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On the Multichannel Rendezvous Problem: Fundamental Limits, Optimal Hopping Sequences, and Bounded Time-To-Rendezvous

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One of the fundamental problems in a cognitive radio network, known as the multichannel rendezvous problem, is for two secondary users to find a common channel that is not blocked by primary users. The basic idea for solving such a problem in most works in the literature is for the two users to select their own channel hopping sequences and then rendezvous when they both hop to a common unblocked channel at the same time. In this paper, we focus on the fundamental limits of the multichannel rendezvous problem and formulate such a problem as a constrained optimization problem, where the selection of the random hopping sequences of the two secondary users must satisfy certain constraints. We derive various lower bounds for the expected (resp. maximum) time-to-rendezvous under certain constraints. For some of these lower bounds, we are also able to construct optimal channel hopping sequences that achieve the lower bounds. Inspired by the constructions of quorum systems and relative difference sets, our constructions of the channel hopping sequences are based on the mathematical theories of finite projective planes, orthogonal Latin squares and sawtooth sequences. The use of such theories in the constructions of channel hopping sequences appear to be new and better than other existing schemes in terms of minimizing the expected (resp. maximum) time-to-rendezvous.

Key words: rendezvous search; cognitive radio networks; finite projective planes; orthogonal Latin squares
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1. Introduction The rendezvous search problem that asks how two unit speed searchers, randomly placed in a known search region, can minimize the time required to meet, has been one of the classical problems in the area of operations research (see e.g., the book by [7] and references therein). Anderson and Weber [10] first studied the rendezvous problem on discrete locations. The rendezvous search problem was generalized to the compact metric space in [1]. Since then, the rendezvous search problem on various topologies has been studied extensively in the literature, e.g., the rendezvous search problem on the line [6, 8, 25, 18], the rendezvous search problem on the interval [27], the rendezvous search problem on a graph [5, 2, 22, 35], the two-dimensional rendezvous problem [9, 3, 31, 32], and rendezvous in higher dimensions [4].

Motivated by the recent development in cognitive radio networks, in this paper we study the multichannel rendezvous problem. As the use of spectrum in current wireless networks is regulated by the fixed spectrum allocation, spectrum is not efficiently used. Cognitive radio, as a new mechanism for dynamic spectrum access, has received a lot of attention in the communication and networking literature. In a cognitive radio network, there are two types of spectrum users: *primary users* and *secondary users*. Primary users usually have the licence to use the spectrum assigned to them. On the other hand, secondary users are only allowed to share spectrum with primary users provided that they do not cause any severe interference to the primary users. To do this, secondary users first sense a number of frequency channels. If a channel is not blocked by a primary user, then that channel may be used for secondary users to establish a communication link. One of the fundamental problems in a cognitive radio network is then for two secondary users to find a common *unblocked* channel and such a problem is known as the multichannel rendezvous problem.

There are several protocols and algorithms that have been proposed in the literature for solving the multichannel rendezvous problem (see e.g., [11, 13, 20, 26, 36, 39, 41, 42, 46, 33]). The basic idea behind most of these works is known as the *channel hopping* mechanism, in which each secondary user hops over the channels with respect to time and *eventually* rendezvous the other party at an unblocked channel. The time-to-rendezvous is then one of the key performance metrics in the multichannel rendezvous problem. A good channel hopping mechanism should have a low expected time-to-rendezvous and sometimes preferably a deterministically bounded time-to-rendezvous.

