EE203001 Linear Algebra
Solutions to Homework #10 Spring Semester, 2003

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1. (c) Let $\{e_1, e_2, ..., e_n\}$ be the standard basis of unit coordinate vectors in $\mathbb{R}^n$. Since the linear transformation T is the multiplication by a fixed scalar c, $T(e_1) = c \cdot e_1$, $T(e_2) = c \cdot e_2$, $\ldots$, $T(e_n) = c \cdot e_n$. Thus the matrix representation of this linear transformation relative to standard basis is

$$\begin{bmatrix} c & 0 & 0 & \cdots & 0 \\ 0 & c & 0 & \cdots & 0 \\ 0 & 0 & c & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c \end{bmatrix}.$$

3. (a) $T(3\boldsymbol{i} - 4\boldsymbol{j}) = 3T(\boldsymbol{i}) - 4T(\boldsymbol{j}) = (3\boldsymbol{i} + 3\boldsymbol{j}) - (8\boldsymbol{i} - 4\boldsymbol{j}) = -5\boldsymbol{i} + 7\boldsymbol{j}$.
$T^2(3\boldsymbol{i} - 4\boldsymbol{j}) = T(T(3\boldsymbol{i} - 4\boldsymbol{j})) = T(-5\boldsymbol{i} + 7\boldsymbol{j}) = -5(\boldsymbol{i} + \boldsymbol{j}) + 7(2\boldsymbol{i} - \boldsymbol{j}) = 9\boldsymbol{i} - 12\boldsymbol{j}$.

(b) Since $T(\boldsymbol{i}) = \boldsymbol{i} + \boldsymbol{j}$ and $T(\boldsymbol{j}) = 2\boldsymbol{i} - \boldsymbol{j}$, the matrix representation of T relative to the standard basis $\{\boldsymbol{i}, \boldsymbol{j}\}$ of $\mathbb{R}^2$ is

$$\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}.$$

And $T^2(\boldsymbol{i}) = T(T(\boldsymbol{i})) = T(\boldsymbol{i} + \boldsymbol{j}) = \boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{i} - \boldsymbol{j} = 3\boldsymbol{i}$,
$T^2(\boldsymbol{j}) = T(T(\boldsymbol{j})) = T(2\boldsymbol{i} - \boldsymbol{j}) = (2\boldsymbol{i} + 2\boldsymbol{j}) - (2\boldsymbol{i} - \boldsymbol{j}) = 3\boldsymbol{j}$.
The matrix representation of T^2 relative to the standard basis $\{\boldsymbol{i}, \boldsymbol{j}\}$ of $\mathbb{R}^2$ is

$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}.$$

(c) Since $e_1 = \boldsymbol{i} - \boldsymbol{j}$ and $e_2 = 3\boldsymbol{i} + \boldsymbol{j}$, $\boldsymbol{i} = \frac{e_1+e_2}{4}$ and $\boldsymbol{j} = \frac{e_2-3e_1}{4}$. Thus

$$T(e_1) = T(\boldsymbol{i}-\boldsymbol{j}) = (\boldsymbol{i}+\boldsymbol{j})-(2\boldsymbol{i}-\boldsymbol{j}) = -\boldsymbol{i}+2\boldsymbol{j} = \frac{-(e_1+e_2)+2(e_2-3e_1)}{4} = \frac{-7e_1+e_2}{4},$$

$$T(e_2) = T(3\boldsymbol{i}+\boldsymbol{j}) = 3(\boldsymbol{i}+\boldsymbol{j})+(2\boldsymbol{i}-\boldsymbol{j}) = 5\boldsymbol{i}+2\boldsymbol{j} = \frac{5(e_1+e_2)+2(e_2-3e_1)}{4} = \frac{-e_1+7e_2}{4}.$$

Hence the matrix representation of T relative to basis $\{e_1, e_2\}$ of $\mathbb{R}^2$ is

$$\begin{bmatrix} -\frac{7}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{7}{4} \end{bmatrix}.$$

And

$$
\begin{aligned}
T^2(e_1) &= T(T(\boldsymbol{i}-\boldsymbol{j})) = T(-\boldsymbol{i}+2\boldsymbol{j}) \\
&= -(\boldsymbol{i}+\boldsymbol{j}) + 2(2\boldsymbol{i}-\boldsymbol{j}) \\
&= 3\boldsymbol{i} - 3\boldsymbol{j} \\
&= \frac{3(e_1+e_2) - 3(e_2 - 3e_1)}{4} \\
&= \frac{12}{4} e_1 = 3e_1,
\end{aligned}
$$

$$
\begin{aligned}
T^2(e_2) &= T(T(3\boldsymbol{i}+\boldsymbol{j})) = T(5\boldsymbol{i}+2\boldsymbol{j}) \\
&= 5(\boldsymbol{i}+\boldsymbol{j}) + 2(2\boldsymbol{i}-\boldsymbol{j}) \\
&= 9\boldsymbol{i} + 3\boldsymbol{j} \\
&= \frac{9(e_1+e_2) + 3(e_2 - 3e_1)}{4} \\
&= \frac{12}{4} e_2 = 3e_2.
\end{aligned}
$$

