

On the Noise-Enhancing Ability of Stochastic Hodgkin-Huxley Neuron Systems

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Recently noise has been shown to be useful in enhancing neuron sensitivity by stochastic resonance. In this study, in order to measure the noise-enhancing factor (NEF), a nonlinear stochastic model is introduced for the Hodgkin-Huxley (HH) neuron system with synaptic noise input stimulation and channel noises in the sodium and potassium channels. The enhancing factor of the HH neuron system is measured from the point of view of the noise-exploiting level of nonlinear stochastic H_∞ signal processing. Since the nonlinear stochastic-enhancing measure problem of HH neuron systems requires a solution for the difficulty presented by the Hamilton Jacobi inequality (HJI), a fuzzy interpolation of locally linearized systems is employed to simplify the nonlinear noise-enhancing problems by solving only a set of linear matrix inequalities. The NEF of the HH neuron system is found to be related to the locations of eigenvalues of linearized HH neuron systems and can be estimated through the H_∞ signal processing method. Based on a stochastic fuzzy linearized HH neuron system, we found that channel noises are enhanced by the active eigenvalues of ionic channels while synaptic noises are attenuated by the passive eigenvalues of synaptic process.

1 Introduction ---

The nervous system is constituted of a large number of highly interconnected neurons, which represent the basic biological computational units (Kandel, Schwartz, & Jessell, 2000). Neurons receive electrical input signals from other nerve cells via threadlike extensions of cell membranes called dendrites. Depending on the spatiotemporal distribution of input currents that depolarize membrane voltage, firing threshold (sufficient membrane voltage for depolarization) can be reached, thus triggering an action potential. However, neurons operating in biological systems are subject to noises, which cause various alterations in neural coding (Douglass, Wilkens, Pantazelou, & Moss, 1993; Levin & Miller, 1996; Segundo, Martinez,

Pakdaman, Stiber, & Vibert, 1994). As noises can directly affect the reliability of neurons, researchers have taken great interest in how neurons disregard or even incorporate their intrinsic noises so as to process information reliably. A prerequisite to understanding the mechanisms underlying such noise-induced changes is the characterization of how noise alone affects neuronal behavior. Further incentive to analyze the response of neurons to noise-like stimuli comes from the fact that such signals well mimic the actual inputs some neurons receive in vivo (Juusola & French, 1997; Mainen & Sejnowski, 1995). Finally system-identification studies through white noise analysis have been another application of noiselike inputs in neuroscience (Marmarelis & Marmarelis, 1978; Sakai, 1992). To be more precise, studies of this kind estimate the Wiener kernels from the spike train evoked by noiselike signals (Boskov, Jovic, Jovanovic, Ljubisavljevic, & Anastasijevic, 1994a, 1994b; Bryant & Segundo, 1976; Buno, Bustamante, & Fuentes, 1984; Bustamante & Buno, 1992; French, 1984; French & Wong, 1977; Korenberg, French, & Voo, 1988; Lewis, Henry, & Yamada, 2000). The combined influence of noise and constant stimulations on Hodgkin-Huxley (HH) neuron systems has been studied through time and frequency analysis of membrane-potential dynamics (Kosmidis & Pakdaman, 2006; Takahata, Tanabe, & Pakdaman, 2002; Tateno & Pakdaman, 2004).

Actually, the noises in neurons come not only from input stimulation but also from the activation and inactivation of sodium and potassium currents through nanoscale sodium and potassium ionic channels, respectively. The combined action of synaptic and channel noises has been discussed by Schmid, Goychuk, Hanggi, Zeng, and Jung (2004) and Zeng and Jung (2004). Therefore, in this study, a more general nonlinear stochastic model is introduced for stochastic HH neuron systems with noises in input stimulation and activation and inactivation of sodium and potassium channels (i.e., both synaptic noises and channel noises are considered in a stochastic HH neuron model). Recently, based on stochastic HH neuron model, it has been found that noises are enhanced by stochastic resonance via intense numerical simulations (Butson & Clark, 2008; Douglass et al., 1993; Fox, 1997; Kandel et al., 2000; Levin & Miller, 1996; Saarinen, Linne, & Yli-Harja, 2006; Schneidman, Freedman, & Segev, 1998; Segundo et al., 1994; Stein, Gossen, & Jones, 2005; White, Rubinstein, & Kay, 2000). Currently, the enhancing mechanism of noises in HH neuron systems still needs to be revealed, especially from the point of view of nonlinear stochastic signal processes. In this situation, based on nonlinear stochastic H_∞ -enhancing theory, the noise-enhancing factor (NEF) of HH neuron systems is measured by the noise-enhancing level from the perspective of nonlinear stochastic systems.

Unlike previous methods, which are based on the linearization method and Wiener kernels (Boskov et al., 1994a, 1994b; Bryant & Segundo, 1976; Buno, Bustamante, & Fuentes, 1984; Bustamante & Buno, 1992; French, 1984; French & Wong, 1977; Korenberg et al., 1988; Lewis et al., 2000; Marmarelis & Marmarelis, 1978; Sakai, 1992), NEF is discussed from the perspective of

nonlinear stochastic process. By solving a constrained optimization problem subject to an HJI, we can measure NEF. However, it is still very difficult to solve HJI analytically or numerically for noise-enhancing measure problems in stochastic HH neuron systems. In recent years, Takagi-Sugeno (TS) fuzzy systems have been used to efficiently interpolate several local linear systems via fuzzy bases to approximate nonlinear systems (Chen, Tsai, & Chen, 2003; Chen, Tseng, & Uang, 1999; Passino & Yurkovich, 1998). Therefore, a T-S fuzzy stochastic system is employed to approximate a nonlinear stochastic HH neuron system by interpolating several linearized HH neuron systems at different operation points of the nonlinear stochastic HH neuron system. Then the problem of measuring the NEF of HH neuron systems can be simplified by solving a constrained optimization problem subject to a set of LMIs developed for measuring the NEF of a set of fuzzy local stochastic linear HH neuron systems—that is, nonlinear HJI constraints are replaced by a set of LMI constraints. These LMIs can be easily solved by an LMI toolbox in Matlab to simplify the procedure of noise-enhancing analysis and the measurement of nonlinear stochastic HH neuron systems. Furthermore, based on the fuzzy approximation method, the analysis of the NEF in nonlinear stochastic HH neuron systems can be investigated from the viewpoint of a set of linear stochastic HH neuron systems. We found that if the eigenvalues of these fuzzy local linear systems are on the far left-hand side of the complex domain (i.e., with more negative real parts or more stability), then the nonlinear HH neuron network attenuates (or filters) more noise. If some eigenvalues are near or on the imaginary axis or even on the right-hand side of complex domain (i.e., with zero or positive real parts), the noises will be resonated or even enhanced by the HH neuron system. As for the fuzzy local linear systems, we found that the eigenvalues of the synaptic process are all on the left-hand-side complex domain. Therefore, the synaptic noises are filtered (attenuated) by the HH neuron system. We also found that many eigenvalues of ionic channels are near or on the imaginary axis. This implies that the HH neuron system has an inherent resonant mechanism that enhances channel noises.

This study makes the following contributions. First, this study considers not only the synaptic noises at input stimulation but also the channel noises at sodium and potassium ionic channels in a nonlinear stochastic HH neuron system. Second, nonlinear H_∞ -filtering or enhancing theory is employed to analyze the NEF of a nonlinear stochastic HH neuron system. Third, the T-S fuzzy model is employed to simplify the noise-enhancing problem of nonlinear stochastic HH neuron systems by substituting a noise-enhancing problem of a set of interpolated linear stochastic neuron systems that can be easily solved by LMI toolbox in Matlab. Finally we found that channel noises are always enhanced when synaptic noises are filtered. Noise enhancement is performed by the active modes of two major processes: sodium activation and potassium activation processes. As constant current level increases, noise enhancement is partly performed by the active modes

of sodium inactivation. Channel noises can be investigated by the locations of the eigenvalues of fuzzy local neuron systems. In general, noise enhancement is performed by the active modes of ionic channels on the $j\omega$ -axis, and the noise filtering is performed by the passive modes of the synaptic process on the left-hand-side complex domain. Unlike conventional methods, which employ Wiener kernel techniques via a linearization method (Dayan & Abbott, 2001; Marmarelis & Marmarelis, 1978), the proposed method can analyze the NEF of stochastic HH neuron systems by the passive or active modes of corresponding state variables from the perspective of nonlinear stochastic systems.