There are many fascinating methods in selecting hopping sequences. In particular, SSCH [11] proposed using prime number modular arithmetic to generate the hopping sequences. In SSCH, each secondary user chooses a hopping seed and an initial hopping channel and hops to the next channel by adding the hopping seed according to the modulo operation. It was argued that hopping sequences with different hopping seeds will rendezvous within a finite period. However, hopping sequences with identical hopping seeds but different initial channels might not rendezvous. For this, SSCH cleverly adds a *parity slot* for those sequences to rendezvous and that prevents logical partition of hopping sequences. In [39], the idea of rotating initial seeds in every period was added into SSCH to increase robustness. By so doing, the rendezvous channels are different in every period to avoid the problem of long term blocking from primary users. By using the Chinese Remainder Theorem, the modular clock algorithm was further generalized in [42] to show that two secondary users will still rendezvous even if they both use different modulo functions. In addition to the prime number modular arithmetic, quorums and difference sets were used to generate hopping sequences in [13, 26]. Difference sets are commonly used to construct quorum systems that guarantee nonempty intersections of sets (and thus rendezvous of users) with phase difference. As such, time synchronization of the channel hopping mechanism is not needed. In particular, the sawtooth sequence in [45] is of particular interest to us. In [20, 40, 33, 42, 12, 17], there are alternative constructions that also do not need time synchronization.

In this paper, we focus on the fundamental limits of the multichannel rendezvous problem. For this, we consider a system with n channels (with $n \geq 2$), indexed from 0 to $n-1$, in the discrete-time setting, where time is indexed from $t = 1, 2, \ldots$. There are two users who would like to rendezvous on a common unblocked channel by hopping over these n channels with respect to time. Denote by $X_1(t)$ (resp. $X_2(t)$) the channel selected by user 1 (resp. user 2) at time t. Let $B(t)$ be the set of channels that are blocked at time t. If a channel is blocked at time t, then the two users will not rendezvous even though they both hop to that channel at time t. Then the time-to-rendezvous, denoted by T, is the first time that these two users select a common unblocked channel, i.e.,

$$T = \inf\{t : X_1(t) = X_2(t) \notin B(t)\}. \tag{1}$$

The multichannel rendezvous problem here is mathematically equivalent to the rendezvous problem on a *labelled* complete graph (Chapter 13 of [7]) as the channels in the multichannel rendezvous problem are *labelled* and known to both users. Since the channels are labelled, both users could simply use the FOCAL strategy [7] and hop to channel 0 when the rendezvous process is started. However, this is not a desirable thing to do in a cognitive radio network as all the secondary users will hop to the same channel and that cause severe interference among themselves. Such a problem is known as the control channel saturation problem (see e.g., [39] and references therein). As such, it is much more desirable to have all the pairs of secondary users rendezvous on different channels.

In practice, there are several constraints for the selection of hopping sequences. In the following, we summarize various verbal descriptions in the literature (see e.g., [13, 42, 46] and references therein) for these constraints.

(i) **(Independence of users)** As both users are not able to communicate with each other at the beginning, they are likely to select their hopping sequences independently.

(ii) **(Efficient and fair use of channels)** In a cognitive radio network, the secondary users should try their best to rendezvous on a common channel that is not blocked by primary users. Ideally, a channel hopping protocol should spread out the rendezvous over all channels evenly [13].

(iii) **(Symmetric users' strategies)** Both users follow the same protocol (or algorithm) to generate their hopping sequences.

(iv) **(Channel blocking)** Primary users should not be affected by secondary users. As such, the behavior of primary users is independent of secondary users.

(v) **(Time synchronization)** Both users may not be synchronized in a cognitive radio network. As such, both users may not start the rendezvous process at the same time.

In this paper, we formalize the above five constraints into the following five mathematical constraints.

(A1) **(Independence constraint)** $\{X_1(t), t \geq 1\}$ and $\{X_2(t), t \geq 1\}$ are two independent random sequences.

(A2) **(Uniform load constraint)** At any time t, each channel is selected with an *equal* probability, i.e., for all t and $i = 0, 1, 2, \ldots, n-1$,

$$\mathsf{P}(X_1(t) = i) = \mathsf{P}(X_2(t) = i) = \frac{1}{n}. \tag{2}$$

(A3) **(Identical distribution constraint)** $\{X_1(t), t \geq 1\}$ and $\{X_2(t), t \geq 1\}$ have the same joint distribution.