Thus the matrix representation of T^2 relative to the basis $\{e_1, e_2\}$ of $\mathbb{R}^2$ is

$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}.$$

8. (a) Since T is a linear transform.
$T(2\boldsymbol{i} - 3\boldsymbol{j}) = 2T(\boldsymbol{i}) - 3T(\boldsymbol{j}) = 2(1,0,1) - 3(-1,0,1) = (5,0,-1)$.
Let $T(\boldsymbol{v}) = O$, $\boldsymbol{v} = a\boldsymbol{i} + b\boldsymbol{j}$ in R^2. Then $a(1,0,1) + b(-1,0,1) = O$.
$\Rightarrow a - b = 0$ and $a + b = 0$. $\Rightarrow a = 0$, $b = 0$.
Thus $\boldsymbol{v} = O$ is the only solution. Then the nullity $= 0$, and the rank $= 2$.

(b) Since $T(\boldsymbol{i}) = (1,0,1)^t$ and $T(\boldsymbol{j}) = (-1,0,1)^t$, the matrix of T relative to the standard bases of R^2 and R^3 is $\begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$.

(c) Note that the null space of T is trivial. Then we can select any basis of R^2 as $B = \{e_1, e_2\}$, says, $e_1 = \boldsymbol{i}$, $e_2 = \boldsymbol{j}$. Then we let $w_1 = T(e_1) = T(\boldsymbol{i}) = (1,0,1)^t$ and $w_2 = T(e_2) = T(\boldsymbol{j}) = (-1,0,1)^t$. It is clear that $\{w_1, w_2\}$ is a linearly independent set. We then let $w_3 = (0,1,0)^t$. Then $B' = \{w_1, w_2, w_3\}$ is a basis of R^3 and the matrix representation of T relative to B and B' is $[T]_{B \to B'} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

16. $D(\sin x) = \cos x$,
$D(\cos x) = -\sin x$,
$D(x \sin x) = \sin x + x \cos x$,
$D(x \cos x) = \cos x - x \sin x$,

$D^2(\sin x) = D(\cos x) = -\sin x,$
$D^2(\cos x) = D(-\sin x) = -\cos x,$
$D^2(x\sin x) = D(\sin x + x\cos x) = \cos x + \cos x - x\sin x = 2\cos x - x\sin x,$
$D^2(x\cos x) = D(\cos x - x\sin x) = -\sin x - \sin x - x\cos x = -2\sin x - x\cos x.$

The matrices of D and of D^2 relative to the given basis are

$$\begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} -1 & 0 & 0 & -2 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

18.

$$\begin{aligned}
D(e^{2x}\sin 3x) &= 2e^{2x}\sin 3x + 3e^{2x}\cos 3x, \\
D(e^{2x}\cos 3x) &= 2e^{2x}\cos 3x - 3e^{2x}\sin 3x = -3e^{2x}\sin 3x + 2e^{2x}\cos 3x, \\
D^2(e^{2x}\sin 3x) &= D(2e^{2x}\sin 3x + 3e^{2x}\cos 3x) \\
&= 2(2e^{2x}\sin 3x + 3e^{2x}\cos 3x) + 3(2e^{2x}\cos 3x - 3e^{2x}\sin 3x) \\
&= (-5)e^{2x}\sin 3x + 12e^{2x}\cos 3x, \\
D^2(e^{2x}\cos 3x) &= D(2e^{2x}\cos 3x - 3e^{2x}\sin 3x) \\
&= 2(2e^{2x}\cos 3x - 3e^{2x}\sin 3x) - 3(2e^{2x}\sin 3x + 3e^{2x}\cos 3x) \\
&= (-12)e^{2x}\sin 3x - 5e^{2x}\cos 3x.
\end{aligned}$$

The matrices representations of D and of D^2 relative to the given bases are

$$\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} \text{ and } \begin{pmatrix} -5 & -12 \\ 12 & -5 \end{pmatrix}.$$

19. Let $\mathcal{B} = \{1, x, x^2, x^3\}$.

(a).

$$\begin{matrix} T(1) = 0, \\ T(x) = x, \\ T(x^2) = 2x^2, \\ T(x^3) = 3x^3, \end{matrix} \Rightarrow \text{ the matrix representaions of } T \text{ relative to } \mathcal{B} \text{ is } \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

(b).

$DT(1) = D(0) = 0,$
$DT(x) = D(x) = 1,$
$DT(x^2) = D(2x^2) = 4x,$
$DT(x^3) = D(3x^3) = 9x^2.$

$$\Rightarrow \text{ the matrix representations of } T \text{ relative to } \mathcal{B} \text{ is } \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(c).

