EE203001 Linear Algebra
Solutions for Homework #1 Spring Semester, 2003

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2. Let $V = \{\text{all vector } (x, y, z) \text{ in } R^3 \text{ with } x = 0 \text{ or } y = 0\}$. Choose a $v_1 = (x_1, 0, z_1) \in V$ with $x_1 \neq 0$ and a $v_2 = (0, y_2, z_2) \in V$ with $y_2 \neq 0$. Since $v_1 + v_2 = (x_1, y_2, z_1 + z_2)$ $\notin V$, Axiom 1 fails to hold and V is not a linear space.

6. Let V be the set of all vectors $v = (x, y, z)$ in $\mathbb{R}^3$ such that

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = vH = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} = O.$$

Let $v = (x, y, z)$, $v_1 = (x_1, y_1, z_1)$, $v_2 = (x_2, y_2, z_2)$, and $v_3 = (x_3, y_3, z_3)$ be elements in V and a , b be real numbers.

(1) Axiom 1.
Since $(v_1 + v_2)H = v_1 H + v_2 H = O, \quad v_1 + v_2$ is in V.

(2) Axiom 2.
Since $(av)H = a(vH) = aO = O, \quad av$ is in V.

(3) Axiom 3.
$v_1 + v_2 = (x_1 + x_2, y_1 + y_2, z_1 + z_2) = (x_2 + x_1, y_2 + y_1, z_2 + z_1) = v_2 + v_1.$

(4) Axiom 4.

$$\begin{aligned} (v_1 + v_2) + v_3 &= ((x_1 + x_2) + x_3, (y_1 + y_2) + y_3, (z_1 + z_2) + z_3) \\ &= (x_1 + (x_2 + x_3), y_1 + (y_2 + y_3), z_1 + (z_2 + z_3)) \\ &= v_1 + (v_2 + v_3). \end{aligned}$$

(5) Axiom 5.
$v + O = (x, y, z) + (0, 0, 0) = (x, y, z) = v.$

(6) Axiom 6.
$v + (-1)v = (x - x, y - y, z - z) = O.$

(7) Axiom 7.
$a(bv) = a(bx, by, bz) = (a(bx), a(by), a(bz)) = ((ab)x), (ab)y), (ab)z) = (ab)(x, y, z) = (ab)v.$

(8) Axiom 8.

$$\begin{aligned} a(v_1 + v_2) &= a(x_1 + x_2, y_1 + y_2, z_1 + z_2) \\ &= (ax_1 + ax_2, ay_1 + ay_2, az_1 + az_2) \\ &= (ax_1, ay_1, az_1) + (ax_2, ay_2, az_2) \\ &= a(x_1, y_1, z_1) + a(x_2, y_2, z_2) \\ &= av_1 + av_2. \end{aligned}$$

(9) Axiom 9.

$$\begin{aligned}(a+b)v &= ((a+b)x,(a+b)y,(a+b)z)\\ &= (ax+bx,ay+by,az+bz)\\ &= (ax,ay,az)+(bx,by,bz)\\ &= a(x,y,z)+b(x,y,z)\\ &= av+bv.\end{aligned}$$

(10) Axiom 10.
$1v=(1x,1y,1z)=(x,y,z)=v$.

21. Let x, y and z be positive real numbers in $V=\mathbb{R}^+$. Let a, b be real numbers. Denote the addition by $\oplus$ and the scalar multiplication by $\odot$.

(1) Axiom 1.
Since x, y are positive, $x\oplus y=xy$, is positive, too.
Then $x\oplus y$ is in V.

(2) Axiom 2.
Since x is positive and a is real, $a\odot x=x^a$ is positive.
Then $a\odot x$ is in V.

(3) Axiom 3.
$x\oplus y=xy=yx=y\oplus x$.

(4) Axiom 4.
$(x\oplus y)\oplus z=xy\oplus z=(xy)z=x(yz)=x\oplus yz=x\oplus(y\oplus z)$.

(5) Axiom 5.
$x\oplus 1=1x=x$.

(6) Axiom 6.
$x\oplus(-1)\odot x=x\oplus x^{-1}=xx^{-1}=1$.

(7) Axiom 7.
$a\odot(b\odot x)=a\odot x^b=(x^b)^a=x^{ab}=(ab)\odot x$.

(8) Axiom 8.
$a\odot(x\oplus y)=a\odot(xy)=(xy)^a=x^ay^a=(a\odot x)(a\odot y)=(a\odot x)\oplus(a\odot y)$.

(9) Axiom 9.
$(a+b)\odot x=x^{a+b}=x^ax^b=(a\odot x)(b\odot x)=(a\odot x)\oplus(b\odot x)$.

(10) Axiom 10.
$1\odot x=x^1=x$.

22. (a) $\boxed{\text{Step 1}}$:
Let 0 be a zero element in V as stated in Axiom 5.
$0x+0x=(0+0)x$ (by Axiom 9) $=0x$.
Add $(-1)(0x)$, which is a negative of the element $0x$, to both sides, we have
$(0x+0x)+(-1)(0x))=0x+(-1)(0x)$
On the right-hand side, we have

$0x+(-1)(0x)=0$ (by Axiom 6).

On the left-hand side, we have

$(0x + 0x) + (-1)(0x) = 0x + (0x + (-1)(0x))$ (by Axiom 4)
$= 0x + 0$ (by Axiom 6)
$= 0x$ (by Axiom 5).