(A4) **(Channel blocking constraint)** The two hopping sequences $\{X_1(t), t \geq 1\}$ and $\{X_2(t), t \geq 1\}$ are independent of $\{B(t), t \geq 1\}$. Also, $\mathsf{E}[|B(t)|] \geq b$ for all t and some nonnegative number b, where $|B(t)|$ is the number of channels that are blocked at time t.

(A5) **(Stationarity constraint)** $\{X_1(t), t \geq 1\}$ and $\{X_2(t), t \geq 1\}$ are *stationary* sequences, i.e., the joint distributions of $\{X_1(t), t \geq 1\}$ and $\{X_2(t), t \geq 1\}$ are invariant to any time shift.

The independence constraint in (A1) seems plausible as both users are not able to communicate with each other at the beginning and they are likely to select their hopping sequences independently. The second constraint in (A2) is for the efficient usage of channels as described in [13, 46]. It is defined in [13] that the *(system) load* of a channel hopping mechanism is the maximum probability (over space and time) that a user hops to a particular channel at a particular time. Clearly, the minimum load of a channel hopping system with n channels is $1/n$ (as each user hops to one of the n channels in every step). On the other hand, the maximum load could be 1. This happens when both users select the degenerate random sequence with $\mathsf{P}(X_1(t) = 1) = \mathsf{P}(X_2(t) = 1) = 1$ for all t. In this trivial scenario, both users select the same hopping sequence and they also rendezvous at time 1 with probability 1. The problem with such a degenerate random sequence is that both users always rendezvous at the same channel and that causes the so-called control channel saturation problem mentioned before. As such, it is of importance to impose a constraint on the load of a channel hopping mechanism. In this paper, we impose the uniform load constraint in (A2) to achieve maximum efficiency for the usage of channels. The third constraint in (A3) is a natural one if both users follow the same protocol (or algorithm) to generate their hopping sequences. The channel blocking constraint in (A4) is used for cognitive radio networks, where there are channels blocked by primary users. If a channel is blocked at time t, then the two users will not rendezvous even though they hop to that channel at time t. The stationary constraint in (A5) appears to be a technical one at the first look, but it is a condition for avoiding time synchronization in the discrete-time setting. If both hopping sequences are stationary, then the joint distributions are invariant to arbitrary time shift. As such, it does not matter whether both users start the rendezvous process at the same time and there is no need for time synchronization. Note that for (A5) to be valid, one still needs to assume that both users follow the discrete-time setting. In other words, there is a global clock and time t_1 of user 1 (resp. t_2 of user 2) might be time t_1' (resp. t_2') of the global clock with t_1 and t_1' (resp. t_2 and t_2') being integers.

In comparison with the classical symmetric rendezvous search in [7], the multichannel rendezvous problem imposes the uniform load constraint in (A2) and thus the FOCAL strategy [7] are ill-suited for applications in cognitive radio networks. Also, the symmetric users' strategies in the multichannel rendezvous problem does not imply that the users' strategies are symmetric with respect to *channels (locations)*. For instance, for the two-channel case, i.e., $n=2$, each user could choose with probability 1/4 from the four periodic sequences with period 3: 000, 011, 101 and 110. Such a choice clearly satisfies the uniform load constraint in (A2). However, it is not symmetric with respect to these two channels and this is quite different from symmetric rendezvous search for *unlabelled* locations in [10]. The constraint (A4) for channel blocking is basically the same as noisy rendezvous in the literature. The additional constraint (A5) on time synchronization seems to be new. With all these, one might be able to further classify the rendezvous search problem by whether the *two users* are labelled, the *channels (locations)* are labelled, and *time* is labelled.