$TD(1) = T(0) = 0,$
$TD(x) = T(1) = 0,$
$TD(x^2) = T(2x) = 2x,$
$TD(x^3) = T(3x^2) = 6x^2.$

$\Rightarrow$ the matrix representaion of T relative to $\mathcal{B}$ is $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

(d). By (b), (c), and the linearity of linear operators, we have
$(TD - DT)(1) = 0,$
$(TD - DT)(x) = -1,$
$(TD - DT)(x^2) = -2x,$
$(TD - DT)(x^3) = -3x^2$. Thus the matrix represention of $(TD - DT)$ relative to $\mathcal{B}$ is $\begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

(e). $\begin{aligned} &T^2(1) = T(0) = 0, \\ &T^2(x) = T(x) = x, \\ &T^2(x^2) = T(2x^2) = 4x^2, \\ &T^2(x^3) = T(3x^3) = 9x^3. \end{aligned}$

$\Rightarrow$ the matrix representation of T relative to $\mathcal{B}$ is $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 9 \end{pmatrix}$.

(f).

$$\begin{cases} D^2(1) = 0, \\ D^2(x) = 0, \\ D^2(x^2) = 2, \\ D^2(x^3) = 6x. \end{cases} \text{and} \begin{cases} T^2(1) = 0, \\ T^2(x) = x, \\ T^2(x^2) = 4x^2, \\ T^2(x^3) = 9x^3. \end{cases}$$

$$\Rightarrow \begin{cases} T^2D^2(1) = T^2(0) = 0, \\ T^2D^2(x) = T^2(0) = 0, \\ T^2D^2(x^2) = T^2(2) = 0, \\ T^2D^2(x^3) = T^2(6x) = 6x. \end{cases} \text{and} \begin{cases} D^2T^2(1) = D^2(0) = 0, \\ D^2T^2(x) = D^2(x) = 0, \\ D^2T^2(x^2) = D^2(4x^2) = 8, \\ D^2T^2(x^3) = D^2(9x^3) = 54x, \end{cases}$$

$$\Rightarrow \begin{cases} (T^2D^2 - D^2T^2)(1) = 0, \\ (T^2D^2 - D^2T^2)(x) = 0, \\ (T^2D^2 - D^2T^2)(x^2) = -8, \\ (T^2D^2 - D^2T^2)(x^3) = 6x - 54x = -48x. \end{cases}$$

Thus the matrix representation of $T^2D^2 - D^2T^2$ relative to $\mathcal{B}$ is $\begin{pmatrix} 0 & 0 & -8 & 0 \\ 0 & 0 & 0 & -48 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

2. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(a) Because $AB = O$, we have $AB = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ 2c & 2d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Hence $c = d = 0$ and a, b are two arbitrary numbers. So $B = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ with a and b arbitrary.

(b) Because $BA = O$, we have $BA = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & a+2b \\ 0 & c+2d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Hence $a = -2b$ and $c = -2d$ for two arbitrary numbers c and d. So $B = \begin{bmatrix} -2b & b \\ -2d & d \end{bmatrix}$ with b and d arbitrary.

4. (a) Given $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 1 & 1 \\ -4 & 2 & -4 \\ 1 & 2 & 1 \end{bmatrix}$. We have $AB = \begin{bmatrix} -2 & 9 & 3 \\ 6 & 8 & 4 \\ -1 & 11 & 4 \end{bmatrix}$ and $BA = \begin{bmatrix} 7 & 11 & 13 \\ 0 & -6 & -4 \\ 6 & 6 & 9 \end{bmatrix}$. Thus $AB - BA = \begin{bmatrix} -9 & -2 & -10 \\ 6 & 14 & 8 \\ -7 & 5 & -5 \end{bmatrix}$.

7. Let $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, we claim that $A^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}$. We prove it by induction:

(a) When $n = 1$, $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is given.

(b) Assume for $n = k$, $A^k = \begin{bmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{bmatrix}$.

(c) For $n = k + 1$, we have

$$\begin{aligned} A^{k+1} &= A^k A \\ &= \begin{bmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos k\theta\cos\theta - \sin k\theta\sin\theta & -\cos k\theta\sin\theta - \sin k\theta\cos\theta \\ \sin k\theta\cos\theta + \cos k\theta\sin\theta & -\sin k\theta\sin\theta + \cos k\theta\cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos(k+1)\theta & -\sin(k+1)\theta \\ \sin(k+1)\theta & \cos(k+1)\theta \end{bmatrix}. \end{aligned}$$

Therefore, $A^2 = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$ and $A^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}$.

9. Let $A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$. We claim that $A^n = \begin{bmatrix} 1 & 0 \\ -n & 1 \end{bmatrix}$. We prove it by induction.

(a) When $n = 1$, we have $A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ as given.

(b) Assume for $n = k$, $A^k = \begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix}$.

(c) For $n = k + 1$, we have

$$\begin{aligned} A^{k+1} &= A^k A \\ &= \begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ -(k+1) & 1 \end{bmatrix}. \end{aligned}$$

Thus, $A^2 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 2-1 & 0 \\ -2 & 2-1 \end{bmatrix} = 2 \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2A - I$
and $A^{100} = \begin{bmatrix} 1 & 0 \\ -100 & 1 \end{bmatrix}$.