2 Methods

This section presents stochastic HH neuron systems with synaptic noises in input stimulation and channel noises in activation and inactivation of sodium and potassium channels. Then the enhancing factor of a stochastic HH neuron system is measured by the noise-enhancing level from the perspective of nonlinear stochastic H_∞ signal processing. Finally, T-S fuzzy techniques are employed to simplify the measurement procedure of the NEF of nonlinear stochastic HH neuron systems. These steps are discussed in the following sections in detail.

2.1 Stochastic HH Neuron Systems. The dynamics of a stochastic model corrupted by both synaptic noise and channel noises is described by the following equations (Hodgkin & Huxley, 1952):

$$\begin{aligned}
 C_m \frac{dV(t)}{dt} &= g_{Na} m(t)^3 h(t) (V_{Na} - V(t)) + g_K n(t)^4 (V_K - V(t)) \\
 &\quad + g_L (V_L - V(t)) + I + w_1(t) \\
 \frac{dm(t)}{dt} &= \alpha_m(V(t))(1 - m(t)) - \beta_m(V(t))m(t) + w_2(t) \\
 \frac{dh(t)}{dt} &= \alpha_h(V(t))(1 - h(t)) - \beta_h(V)h(t) + w_3(t) \\
 \frac{dn(t)}{dt} &= \alpha_n(V(t))(1 - n(t)) - \beta_n(V)n(t) + w_4(t),
 \end{aligned} \tag{2.1}$$

where the variables $V(t)$, $m(t)$, $h(t)$, and $n(t)$ are the membrane potential, the activation and inactivation of the sodium current, and the activation of the potassium current, respectively. C_m is the membrane capacitance. V_{Na} , V_K , and V_L are, respectively, the reversal potentials of the sodium, potassium, and leak currents, and g_{Na} , g_K , and g_L are the corresponding maximal conductances. I is the constant stimulation. $w_1(t)$ denotes the stimulation noises, that is, synaptic noises. $w_i(t)$, $i = 2, 3, 4$,

denote the stochastic noises of activation and inactivation in sodium and potassium ion channels, that is, channel noises. Suppose synaptic noise with $\langle w_1(t) \rangle =: E(w_1(t)) = 0$, $\langle w_1(s)w_1(t) \rangle =: E(w_1(s)w_1(t)) = \sigma_1^2 \delta(t - s)$, where $\delta(t)$ is the Dirac impulse function. As did Fox and Lu (1994) and Zeng and Jung (2004), we define that channel noises w_2, w_3 , and w_4 are zero-mean white noises with $\langle w_2(t)w_2(t') \rangle = (2/N_{Na})\{\alpha_m(1 - m) + \beta_m m\}/2 \delta(t - t')$, $\langle w_3(t)w_3(t') \rangle = (2/N_{Na})\{\alpha_h(1 - h) + \beta_h h\}/2 \delta(t - t')$ and $\langle w_4(t)w_4(t') \rangle = (2/N_K)\{\alpha_n(1 - n) + \beta_n n\}/2 \delta(t - t')$, respectively. Here N_{Na} and N_K denote the total numbers of sodium and potassium channels. The number of sodium ion channels N_{Na} is kept proportional, $N_{Na} = 60(N_K/18)$, to the number of potassium ion channels (Schmid et al., 2004). In the numerical simulations, we also set $N_K = 10$. In this study, all noise variances are not necessarily known or uncertain. In addition, the auxiliary functions $\alpha_m, \alpha_n, \alpha_h, \beta_m, \beta_h$, and β_n are (Hodgkin & Huxley, 1952)

$$\begin{aligned} \alpha_m(V) &= \frac{2.5 - 0.1V(t)}{e^{(2.5-0.1V(t))} - 1}, & \beta_m(V) &= 4e^{-V(t)/18} \\ \alpha_n(V) &= \frac{0.1 - 0.01V(t)}{e^{(1-0.1V(t))} - 1}, & \beta_n(V) &= 0.125e^{-V(t)/80} \\ \alpha_h(V) &= 0.07e^{-V(t)/20}, & \beta_h(V) &= \frac{1}{e^{(3-0.1V(t))} + 1}. \end{aligned} \tag{2.2}$$

The parameter values used in this study are $C_m = 1.0\mu\text{F}/\text{cm}^2$, $g_{Na} = 120 \text{ mS}/\text{cm}^2$, $g_K = 36 \text{ mS}/\text{cm}^2$, $g_L = 0.3 \text{ mS}/\text{cm}^2$, $V_{Na} = 115 \text{ mV}$, $V_K = -12 \text{ mV}$, and $V_L = 10.613 \text{ mV}$ (Takahata et al., 2002). The parameter values were set so that the resting potential was 0.003487 mV for $I = 0 (\mu\text{A}/\text{cm}^2)$. In the following simulation example, we use different values of the current I so that the resting potential will change with different current I . Suppose the equilibrium points of interest in an HH neuron system without noises are (V^*, m^*, h^*, n^*) . Let us denote

$$X(t) = \begin{bmatrix} V(t) \\ m(t) \\ h(t) \\ n(t) \end{bmatrix}, \quad W(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \\ w_4(t) \end{bmatrix}, \quad X^* = \begin{bmatrix} V^* \\ m^* \\ h^* \\ n^* \end{bmatrix}. \tag{2.3}$$

Then the nonlinear stochastic HH equation can be represented by the following nonlinear stochastic system,

$$\dot{X}(t) = F(X(t)) + HW(t), \tag{2.4}$$

where $H = I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ 0 & \dots & 0 & 1 \end{bmatrix}$ when both synaptic noise and channel noises appear, $H = H_1 = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & \dots & 0 & 0 \end{bmatrix}$ when only synaptic noises appear, and $H = H_2 = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ 0 & \dots & 0 & 1 \end{bmatrix}$ when only channel noises appear.

For convenience in the analysis of noise filtering and enhancing, the origin of the stochastic HH system in equation 2.4 should be shifted to the equilibrium point X^* , that is, $X(t) = X^* + x(t)$. Then we get the shifted system as follows:

$$\dot{x}(t) = F(x(t) + X^*) + HW(t). \tag{2.5}$$

In this situation, the equilibrium point of interest in the stochastic HH neuron system in equation 2.5 is at the origin. If $x = 0$, then $X = X^* = (V^*, m^*, h^*, n^*)^T$, converging to the equilibrium point in the noise-free case.

In numerical simulations, the Box-Muller algorithm is also needed for generating channel noises (Fox & Lu, 1994). Noises are generated at each integration step. In order to guarantee that the variables $m, h,$ and n are always in the range $(0,1)$, one must redo the integration step until the updated values of the variables $m, h,$ and n all stay within $(0,1)$ in the simulation process. The numerical integration of stochastic differential equation 2.1 was performed by the Euler method with the time step 0.01 ms.

2.2 Noise Enhancing of Stochastic HH Neuron Systems. Let us denote the noise-enhancing level ρ of the stochastic neuron system in equation 2.5 as follows:

$$\frac{E \int_0^\infty v^2(t) dt}{E \int_0^\infty W^T(t)W(t) dt} < \rho^2 \quad \text{or} \tag{2.6}$$

$$E \int_0^\infty v^2(t) dt < \rho^2 E \int_0^\infty W^T(t)W(t) dt,$$

for all possible stochastic noises with finite variances, where $v(t) = V(t) - V^*$. The physical meaning of the inequality in equation 2.6 is that the effect of all possible stochastic noises on the membrane potential from the equilibrium V^* is less than ρ^2 from the point of view of mean energy, that is, the gain from $W(t)$ to $v(t)$ is less than ρ from the perspective of mean energy. The ratio in equation 2.6 can also be considered to be the signal-to-noise

ratio from the point of view of mean energy (Butson & Clark, 2008; Stein et al., 2005; White et al., 2000), that is, the signal-to-noise ratio is less than ρ^2 .