Our first objective of this paper is to establish *fundamental limits* for the time-to-rendezvous T under various subsets of the above constraints for selecting the two random hopping sequences $\{X_1(t), t \geq 1\}$ and $\{X_2(t), t \geq 1\}$. There are two specific performance metrics in which we are particularly interested: the *expected* time-to-rendezvous and the *maximum* time-to-rendezvous (the least deterministic upper bound for the time-to-rendezvous). We will first derive lower bounds for these two performance metrics under a certain subset of these constraints, starting from the simplest case with (A1) and (A2) and then adding additional constraints along the way. Our second objective is then to seek methods to construct *optimal random hopping sequences* that achieve these lower bounds under various subsets of these constraints. For certain subsets of constraints, we are able to show these lower bounds are tight. However, there are also cases that we are not able to construct hopping sequences that achieve the lower bounds. For these cases, we report our constructions that have the smallest maximum time-to-rendezvous in the literature.

Our main results and contributions are summarized as follows:

(i) In Section 2, we consider users with potentially different distributions for hopping sequences. Under (A1) and (A2), we show in Theorem 1 that there does not exist hopping sequences such that the time-to-rendezvous can be deterministically bounded by $n-1$ steps. Moreover, the expected time-to-rendezvous is lower bounded by $(n+1)/2$. In Example 1, we show the wait-for-mommy approach guarantees that the maximum time-to-rendezvous is n and it achieves the lower bound for the expected time-to-rendezvous. Hence, the wait-for-mommy approach is optimal under (A1) and (A2).

(ii) In Section 3, we consider users with identical distributions for hopping sequences. Under (A1-3), we show in Theorem 2 of Section 3.1 that there does not exist hopping sequences such that the time-to-rendezvous can be deterministically bounded by n steps. Moreover, the expected time-to-rendezvous is lower bounded by $\frac{n+1}{2} + \frac{1}{2} - \frac{1}{2n}$. In Theorem 3 of Section 3.2, we show if there is a

TABLE 1. Summary of the achievable lower bounds for the time-to-rendezvous T under various constraints

Set of constraints	A1, A2	A1, A2, A3	A1, A2, A4
Det. lower bound on T	n	$n+1$	$n_1^* = n+b+\lceil \frac{b^2}{n-b} \rceil$
Lower bound on $\mathsf{E}[T]$	$(n+1)/2$	$(n+1)/2+1/2-1/2n$	$n_1^* - (\frac{n-b}{n^2})(\frac{n_1^*(n_1^*-1)}{2})$
Tightness	for all n	for all prime power n	$b=1$, $n \geq 3$ and $n \neq 6$
References	Theorem 1	Theorems 2 and 3	Theorems 4 and 5