We conclude that $0x = 0$.

Step 2 :

$x = x + 0x$ (by Axiom 5 and Step 1)
$= x + ((-1) + 1)x = x + ((-1)x + 1x)$ (by Axiom 9)
$= (x + (-1)x) + 1x$ (by Axiom 4)
$= 0 + 1x$ (by Axiom 6)
$= 1x + 0$ (by Axiom 3)
$= 1x$ (by Axiom 5).

(b) Let S be the set of all ordered pairs (x_1, x_2) of real numbers. If we define the addition as $(x_1, x_2)+(y_1, y_2) = (x_1+x_2, y_1+y_2)$, and the multiplication by scalars as $a(x_1, x_2) = (ax_1, 0)$, then the first 9 Axioms, but with Axioms 6 replaced by Axiom 6', hold but the 10th Axiom does not hold.

23 Let $v = (x_1, x_2)$ be an element in S with $x_1, x_2 \in \mathbb{R}$

(a) (1) First we show (0,0) is a zero element in S. Because for any v in S, $v+(0,0) = (x_1, x_2) + (0,0) = (x_1, x_2) = v$, thus (0,0) is a zero element in S.

(2) Since $v + (-1)v = (x_1, x_2) + (-x_1, 0) = (0, x_2) \neq (0,0)$ in general, Axiom 6 fails to hold.

(3) Since $1v = 1(x_1, x_2) = (x_1, 0) \neq (x_1, x_2) = v$ in general, Axiom 10 fails to hold.

(b) (1) If there exists a zero element called O in S, $O = (o_1, o_2)$, then for any element v in S, $v + O = (x_1, x_2) + (o_1, o_2) = (x_1 + o_1, 0)$. But due to the property of a zero element, $v + O = v = (x_1, x_2)$. Thus, there could not exist any zero element and Axiom 5 fails to hold.

(2) Because there is no zero element in S, there would not exist any negative for an element in S. So Axiom 6 fails to hold.

(c) (1) Since $(x_1, x_2) + (y_1, y_2) = (x_1, x_2 + y_2)$, $(y_1, y_2) + (x_1, x_2) = (y_1, y_2 + x_2)$, we have $(x_1, x_2) + (y_1, y_2) \neq (y_1, y_2) + (x_1, x_2)$ in general. The commutative law (Axiom 3) for addition fails to hold.

(2) Since $(x_1, x_2)+(0,0) = (x_1, x_2+0) = (x_1, x_2)$, $(0,0)$ is a zero element of S. But $(x_1, x_2)+(-1)(x_1, x_2) = (x_1, x_2)+(-x_1, -x_2) = (x_1, x_2-x_2) = (x_1, 0) \neq (0,0)$ in general, Axiom 6 fails to hold.

(3) Since $(a + b)(x_1, x_2) = ((a + b)x_1, (a + b)x_2)$, and $a(x_1, x_2) + b(x_1, x_2) = (ax_1, ax_2)+(bx_1, bx_2) = (ax_1, ax_2+bx_2)$, we have $(a+b)(x_1, x_2) \neq a(x_1, x_2)+b(x_1, x_2)$ in general. Thus Axiom 9 fails to hold.

(d) (1) Let $u = (1,0)$, $v = (-1,0)$ and $w = (0,0)$. We have $(u + v) + w = (|1 + (-1)|, |0 + 0|) + w = (0,0) + (0,0) = (0,0)$, but $u + (v + w) = u + (|(-1) + 0|, |0+0|) = u+(1,0) = (1,0)+(1,0) = (2,0)$. Thus Axiom 4, the associative law for addition, fails to hold.

(2) Suppose there is an element $O = (o_1, o_2)$ such that $w + O = w$, $\forall w \in S$. Then with $w = (0, 0)$, we have $(0, 0) = (0, 0) + O = (|0 + o_1|, |0 + o_2|) = (|o_1|, |o_2|)$, Thus O must be $(0, 0)$. But with $w = (-10)$, we have $(-1, 0) = (-1, 0) + O = (|-1 + 0|, |0 + 0|) = (1, 0)$, a contradiction. We conclude that S has no zero element and Axiom 5 fails to hold and neither does Axiom 6.

(3) Let $a > 0$, $u = (1, 0)$, and $v = (-1, 0)$. Then $a(u+v) = a(|1+(-1)|, |0+0|) = a(0, 0) = (|a\cdot 0|, |a\cdot 0|) = (0, 0)$, and $au+av = (|a\cdot 1|, |a\cdot 0|)+(|a\cdot(-1)|, |a\cdot 0|) = (a, 0) + (a, 0) = (|a + a|, |0 + 0|) = (2a, 0)$. We have $a(u + v) \neq au + av$, and Axiom 8 fails to hold.

(4) Let $a = 1$, $b = -1$ and $u = (1, 0)$. Then $(a+b)u = 0u = (|0\cdot 1|, |0\cdot 0|) = (0, 0)$, but $au+bu = (|1\cdot 1|, |1\cdot 0|)+(|-1\cdot 1|, |-1\cdot 0|) = (1, 0)+(1, 0) = (|1+1|, |0, 0|) = (2, 0)$. Thus Axiom 9 fails to hold.

(5) Let $u = (-1, 0)$. Then $1u = (|1 \cdot (-1), |1 \cdot 0|) = (1, 0) \neq u$. Axiom 10 fails to hold.