Remark 1. (1) If only the effect of synaptic noise $w_1(t)$ is considered (Takahata et al., 2002), then $H = H_1$ in equations 2.4 and 2.5. If only the effect of three channel noises is considered, then $H = H_2$ in equations 2.4 and 2.5. If the effects of both synaptic noise and channel noises are considered, then $H = I$. (2) In order to avoid a divergent integral in equation 2.6, the inequality could be changed to $\lim_{T \rightarrow \infty} \frac{1}{T} E \int_0^T v^2(t) dt / \lim_{T \rightarrow \infty} \frac{1}{T} E \int_0^T W^T(t)W(t) dt < \rho^2$. Since both the numerator and denominator are divided by T and can be canceled by each other, this new definition is equivalent to equation 2.6 and does not affect the derived results in the sequel. Actually, it can be considered as the ratio of variances of $v(t)$ and $W(t)$.

Remark 2. If the initial condition $v(0)$ is not zero, then the inequality in equation 2.6 should be modified as follows (Chen, Tseng, & Uang, 2000; Zhang, Chen, & Tseng, 2005),

$$E \int_0^\infty v^2(t) dt < E\Phi(v(0)) + \rho^2 E \int_0^\infty W^T(t)W(t) dt, \tag{2.7}$$

for some positive function $\Phi(\cdot)$, that is, the effect of the energy of the initial $v(0)$ should also be considered in the above inequality. For convenience, the above inequality is transformed to the following equivalent inequality by $v(t) = [1 \ 0 \ 0 \ 0]^T x(t)$:

$$E \int_0^\infty x^T(t)Qx(t) dt < E\Phi(x(0)) + \rho^2 E \int_0^\infty W^T(t)W(t) dt \tag{2.8}$$

where $Q = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} [1 \ 0 \ 0 \ 0] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Remark 3. If we want to discuss the effect of noises $W(t)$ on $x(t)$ (i.e., not only on $v(t)$ but also on the other state variables of the neuron system), then we could let $Q = I$ in equation 2.8.

Based on the analysis above, we get the following result:

Proposition 1. For the stochastic HH neuron systems in equation 2.5, if the following HJI has a positive solution $\Phi(x) > 0$,

$$\begin{aligned} & \left(\frac{\partial \Phi(x(t))}{\partial x} \right)^T F(x(t) + X^*) + x(t)^T Qx(t) \\ & + \frac{1}{4\rho^2} \left(\frac{\partial \Phi(x(t))}{\partial x} \right)^T HH^T \frac{\partial \Phi(x(t))}{\partial x} < 0, \end{aligned} \quad (2.9)$$

then the noise-enhancing level ρ in equations 2.6, 2.7, and 2.8 for stochastic HH neuron systems in equation 2.4 is achieved.

Proof: See appendix A.

From equation 2.6, it is evident that ρ is the upper bound of noise enhancing. The NEF ρ_0 is defined as follows:

$$\begin{aligned} \rho_0 &= \min_{\Phi(x)} \rho \\ &\text{subject to equation 2.9,} \end{aligned} \quad (2.10)$$

that is, the NEF is defined as the minimum noise-enhancing level of the stochastic neuron system. This is the so-called H_∞ noise-filtering or noise-enhancing problem (Chen et al., 2003; Zhang et al., 2005). In general, if $\rho_0 < 1$, we call it noise filtering (i.e., the noises are attenuated by the HH neuron system). If $\rho_0 \geq 1$, we call it noise enhancing by the HH neuron system. Therefore, the measurement of the NEF for the stochastic neuron system solves the HJI-constrained optimization problem in equation 2.10.

Remark 4. ρ in equation 2.6 can be considered as the upper bound of NEF ρ_0 , which is defined as $\rho_0^2 = \max_{W(t) \in \mathcal{L}_2} E \int_0^\infty v^2(t) dt / E \int_0^\infty W^T(t)W(t) dt$, where \mathcal{L}_2 denotes the set of all possible zero-mean white noises with finite variances—the worst-case signal-to-noise ratio of the neuron system for all possible stochastic noises from the mean energy point of view. From the system point of view, ρ_0 can be considered as amplification or attenuation of stochastic neuron system from $W(t)$ to $v(t)$ (Boyd, 1994). In the linear system case, the filtering ability ρ_0 is equivalent to the largest peak of transfer function from noise input $W(t)$ to output $v(t)$, which is related to the eigenvalues of the system matrix A of the linear system (Boyd, 1994). Obviously our method is an extension of linear filtering theory to nonlinear filtering theory. Therefore, the minimization of the noise-enhancing and noise-filtering level ρ^2 in equation 2.10 is to minimize its upper bound ρ^2 to achieve the NEF ρ_0^2 from the suboptimal perspective. That is, we do not solve ρ_0 from the optimization problem $\max_{W(t) \in \mathcal{L}_2} E \int_0^\infty v^2(t) dt / E \int_0^\infty W^T(t)W(t) dt$ directly but solve this problem from equations 2.6 and 2.10 indirectly from the suboptimal perspective instead.

2.3 Fuzzy Interpolation Method for Measurement of NEF. In general, it is very difficult to solve nonlinear HJI for the noise-enhancing problem in equations 2.9 and 2.10. In order to simplify the noise-enhancing problem, T-S fuzzy model (Chen et al., 1999, 2000; Hsiao, Chen, Liang, Xu, & Chiang, 2005; Hsiao, Hwang, Chen, & Tsai, 2005; Lian, Chiu, Chiang, & Liu, 2001a, 2001b; Takagi & Sugeno, 1985) is employed to approximate the nonlinear stochastic HH neuron system in equation 2.5 by interpolating several linearized HH neuron systems at different operation sets so that the nonlinear noise-enhancing problem can be transformed into a fuzzy noise-enhancing problem (Chen et al., 2003). Using such a fuzzy interpolation approach, the HJI in equations 2.9 and 2.10 can be replaced by a set of LMIs. In this situation, the measurement problem of the nonlinear NEF of stochastic HH neuron systems can be solved easily by using the fuzzy noise-filtering or -enhancing method.

The T-S fuzzy model is a piecewise interpolation of several locally linearized models through membership functions at different operation sets. The fuzzy model is described by fuzzy if-then rules. The i th rule of the fuzzy model for nonlinear stochastic neuron systems in equation 2.5 is given in the following form (Chen et al., 2003, 1999; Passino & Yurkovich, 1998):

$$\begin{aligned} \text{Rule } i: & \text{ If } x_1(t) \text{ is } F_{i1}, x_2(t) \text{ is } F_{i2}, x_3(t) \text{ is } F_{i3}, x_4(t) \text{ is } F_{i4} \\ & \text{ then } \dot{x}(t) = A_i x(t) + HW(t), \quad i = 1, \dots, L, \end{aligned} \quad (2.11)$$

where $F_{i,j}$ are the fuzzy sets, A_i are the known constant matrices, and L is the number of if-then rules. The fuzzy system in equation 2.11 is inferred as follows (Chen et al., 2003, 1999; Passino & Yurkovich, 1998; Zhang et al., 2005):

$$\begin{aligned} \dot{x}(t) &= \frac{\sum_{i=1}^L \mu_i(x) [A_i x(t) + HW(t)]}{\sum_{k=1}^L \mu_k(x)} \\ &= \sum_{i=1}^L h_i(x) [A_i x(t) + HW(t)], \end{aligned} \quad (2.12)$$

where $\mu_i(x) = \prod_{j=1}^4 F_{ij}(x_j(t))$, $h_i(x) = \mu_i(x(t)) / \sum_{k=1}^L \mu_k(x(t))$, and $F_{ij}(x_j(t))$ are the grade membership function of $x_j(t)$ in the fuzzy set F_{ij} . Examples of the two fuzzy membership functions in each state variable $x_j(t)$ and the case of membership functions for 16 fuzzy rules are shown in Figure 1, which will be discussed in the simulation example in the sequel.