finite projective plane of order n, then one can construct random hopping sequences (see Algorithm 1 for more details) such that the maximum time-to-rendezvous is $n+1$ and they also achieve the lower bound for the expected time-to-rendezvous. Such a construction is optimal and it is nearly twice better than SYNC-ETCH [46]. The use of finite projective planes for constructing optimal hopping sequences for the multichannel rendezvous problem appears to be new (to the best of our knowledge) though finite projective planes were previously used in [34] to establish quorum systems for mutual exclusion in decentralized systems and in [19] for the construction of optical orthogonal codes. Since there is a systematic method to construct a finite projective plane of order n when n is a prime power, it is much more general than the prime number modular arithmetic in [11, 39, 42]. We note that QCH [13] does not satisfy the uniform load constraint in (A2) and it tends to rendezvous at the same channel in a frame. Since finite projective planes can also be used for constructing quorum systems, our construction can be viewed as a better way to map points in finite projective planes to spread out the rendezvous over all channels evenly. In fact, we also proposed CACH in [47] that outperforms QCH in terms of reducing load while keeping the same maximum time-to-rendezvous.
(iii) In Section 4, we further take the channel blocking constraint in (A4) into account. Under (A1), (A2) and (A4), we show in Theorem 4 of Section 4.1 that there does not exist hopping sequences such that the time-to-rendezvous can be deterministically bounded by n_1^*-1 steps, where $n_1^* = n+b+\lceil \frac{b^2}{n-b} \rceil$. Moreover, the expected time-to-rendezvous is lower bounded by $n_1^* - (\frac{n-b}{n^2})(\frac{n_1^*(n_1^*-1)}{2})$. For the case with a single permanently blocked channel, i.e., $b=1$, we show how to construct random sequences by using two orthogonal Latin squares to achieve the lower bound (see Algorithm 2′ in Section 4.2 for more details). Hence, such a construction is optimal under (A1), (A2) and (A4) for $b=1$. Orthogonal Latin squares are closely related to finite projective planes [14]. Though orthogonal Latin squares were previously used in [28] for scheduling in multihop radio networks, its use for constructing optimal hopping sequences for the multichannel rendezvous problem appears to be new (to the best of our knowledge). For the case with multiple permanently blocked channels, i.e., $b>1$, we show in Theorem 6 in Section 4.3 that one can construct random hopping sequences (see Algorithm 3 for more details) that satisfy the constraints in (A1-4) and the maximum time-to-rendezvous is $(n+1)(b+1)$. Our construction is based on a rotated version of the hopping sequence from a finite projective plane.
(iv) In Section 5, we further take time synchronization in (A5) into account. Under (A1), (A2), (A3) and (A5), we use the sawtooth sequence in [45] to show in Theorem 7 of Section 5.1 that one can construct random hopping sequences (see Algorithm 4 for more details) that satisfy these four constraints and the maximum time-to-rendezvous is $2n+1$. For the setting with a single permanently blocked channel, we propose interleaving a sawtooth sequence and an inverted sawtooth sequence in time (see Algorithm 5 in Section 5.2 for more details) so that any two sequences are guaranteed to rendezvous within $4n+2$ steps.

These results are further summarized in Tables 1 and 2, where we summarize the achievable lower bounds for the time-to-rendezvous T under various constraints in Table 1 and the smallest deterministic upper bounds for the time-to-rendezvous T under various constraints in Table 2.

Table 2. Summary of the smallest deterministic upper bounds for the time-to-rendezvous T under various constraints

Set of constraints	A1-A4	A1-A3, A5	A1-A5, $b=1$	A1-A5, $b=n-1$
Det. upper bound on T	$(n+1)(b+1)$	$2n+1$	$4n+2$	$n(3n-1)$
Conditions	for all prime power n	for all n	for all even n	for all prime n
References	Theorem 6	Theorem 7	Theorem 8	[40]

2. Users with potentially different distributions for hopping sequences In this section, we show in Theorem 1 that under the constraints in (A1) and (A2), there does not exist random sequences such that the time-to-rendezvous can be deterministically bounded within $n-1$ steps. Moreover, the expected time-to-rendezvous under the constraints (A1) and (A2) is not lower than $(n+1)/2$. The proof of Theorem 1 is is basically the same as that in the first lemma of [10] and is thus omitted.

Theorem 1. *Under the constraints in (A1) and (A2),*

$$\mathsf{P}(T \geq n) \geq \frac{1}{n} > 0, \tag{3}$$

and

$$\mathsf{E}[T] \geq \frac{n+1}{2}. \tag{4}$$

Though our rendezvous problem is different from the *unlabelled* rendezvous problem in [10], the lower bound for the expected time-to-rendezvous and its proof are the same as that in the first lemma of that paper. Interesting enough, the well-known “wait-for-mommy” approach in [10, 42] can also be used for constructing two random sequences $\{X_1(t), t \geq 1\}$ and $\{X_2(t), t \geq 1\}$ that satisfy (A1) and (A2) and achieve the lower bounds in Theorem 1. We illustrate this in the following example.

