-bc)I$.

For $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, If $a \neq 0$, then we can apply Gauss-Jordan process as follows:

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ c & d & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & \frac{ad-bc}{a} & -\frac{c}{a} & 1 \end{array}\right].$$

Thus $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is nonsingular if and only if $ad-bc \neq 0$ and then applying Gauss-Jordan process again, we have

$$\left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right] \Rightarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

which says that its inverse is $\frac{1}{(ad-bc)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

If $a = 0$, we have $\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cc|cc} c & d & 0 & 1 \\ 0 & b & 1 & 0 \end{array}\right]$ by interchanging two equations.

To be able to apply Gauss-Jordan process further, i.e., $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is nonsingular if and only if $c \neq 0$, $b \neq 0$(i.e. $ad - bc \neq 0$), and then

$$\left[\begin{array}{cc|cc} 1 & \frac{d}{c} & 0 & \frac{1}{c} \\ 0 & 1 & \frac{1}{b} & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|cc} 1 & 0 & -\frac{d}{bc} & \frac{1}{c} \\ 0 & 1 & \frac{1}{b} & 0 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

since $a = 0$, which says that its inverse is

$$\frac{1}{(ad-bc)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

14.

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ -2 & 5 & -4 & 0 & 1 & 0 \\ 1 & -4 & 6 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 5 & 2 & 0 \\ 0 & 1 & -2 & 2 & 1 & 0 \\ 0 & 0 & 1 & 3 & 2 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 14 & 8 & 3 \\ 0 & 1 & 0 & 8 & 5 & 2 \\ 0 & 0 & 1 & 3 & 2 & 1 \end{array}\right]$$

Thus the inverse is $\begin{bmatrix} 14 & 8 & 3 \\ 8 & 5 & 2 \\ 3 & 2 & 1 \end{bmatrix}$.

2. (a). We have proved that $AB = -BA$ implies $A^2B = BA^2$ in 15(a), p.147. Thus $A^2B^3 = (A^2B)B^2 = (BA^2)B^2 = B(A^2B)B = B(BA^2)B = BB(A^2B) = BBBA^2 = B^3A^2$.

(b). Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. A and B are nonsingular, but $A+B = O$ is singular.

(c). If A and B are nonsingular, then $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = BB^{-1} = I$, i.e., AB is nonsingular.

(d) The statement is true. Since AB is nonsingular, if C is the inverse of AB, then $C(AB) = I = (AB)C$. Thus (CA) is a leftinverse of B and then B is nonsingular. Also (BC) is a rightinverse of A and then A is nonsingular by Theorem 4.20.

(e) The statement is not true.
Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$, then $A + B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. And the inverse of $A + B$ is $\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. But $A - B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, it is singular.

(f) The statement is not true. Let $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$, then $A^2 = O$. Thus if $A^2 = O$, A may not be O.

(g) We have $(A+I)(-A+I) = -A^2 + A - A + I = I$ since $A^2 = O$. Thus $(A+I)$ has $(-A+I)$ as its inverse. Hence $(A+I)$ is nonsigular.

(h) We have $(A-I)(-A^2-A-I) = -A^3 - A^2 - A + A^2 + A + I = I$ since $A^3 = O$. Thus $(A-I)$ has $(-A^2-A-I)$ as its inverse. Hence $(A-I)$ is nonsigular.

(i) $(A+2I)^2 = O \Rightarrow (A+2I)(A+2I) = O \Rightarrow A^2+4A+4I = O \Rightarrow A(A+4I) = -4I$. Thus A has $-(A+4I)/4$ as its inverse. Hence A is nonsigular.

(j) We prove it by induction.

i. When $k = 1$, $AB = BA$ as given.

ii. Assume $AB^k = B^kA$ when $n = k$.

iii. For $n = k+1$, we have $AB^{k+1} = AB^kB = B^kAB = B^kBA = B^{k+1}A$. Thus $AB^k = B^kA$ for every integer $k \geq 1$

4. Assume $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & b(a+d) \\ c(a+d) & bc+d^2 \end{bmatrix}$. Because $A^2 = A$, we have four equations:

(a) $a^2 + bc = a$.

(b) $b(a+d) = b$.

(c) $c(a+d) = c$.

(d) $bc + d^2 = d$.

(1) If $(a+d) = 1$, we have $a^2 + bc = a$, $b = b$, $c = c$, and $bc + (1-a)^2 = (1-a) \Rightarrow bc + 1 - 2a + a^2 = 1 - a \Rightarrow bc + a^2 = a$. In this case, $A = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$, where b and c are arbitrary and a is any solution of the quadratic equation $a^2 - a + bc = 0$.

(2) If $(a+d) \neq 1$, we have $b = 0$ and $c = 0$. Thus, $a^2 = a$ and $d^2 = d$. In this case, $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Be careful that $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is illegal because $(a+d) \neq 1$.

7. First we define the notation $[A]_{ij} = a_{ij}$ for matrix A. As for the definition of the transpose, we have $[A^t]_{ij} = [A]_{ji}$ and $a^t_{ij} = a_{ji}$.

(a) Because $[(A^t)^t]_{ij} = [A^t]_{ji} = [A]_{ij}$, we have $(A^t)^t = A$.

(b) Because $[(A+B)^t]_{ij} = [(A+B)]_{ji} = a_{ji} + b_{ji} = [A]_{ji} + [B]_{ji} = [A^t]_{ij} + [B^t]_{ij} = [A^t + B^t]_{ij}$, we have $(A+B)^t = A^t + B^t$.

(c) Because $[(cA)^t]_{ij} = [cA]_{ji} = ca_{ji} = c(a_{ji}) = c[A]_{ji} = c[A^t]_{ij} = [cA^t]_{ij}$, we have $(cA)^t = cA^t$.

(d) Assume A is $m \times n$ and B is $n \times p$. Define matrix $C = AB$. Then $[(AB)^t]_{ij} = [C^t]_{ij} = [C]_{ji} = c_{ji} = \sum_{x=1}^n a_{jx}b_{xi} = \sum_{x=1}^n b_{xi}a_{jx} = \sum_{x=1}^n b^t_{ix}a^t_{xj} = [B^tA^t]_{ij}$. Thus $(AB)^t = B^tA^t$.

(e) If A is nonsigular, A^{-1} exists. Then $(A^{-1})^t(A^t) = (AA^{-1})^t = I^t = I$ by (d).Thus we have $(A^t)^{-1} = (A^{-1})^t$ if A is nonsigular.

8. Define $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

(a)

$$
\begin{aligned}
AA^t &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \\
&= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= I
\end{aligned}
$$

(b) If A is a real orthogonal $n \times n$ matrix, $AA^t = I$. This means the dot product of different rows equals zero and that of the same row equals 1. Hence its rows form an orthonormal set.