When we assume $\mu_i(x) > 0$, then the bases of fuzzy interpolations satisfy the following constraints:

$$h_i(x) > 0, \quad \sum_{i=1}^L h_i(x) = 1. \quad (2.13)$$

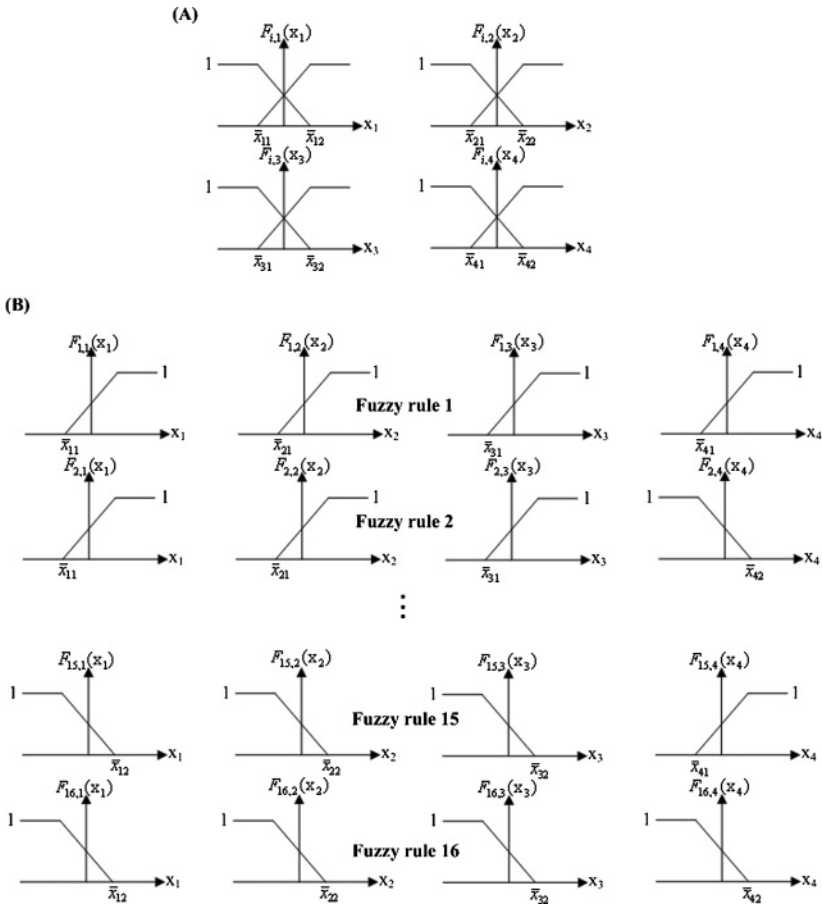


Figure 1: (A) Fuzzy membership functions $F_{ij}(x_j)$ of the fuzzy sets F_{ij} in the example. (B) The fuzzy membership functions for 16 fuzzy rules in equation 2.19.

The T-S fuzzy model in equation 2.12 interpolates L local linear stochastic neuron systems to approximate the nonlinear stochastic HH neuron system in equation 2.5 via fuzzy basis functions $h_i(x)$ at different operation sets. We could specify parameter A_i easily so that $\sum_{i=1}^L h_i(x)A_i x(t)$ in equation 2.12 can approximate $F(x)$ in equation 2.5 using the fuzzy identification method (Takagi & Sugeno, 1985), that is,

$$\begin{aligned} \dot{x}(t) &= F(x(t) + X^*) + HW \\ &= \sum_{i=1}^L h_i(x)[A_i x(t) + HW(t)] + \Delta f(x), \end{aligned} \tag{2.14}$$

where $\Delta f(x) = F(x(t) + x^*) - \sum_{i=1}^L h_i(x)A_i x(t)$ denotes the fuzzy approximation error. Let us suppose the approximation error is bounded by

$$\|\Delta f(x)\| \leq \alpha \|x(t)\|, \tag{2.15}$$

where $\|x(t)\| =: \sqrt{x_1^2(t) + x_2^2(t) + x_3^2(t) + x_4^2(t)}$, that is, the approximation error $\Delta f(x)$ is bound by a linear sector $[-\alpha x(t), \alpha x(t)]$ (Cao, Rees, & Feng, 2001). In general, if the number of membership functions becomes larger, the approximation error will become smaller, that is, α in equation 2.15 will become smaller. However, it will lead to a complex fuzzy system and increase the computation complexity. There is a trade-off between computation complexity and approximation error.

After the nonlinear stochastic HH neuron system in equation 2.5 is approximated by the T-S fuzzy system in equation 2.14, the nonlinear noise-enhancing (noise-filtering) problem in equations 2.5 and 2.8 can be replaced by solving the following fuzzy noise-enhancing (noise-filtering) problem for equations 2.14 and 2.6.

Proposition 2. *For the fuzzy neuron system, equation 2.14, if the following inequalities have a solution $P > 0$ and $\gamma \geq 0$ for a prescribed ρ ,*

$$P A_i + A_i^T P + Q + P \left(\frac{1}{\gamma^2} I + \frac{H H^T}{\rho^2} \right) P + \gamma^2 \alpha^2 I < 0, \quad i = 1, \dots, L, \tag{2.16}$$

then the noise-enhancing level ρ in equation 2.6 is achieved.

Proof: See appendix B.

By fuzzy approximation, obviously, HJI in equation 2.9 can be approximated by a set of algebraic inequalities in equation 2.16. Since ρ^2 is the upper bound of the noise-enhancing (noise-filtering) level in equation 2.6, the NEF still has to minimize ρ^2 as in equation 2.10, as follows:

$$\rho_0^2 = \min_{P>0, \gamma \geq 0} \rho^2 \tag{2.17}$$

subject to equation 2.16,

that is, the measurement problem of NEF in equation 2.10 can be replaced by a measurement problem of NEF equivalent to equation 2.17.

By the Schur complement (Boyd, 1994), the constrained optimization problem for measuring the NEF in equation 2.17 is equivalent to the

following LMI-constrained optimization problem:

$$\rho_0^2 = \min_{P>0, \gamma \geq 0} \rho^2, \quad (2.18)$$

$$\text{subject to } \begin{bmatrix} PA_i + A_i^T P + Q + \gamma^2 \alpha^2 I & PH & P \\ & H^T P & -\rho^2 I & 0 \\ & P & 0 & -\gamma^2 I \end{bmatrix} < 0.$$

$$i = 1, \dots, L$$

Remark 5. (1) The fuzzy basis function $h_i(x)$ in equations 2.12 and 2.13 for fuzzy interpolation can be replaced by other interpolation functions, for example, cubic spline functions. (2) By fuzzy approximation, the nonlinear HJI-constrained optimization for measuring the NEF is replaced by LMI-constrained optimization in equation 2.18, which can be easily solved by the LMI toolbox in Matlab (Boyd, 1994). (3) The constrained optimization problem in equation 2.18 could be solved by decreasing ρ^2 until no positive solution $P > 0$ and $\gamma \geq 0$ exists to obtain the minimum ρ_0^2 . (4) Unlike conventional noise-enhancing analysis and measurements, the proposed method in equation 2.18 can estimate the NEF from the system properties of fuzzy local linear neuron systems (i.e., A_i and H). Obviously the locations of the eigenvalues of A_i still play important roles in the noise enhancing or noise filtering of HH neuron systems. This will be discussed in the next section. This is like a linear lowpass, bandpass, or highpass filter, of which the noise filter ability is dependent on the transfer function or the system matrix A in state-space model. The proposed measure method is an extension of linear filtering theory to the nonlinear stochastic filtering theory only if the neuron system is considered as a nonlinear filter.

According to the analysis above, the measurement of NEF of stochastic HH neuron systems is summarized by the following procedure:

1. Give a nonlinear stochastic model for the HH neuron system with an equilibrium point of interest shifted to the origin as equation 2.5.
2. Select a T-S fuzzy system to approximate the nonlinear stochastic HH system as equation 2.14.
3. Solve the constrained optimization in equation 2.18 to obtain the NEF of ρ_0 .

2.4 Computational Simulation Example. In the analysis of Lee, Neiman, and Kim (1998), the external stimulus I was taken to be a time-independent DC current, which served as a bifurcation parameter of the HH neuron system. In the parameter region $I < 6.2$ ($\mu\text{A}/\text{cm}^2$), the system possesses a global attractor with a fixed point, while for $6.2 < I < 9.8$ ($\mu\text{A}/\text{cm}^2$), the system possesses two coexisting stable attractors with a fixed

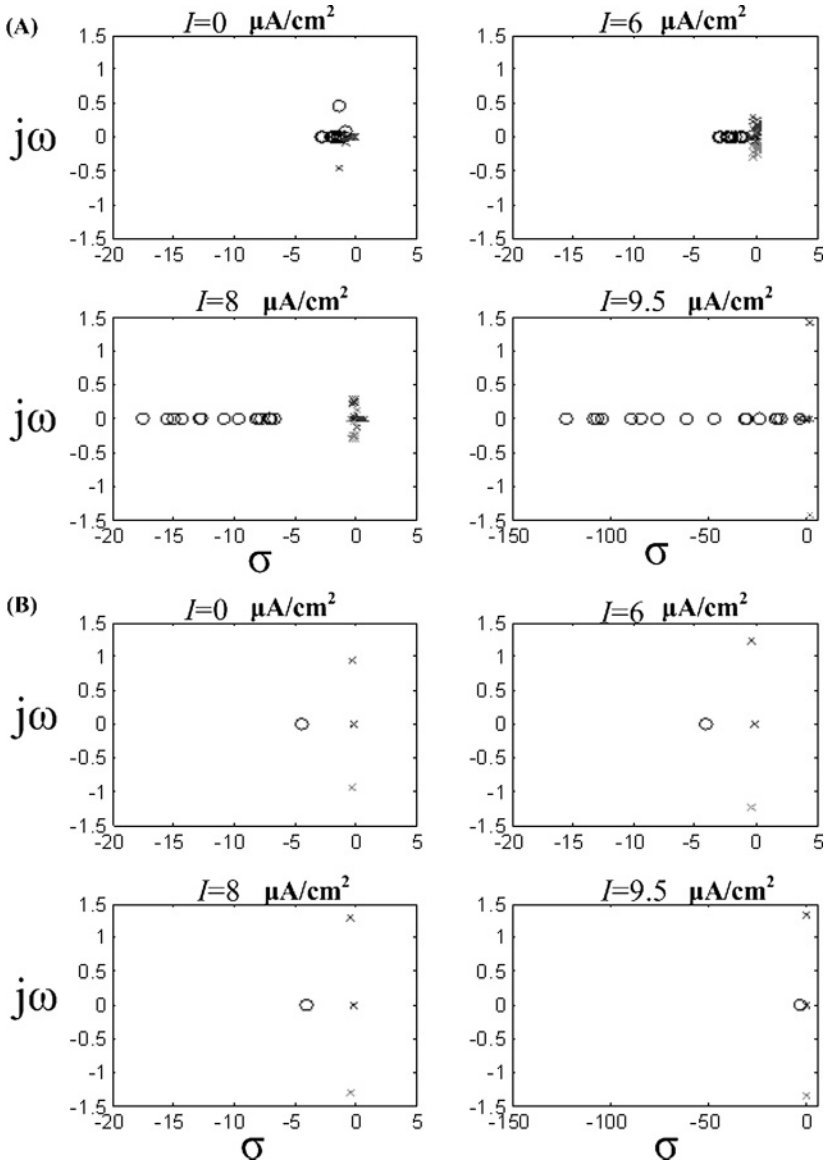
point and a limit cycle. With noise taken into account, the system fluctuates and displays trains of a few short periodic oscillations around the fixed point. In this study, the focus of interest is on measuring the NEF of the HH neuron system around the fixed point. Therefore, in our numerical experiments, we use $I = 0, 6, 8,$ and 9.5 ($\mu\text{A}/\text{cm}^2$) as the constant stimulation current levels in the stochastic HH neuron system.

Consider the stochastic HH neuron system in equation 2.1. We want to measure the NEF of the stochastic HH neuron system at different constant stimulation current levels. The equilibrium point of the stochastic HH neuron system is dependent on constant stimulation current I . In this example, four constant stimulation currents— $I = 0, 6, 8$ and 9.5 ($\mu\text{A}/\text{cm}^2$)—are considered respectively in the measurement of NEF. First, according to the above procedure, we find the equilibrium point at $(V^*, m^*, h^*, n^*) = (0.003486713806271 \text{ (mV)}, 0.052982334776129, 0.596553628773007, 0.318074861569072)$ when $I = 0$ ($\mu\text{A}/\text{cm}^2$); at $(V^*, m^*, h^*, n^*) = (3.759215184333003 \text{ (mV)}, 0.081596330872954, 0.461731687663958, 0.376538405945545)$ when $I = 6$ ($\mu\text{A}/\text{cm}^2$); at $(V^*, m^*, h^*, n^*) = (4.646567615658967 \text{ (mV)}, 0.090067227493615, 0.390635159502379, 0.430453888649725)$ when $I = 8$ ($\mu\text{A}/\text{cm}^2$); and at $(V^*, m^*, h^*, n^*) = (5.241964767043118 \text{ (mV)}, 0.096159184979368, 0.400127010471808, 0.409795996369423)$ when $I = 9.5$ ($\mu\text{A}/\text{cm}^2$) in the noise-free case. Then we shift the origin to the equilibrium $X^* = (V^*, m^*, h^*, n^*)^T$ of interest. After we get the shifted nonlinear stochastic HH model in equation 2.5 with the origin as the equilibrium point of interest, based on the membership functions in Figure 1, we get the following fuzzy system to approximate the shifted nonlinear stochastic system in equation 2.5.

$$\begin{aligned} &\text{If } x_1(t) \text{ is } F_{i1}, x_2(t) \text{ is } F_{i2}, x_3(t) \text{ is } F_{i3}, x_4(t) \text{ is } F_{i4} \\ &\text{then } \dot{x}(t) = A_i x(t) + HW(t), \quad i = 1, 2, \dots, 16, \end{aligned} \quad (2.19)$$

where the system parameters A_i are estimated by fuzzy identification methods (Passino & Yurkovich, 1998; Takagi & Sugeno, 1985) in the supplementary material (<http://www.mitpressjournals.org/doi/suppl/10.1162/neco.2010.07-09-1057>) and the number of fuzzy rules $L = 16$ is given in this example.

First, we consider the effects of synaptic noise (in the case of $H = H_1$) and channel noises (in the case of $H = H_2$) on the stochastic HH neuron system, respectively. By solving the LMI-constrained optimization in equation 2.18, we estimate (1) the NEF $\rho_0 = 0.060$ under only synaptic noises and the NEF $\rho_0 = 7.301$ under only channel noises when $I = 0$ ($\mu\text{A}/\text{cm}^2$); (2) the NEF $\rho_0 = 0.072$ under only synaptic noises and the NEF $\rho_0 = 21.890$ under only channel noises when $I = 6$ ($\mu\text{A}/\text{cm}^2$); (3) the NEF $\rho_0 = 0.089$ under only synaptic noises and the NEF $\rho_0 = 21.898$ under only channel noises when $I = 8$ ($\mu\text{A}/\text{cm}^2$); and (4) the NEF $\rho_0 = 0.555$ under only synaptic noises and



the NEF $\rho_0 = 20.396$ under only channel noises when $I = 9.5 (\mu\text{A}/\text{cm}^2)$ (see Figures 4A and 4B). Obviously the synaptic noises are filtered by the HH neuron system, and the channel noises are enhanced by the HH neuron system. The locations of the eigenvalues of A_i in the fuzzy locally linearized systems in the cases above are plotted in Figure 2A. From Figure 2A, in the

four cases (i.e., $I = 0, 6, 8,$ and $9.5 \text{ } (\mu\text{A}/\text{cm}^2)$), all eigenvalues of synaptic processes x_1 are located on the left-hand side of the $j\omega$ -axis (i.e., they are of passive modes), and some eigenvalues of channel processes $x_2, x_3,$ and x_4 are near or on the $j\omega$ -axis (i.e., they are of the active modes). Therefore, the synaptic noises ω_1 are filtered (attenuated) by the passive modes of synaptic processes, whereas the channel noises $\omega_2, \omega_3,$ and ω_4 are enhanced by the active modes of ionic channel processes. These facts are consistent with the observation that the HH system under only synaptic noises spends most of its time fluctuating around the equilibrium point and displays trains with a few short periodic oscillations called *rigid excitation* (Basalyga and Salinas, 2006; Lee et al., 1998). Furthermore, for confirmation, the synaptic noise-induced attenuation and the channel noise-induced enhancement are shown respectively in Figures 3A and 3B by Monte Carlo simulation. Obviously the mean firing rate of the spike trains of membrane potential is induced by active modes of channel processes, and the synaptic noises

Figure 2: (A) The locations of eigenvalues of A_i in the fuzzy locally linearized systems in the S -domain with $S = \sigma + j\omega$ in the four cases $I = 0, 6, 8,$ and $9.5 \text{ } \mu\text{A}/\text{cm}^2$. The eigenvalues corresponding to the synaptic processes and ionic channel processes are respectively denoted by circles and crosses. The eigenvalues of the ionic channel processes for sodium activation m , potassium activation n , and sodium inactivation h are denoted by red cross, green cross and blue cross, respectively. The state variables x_1 of synaptic processes in the four cases have no eigenvalues located on the $j\omega$ -axis; the state variables x_2 of sodium activation processes in the four cases have 0, 8, 3, and 1 eigenvalues, respectively, located on the $j\omega$ -axis; the state variables x_3 of sodium inactivation processes in the four cases have 0, 1, 1, and 3 eigenvalues, respectively, located on the $j\omega$ -axis; and the state variables x_4 of potassium activation processes in the four cases have 0, 9, 4, and 4 eigenvalues respectively located on the $j\omega$ -axis. Therefore, the channel noises are enhanced by the active modes of ionic channel processes when the synaptic noises are filtered by the passive modes of the synaptic processes. (B) The locations of eigenvalues of linearized deterministic HH neuron system in the S -domain with $S = \sigma + j\omega$ in the four cases $I = 0, 6, 8,$ and $9.5 \text{ } \mu\text{A}/\text{cm}^2$. The symbols of state variables are the same as those shown in *A*. In the four constant current cases of linearized deterministic HH neuron system, the state variables x_2 and x_4 are both located on the $j\omega$ -axis. Although *A* and *B* are partly similar, the characteristics of nonlinear stochastic neuron system approximated by stochastic fuzzy system cannot be interpreted completely by the linearized deterministic case of HH neuron system. From the above two results, we infer that noise enhancement is performed by the active modes of two major processes: sodium activation and potassium activation processes. As the constant current level increases, noise enhancement is partly performed by the active modes of sodium inactivation. (For the color version, see Figure S1 in the supplementary material, available online at <http://www.mitpressjournals.org/doi/suppl/10.1162/neco.2010.07-09-1057>.)

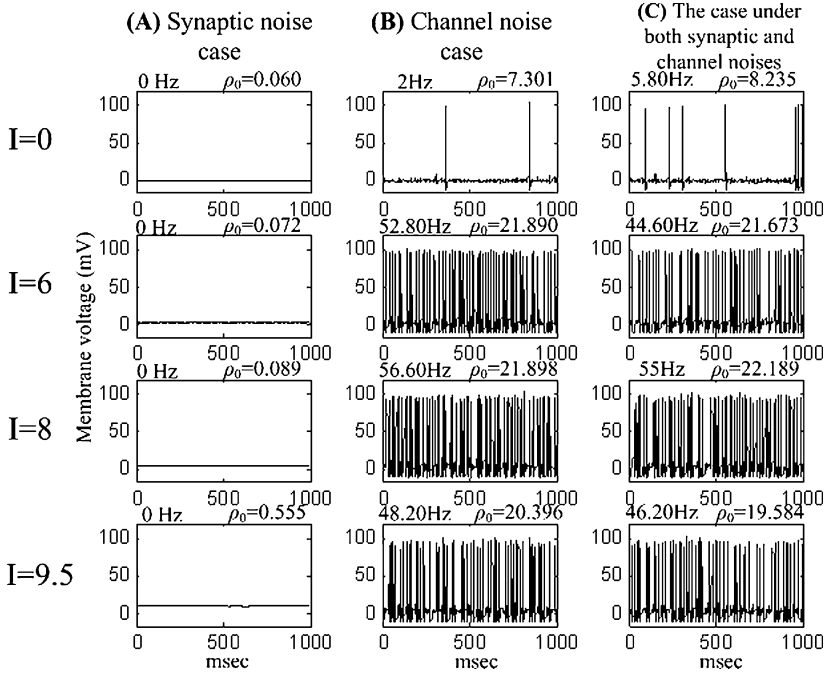


Figure 3: Noise-enhancing (or noise-filtering) abilities, membrane voltages $v(t)$, and their resonant frequencies are obtained from Monte Carlo simulations with different constant currents ($I = 0, 6, 8, 9.5 \mu\text{A}/\text{cm}^2$) under three cases of environmental noises: (A) under only synaptic noises, (B) under only channel noises, and (C) under both synaptic and channel noises. Obviously synaptic noises are attenuated as in Figure 3A by passive modes of synaptic processes of HH neuron systems as shown in Figure 2A. The channel noises are enhanced in B by active eigenvalues of ionic channel processes as in Figure 2A. The affection on the NEF in C is dominated by the presence of channel noises. For the computations of mean firing rates, the occurrences of action potentials are defined by upward crossings of voltage threshold 60 (mV) if it had previously crossed the equilibrium point of membrane potentials from below (Dayan and Abbott, 2001; Koch, 1999).

are attenuated by passive modes of synaptic processes. In general, the simulation results by Monte Carlo simulation match the above estimated results of the proposed method in equation 2.18 quite well. From a systematic point of view, synaptic processes can attenuate synaptic noises in HH systems by locating all eigenvalues on the left-hand side of $j\omega$ -axis (see the black circles in Figure 2A). The ionic channel processes play a major role in accelerating the generation of probabilistic spiking in the stochastic HH system by locating some eigenvalues of sodium activation, m (see Figure 2A, red cross), and potassium activation, n (see Figure 2A, green

cross), on the $j\omega$ -axis. As the constant current level increases, probabilistic spiking is accelerated by locating not only sodium activation and potassium activation but also sodium inactivation, h (see Figure 2A, blue cross), on the $j\omega$ -axis. Therefore, stochastic ionic channel processes may boost neuronal response to weak stimuli and are responsible for subthreshold oscillations and spiking behavior (Lee & Kim, 1999; White, Budde, & Kay, 1995).

Finally, this study discusses the total effect of both synaptic and channel noises on HH neuron systems. Then by solving the LMI-constrained optimization in equation 2.18 under both synaptic and channel noises (i.e., in the case of $H = I$, we estimate the total NEF $\rho_0 = 8.235$ when $I = 0$ ($\mu\text{A}/\text{cm}^2$); $\rho_0 = 21.673$ when $I = 6$ ($\mu\text{A}/\text{cm}^2$); $\rho_0 = 22.189$ when $I = 8$ ($\mu\text{A}/\text{cm}^2$); and $\rho_0 = 19.584$ when $I = 9.5$ ($\mu\text{A}/\text{cm}^2$) (see Figure 4B). Obviously total noises are enhanced by the HH neuron systems in the four constant stimulation current cases. Under these circumstances, the environmental noises can accelerate spike generation as shown in Figure 3C by Monte Carlo simulation. Since there are several eigenvalues near the $j\omega$ -axis (see Figure 2A), the HH neuron system is very sensitive to the environmental noises from the point of view of the system. From the Monte Carlo simulation in Figure 3C, the total noise-enhancing level of the HH neuron system can be approximately measured as follows,

$$\rho^2 \approx \frac{\frac{1}{50} \sum_{k=1}^{50} \int_0^\infty v_k^2(t) dt}{\frac{1}{50} \sum_{k=1}^{50} \int_0^\infty W_k^T(t) W_k(t) dt} = \begin{cases} (8.0837)^2 & \text{if } I = 0 \mu\text{A}/\text{cm}^2 \\ (21.1021)^2 & \text{if } I = 6 \mu\text{A}/\text{cm}^2 \\ (21.9579)^2 & \text{if } I = 8 \mu\text{A}/\text{cm}^2 \\ (19.3566)^2 & \text{if } I = 9.5 \mu\text{A}/\text{cm}^2 \end{cases} ,$$

where k denotes the number of random trials (see Figure 4B).

Remark 6. In traditional studies (Jung & Shuai, 2001; Schmid et al., 2004; Zeng & Jung, 2004), the measurement of variability is statistically solved by several random trials of stochastic ODE, but in this study, we propose a systematic method to estimate the noise enhancement and suppression factor of stochastic HH neuron systems from the system amplification or attenuation point of view by solving an LMI-constrained optimization problem in equation 2.18 instead. Monte Carlo simulation is used only to produce the noise-enhancing or noise-filtering level and spike frequency to validate our results.

From the simulation examples in Figure 3, it is apparent that the NEF estimated by the proposed H_∞ -enhancing (filtering) theory can be approached by increasing (decreasing) the noise-induced firing rate. Obviously the HH neuron system is sensitive to noises, which easily generate “spontaneous” spikes (Gammaitoni, Hanggi, Jung, & Marchesoni, 1998; Levin & Miller, 1996; Nozaki, Mar, Grigg, & Collins, 1999; Schneidman et al., 1998). This

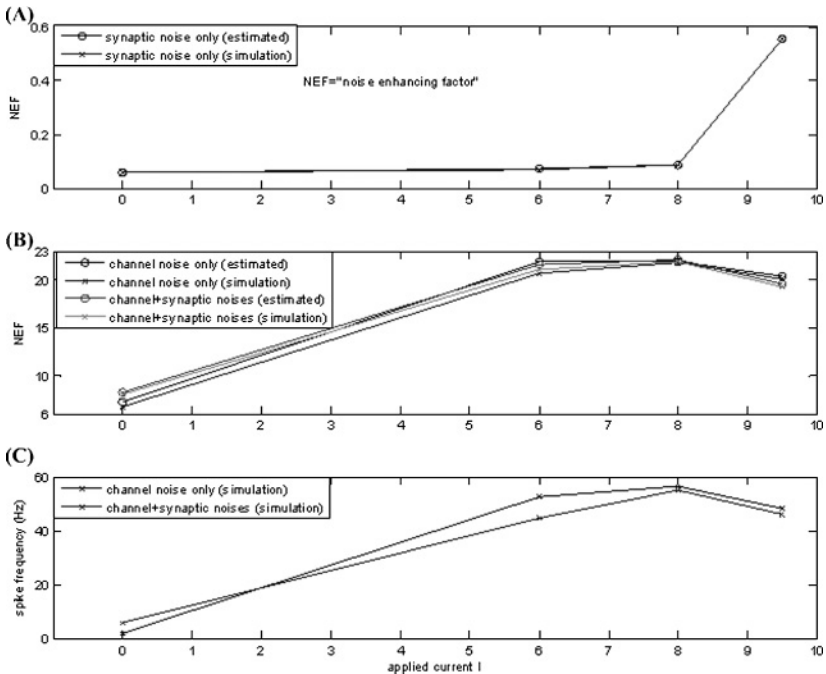


Figure 4: (A) The comparison of the estimated NEF (black line) and noise-enhancing/filtering level (blue line) under only synaptic noises in different constant currents. (B) The comparison of the estimated NEF (black and red lines) and noise-enhancing/filtering level (blue and green lines) under only channel noises and both channel and synaptic noises in different constant currents. The noise-enhancing level of the HH neuron system is approximately measured by the Monte Carlo simulation under only channel noises (blue line) and both channel and synaptic noises (green line) in different constant currents. (C) The measured spike frequency by the Monte Carlo simulation under only channel noises (black line) and both channel and synaptic noises (blue line) in different constant currents. The Monte Carlo simulation can validate our results very well. Therefore, we infer that NEF can be seen as an index in both noise-enhancing level and spike frequency (For color version see Figure S2 in the supplementary material, available online at <http://www.mitpressjournals.org/doi/suppl/10.1162/neco.2010.07-09-1057>.)

fact can be proven by nonlinear H_{∞} -enhancing (filtering) theory and confirmed by computer simulation (Schneidman et al., 1998). Although the spikes are easily generated, the firing rate of the spike trains has to reach a rate threshold to pass the information to downstream neurons (Lo & Wang, 2006; Luce, 1986; Usher & McClelland, 2001). From Figure 3, we show that the mean firing rates are correlated with NEFs. For example, both the mean

firing rate and the NEFs are small under only synaptic noises present with the lack of active modes in the synaptic process (see Figure 3A) and large under only channel noises and both synaptic and channel noises that result from enhancement of active modes in channel processes (Figures 3B and 3C) in HH neuron systems. Furthermore, by comparing the estimated NEF (see Figure 4B; black and red lines) solved by the LMI-constrained optimization problem in equation 2.18 with noise-enhancing or noise-filtering level (see Figure 4B; blue and green lines) and spike frequency (Figure 4C) measured by the Monte Carlo simulation, the NEF ρ_0 can be seen as an index to pass neural information to downstream neurons.

3 Discussion and Conclusion

Biological neurons have been found to be noisy in both the generation of action potentials and the transmission of synaptic signals. This is because the biophysical mechanisms involved are stochastic in nature. Among the noises in neurons, channel noises come mainly from the random openings of ion channels, while synaptic noises involve multiple mechanisms such as quantal releases of neural transmitters (White et al., 2000). As noises directly affect the reliability of neurons, scholars have taken great interest in how neurons disregard or even incorporate their intrinsic and extrinsic noises, so as to process information reliably (Butson & Clark, 2008; Schneidman et al., 1998; Stein et al., 2005; White et al., 2000).

Some studies based on stochastic neuron models have indicated that noise is crucial for enriching a single neuron's behavioral variety (Saarinen et al., 2006; Tuckwell, 2005), as well as for high-level information processing (Salinas, 2006; Tuckwell, 2005). For example, for the fusimotor system, noise-induced resonance is used to increase the sensitivity of muscle spindle to stretching (Cordo et al., 1996). For other neuron systems, noise-induced resonance has been found useful for enhancing the sensitivity of sensory neurons in detecting weak periodic signals (Stein et al., 2005; Wiesenfeld & Moss, 1995) or enhancing the sensitivity of mechano-electrical transduction (Jaramillo & Wiesenfeld, 1998). These intriguing findings suggest that neurons can utilize rather than suppress noises during computing.

In traditional studies, the measurement of variability is statistically solved by several random trials of stochastic ODE, and they discussed the effect of total numbers of sodium and potassium channels on the variations of interspike interval (Jung & Shuai, 2001; Schmid et al., 2004; Zeng & Jung, 2004). In this study we propose a systematic method to measure the noise enhancement and suppression factor of stochastic HH neuron systems from the worst-case signal-to-noise energy ratio point of view. The worst-case signal-to-noise energy ratio ρ_0 is equivalent to the system gain (i.e., system amplification or attenuation ability; in the linear system case, ρ_0 is related to the largest peak of transfer function of the linear system; Boyd, 1994) from $W(t)$ to $v(t)$, which is dependent on the system characteristics,

like eigenvalues, from the system theory point of view (Boyd, 1994). In the linear filter case, the lowpass, bandpass, or highpass filtering ability is rather dependent on the transfer function or the eigenvalues of system matrix of the linear filter than dependent on the external noises. Our method is the extension of the linear filtering theory to the nonlinear filtering theory only if the neuron system is considered as a nonlinear filter. Monte Carlo simulation is used only to produce noise-enhancing or noise-filtering level and spike frequency to validate our results. The estimation of NEF is to measure the signal-to-noise ratio—the system gain from all possible input noises $W(t)$ to output membrane potential $v(t)$, from the perspective of mean energy. From our result, the measure of the noise enhancement or suppression factor is determined by the system characteristics, for example, the eigenvalues of fuzzy linearized neuron systems. Therefore, noise variance does not affect on the noise enhancement or suppression factor ρ_0 of the derived results, because we have considered all possible noises with bounded variances in the analysis process. The input noise cannot affect the intrinsic system characteristics of the neuron system. Although the total numbers of sodium and potassium channels N_{Na} and N_K will affect the variances of channel noises, the NEF from the perspective of the mean energy ratio is correlated with the system characteristic of the neuron system rather than with the variances of channel noises if they are finite.

According to our results in the estimation of the NEF of stochastic HH neuron systems in Figure 4B, the NEF is increased with externally applied current and slightly decreased in the case $I = 9.5 \mu\text{A}/\text{cm}^2$ under both synaptic and channel noises. The same result is also shown in Schmid et al. (2004). The mean spiking rate under $I = 10 \mu\text{A}/\text{cm}^2$ is N times larger than that under $I = 0 \mu\text{A}/\text{cm}^2$ in different numbers of ion channels.

In this study, a nonlinear stochastic HH neuron model is developed to formulate HH neuron systems with synaptic noises and channel noises. In order to exploit the NEF of the neuron system, according to the nonlinear stochastic H_∞ -enhancing (filtering) theory, the estimation problem of noise-induced enhancing level is formulated as in proposition 1. In order to avoid the difficulty of solving HJI for estimating NEF, a T-S fuzzy system is employed to approach the nonlinear stochastic HH neuron model with the interpolation of several local linear stochastic models. Hence, solving HJI for estimating the noise enhancement problem in the HH neuron system is simplified by solving a set of LMIs as in proposition 2, which can be efficiently solved by the LMI toolbox in Matlab. As ρ_0 increases, the affection of the noises on HH neuron systems increases. Under such circumstances, some states of the neuron system may be induced by noises and are related to the generation of probabilistic spiking. The accelerated spike generation that enhances the noise-induced firing rate of spike trains is related to the active eigenvalues on the $j\omega$ -axis of fuzzy linearized HH neuron systems. In general, synaptic noises are more attenuated by passive eigenvalues of synaptic processes, whereas channel noises are more enhanced by the active

eigenvalues of stochastic ionic channel processes. Unlike conventional stochastic simulation methods, we propose a method of theoretical analytical estimation based on nonlinear stochastic H_∞ -enhancing (filtering) theory from the system perspective—that is, the noise-enhancing or noise-filtering ability would lie in the intrinsic system characteristics of neuron system rather than in the external noises. The proposed method of NEF measurement can be confirmed by the measurement of Monte Carlo stochastic simulation and can be also seen as an index to pass neural information to downstream neurons.

Appendix A: Proof of Proposition 1 _____

Before the proof of propositions, the following fact (fact 1) is necessary (Boyd, 1994):

$$X^T P Y + Y^T P X \leq \frac{1}{\alpha} X^T P X + \alpha Y^T P Y$$

for vectors X, Y , a constant $\alpha > 0$, and a positive-definite matrix $P > 0$ with approximate dimensions.

Proof of Proposition 1. Consider the following equivalent equation:

$$\begin{aligned} E \int_0^\infty v^2(t) dt &= E\Phi(x(0)) - E\Phi(x(\infty)) \\ &+ E \int_0^\infty (x^T(t)Qx(t) dt + d\Phi(x(t))) \end{aligned} \tag{A.1}$$

where $\Phi(x(t)) > 0$.

By chain rule, we get (Gardiner, 1983; Khas'minskii, 1980)

$$d\Phi(x(t)) = \frac{\partial\Phi(x(t))^T}{\partial x} (F(x(t) + X^*) + HW(t)) dt. \tag{A.2}$$

Substituting the above equation into A.1, by the fact that $\Phi(x(\infty)) > 0$, we get

$$\begin{aligned} E \int_0^\infty v^2(t) dt &\leq E\Phi(x(0)) + E \int_0^\infty \left(x^T(t)Qx(t) \right. \\ &\left. + \frac{\partial\Phi(x(t))^T}{\partial x} (F(x(t) + X^*) + HW(t)) \right) dt. \end{aligned} \tag{A.3}$$

By fact 1 with $\alpha = \rho^2$, $X = (1/2) \cdot H^T \partial \Phi(x(t))/\partial x$, and $Y = W(t)$, we have

$$\begin{aligned} \frac{\partial \Phi(x(t))^T}{\partial x} HW(t) &= \frac{1}{2} \frac{\partial \Phi(x(t))^T}{\partial x} HW(t) + \frac{1}{2} W(t)^T H^T \frac{\partial \Phi(x(t))}{\partial x} \\ &\leq \frac{1}{4\rho^2} \frac{\partial \Phi(x(t))^T}{\partial x} HH^T \frac{\partial \Phi(x(t))}{\partial x} + \rho^2 W(t)^T W(t). \end{aligned} \tag{A.4}$$

Therefore, we can obtain

$$\begin{aligned} E \int_0^\infty v^2(t) dt &\leq E\Phi(x(0)) \\ &\quad + E \int_0^\infty \left[x^T(t) Q x(t) + \frac{\partial \Phi(x(t))^T}{\partial x} F(x(t) + X^*) + \right. \\ &\quad \left. + \frac{1}{4\rho^2} \frac{\partial \Phi(x(t))^T}{\partial x} HH^T \frac{\partial \Phi(x(t))}{\partial x} + \rho^2 W(t)^T W(t) \right] dt. \end{aligned} \tag{A.5}$$

By the inequality in equation 2.9, we get

$$E \int_0^\infty v^2(t) dt < E\Phi(x(0)) + \rho^2 E \int_0^\infty W(t)^T W(t) dt. \tag{A.6}$$

Obviously, the noise-enhancing level in equation 2.8 is achieved.

Appendix B: Proof of Proposition 2 _____

Proof of Proposition 2. Consider the following equivalent equation:

$$\begin{aligned} E \int_0^\infty v^2(t) dt &= E\Phi(x(0)) - E\Phi(x(\infty)) \\ &\quad + E \int_0^\infty (x^T(t) Q x(t) dt + d\Phi(x(t))), \end{aligned} \tag{B.1}$$

where $\Phi(x(t)) = x(t)^T P x(t)$ for some $P = P^T > 0$.

By chain rule (Khas'minskii, 1980) and equation 2.14, we get

$$\begin{aligned} E [d\Phi(x(t))] &= E \frac{\partial \Phi(x(t))^T}{\partial x} \left\{ \sum_{i=1}^L h_i(x) [A_i x(t) + HW(t) + \Delta f(x)] \right\} dt \\ &= E \left\{ \sum_{i=1}^L h_i(x) [x(t)^T (P A_i + A_i^T P) x(t) + x(t)^T P H W(t) \right. \end{aligned}$$

$$\begin{aligned}
 &+ W(t)^T H^T P x(t) \\
 &+ x(t)^T P \Delta f(x) + \Delta f(x)^T P x(t)] dt \}. \tag{B.2}
 \end{aligned}$$

Substituting the above equation into B.1, by the fact that $\Phi(x(t)) = x(t)^T P x(t)$, we get,

$$\begin{aligned}
 E \int_0^\infty v^2(t) dt &\leq E\Phi(x(0)) \\
 &+ E \int_0^\infty \left\{ \sum_{i=1}^L h_i(x) [x(t)^T (P A_i + A_i^T P + Q)x(t) \right. \\
 &+ x(t)^T P H W(t) + W(t)^T H^T P x(t) + x(t)^T P \Delta f(x) \\
 &\left. + \Delta f^T(x) P x(t) \right\} dt. \tag{B.3}
 \end{aligned}$$

By fact 1, we get

$$x(t)^T P H W(t) + W(t)^T H^T P x(t) \leq \frac{1}{\rho^2} x(t)^T P H H^T P x(t) + \rho^2 W(t)^T W(t) \tag{B.4}$$

and

$$x(t)^T P \Delta f(x) + \Delta f^T(x) P x(t) \leq \frac{1}{\gamma^2} x(t)^T P P x(t) + \gamma^2 \Delta f^T(x) \Delta f(x). \tag{B.5}$$

By equations B.4, B.5, and 2.15, we get

$$\begin{aligned}
 E \int_0^\infty v^2(t) dt &\leq E\Phi(x(0)) \\
 &+ E \sum_{i=1}^L \int_0^\infty h_i \left[x^T(t) \left(P A_i + A_i^T P + Q + P \left(\frac{1}{\gamma^2} I + \frac{1}{\rho^2} H H^T \right) P \right. \right. \\
 &\left. \left. + \gamma^2 \alpha^2 I \right) x(t) + \rho^2 W^T(t) W(t) \right] dt. \tag{B.6}
 \end{aligned}$$

By the inequalities in equation 2.16, we get

$$E \int_0^\infty v^2(t) dt < E\Phi(x(0)) + \rho^2 E \int_0^\infty W^T(t) W(t) dt. \tag{B.7}$$

Obviously, the noise-enhancing level in equation 2.8 is achieved. Furthermore, by Shur complement (Boyd, 1994), the inequalities in equation 2.16 are equivalent to the LMIs in equation 2.18.

Acknowledgments

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