Example 1. **(Wait for mommy)** *Let U_1 and U_2 be two uniform random variables over $[0, 1, \ldots, n-1]$ that are independent of each other. Let $\{X_1(t) = U_1, t \geq 1\}$ and $\{X_2(t) = ((U_2 + (t-1)) \bmod n), t \geq 1\}$. Clearly, (A1) and (A2) are satisfied. Also, note that user 1 (the child) fixes his/her channel once he/she makes the channel selection at time 1. On the other hand, user 2 (the mother) cycles through all the n channels periodically with period n. As such, they rendezvous before the end of a period, i.e., $T \leq n$. Moreover, we have $T = ((U_1 - U_2) \bmod n) + 1$ and thus T is also uniformly distributed over $[1, \ldots, n]$, i.e., for $\tau = 1, \ldots, n$,*

$$\mathsf{P}(T = \tau) = \frac{1}{n}. \tag{5}$$

This then leads to $\mathsf{E}[T] = \frac{n+1}{2}$.

3. Users with identical distributions for hopping sequences In this section, we further take (A3) into consideration. Under (A3), both users have identical distributions for their hopping sequences.

3.1. Lower bounds In this section, we show in the following theorem that there does not exist random sequences such that the time-to-rendezvous can be deterministically bounded by n steps under the constraints (A1), (A2) and (A3). Moreover, the expected time-to-rendezvous under these three constraints is not lower than $\frac{n+1}{2} + \frac{1}{2} - \frac{1}{2n}$. The proof of Theorem 2 is deferred to the end of this section.

THEOREM 2. *Under the constraints in (A1), (A2) and (A3), we have for $\tau \geq 2$,*

$$\mathsf{P}(T \geq \tau) \geq (1 - \frac{\tau - 1}{n} + \frac{(\tau - 2)}{n^2})^+, \tag{6}$$

where $x^+ = \max(0, x)$. As such, it follows that for $n \geq 2$,

$$\mathsf{P}(T \geq n+1) \geq \frac{1}{n}(1 - \frac{1}{n}) > 0, \tag{7}$$

and that

$$\mathsf{E}[T] \geq \frac{n+1}{2} + \frac{1}{2} - \frac{1}{2n}. \tag{8}$$

The rest of this section is devoted to the proof of Theorem 2. For the proof of Theorem 2, we first generalize the union bound in Lemma 1. Such a generalization is crucial to our proof and it might be of independent interest to probability theory as it is tighter than the classical union bound. Note that our generalized union bound is different from the Boole-Bonferroni family of inequalities [38] (which are also known as the generalized union bound).

LEMMA 1. (**Generalized union bound**) *For any $n \geq 2$ events, $E_1, E_2, \ldots, E_n$,*

$$\mathbf{1}_{\cup_{i=1}^n E_i} \leq \sum_{i=1}^n \mathbf{1}_{E_i} - \sum_{i=1}^{n-1} \mathbf{1}_{E_i \cap E_{i+1}}, \tag{9}$$

where $\mathbf{1}_E$ is the indicator random variable that takes value 1 if the event E is true and 0 otherwise. Thus,

$$\mathsf{P}(\cup_{i=1}^n E_i) \leq \sum_{i=1}^n \mathsf{P}(E_i) - \sum_{i=1}^{n-1} \mathsf{P}(E_i \cap E_{i+1}). \tag{10}$$

Proof. We prove (9) by induction. For $n = 2$, this is simply the inclusion-exclusion principle as

$$\mathbf{1}_{E_1 \cup E_2} = \mathbf{1}_{E_1} + \mathbf{1}_{E_2} - \mathbf{1}_{E_1 \cap E_2}.$$

Now we assume that (9) holds for some $n \geq 2$ as the induction hypothesis. Note that

$$\begin{aligned} \mathbf{1}_{\cup_{i=1}^{n+1} E_i} &= \mathbf{1}_{\{\cup_{i=1}^n E_i\} \cup E_{n+1}} \\ &= \mathbf{1}_{\cup_{i=1}^n E_i} + \mathbf{1}_{E_{n+1}} - \mathbf{1}_{\{\cup_{i=1}^n E_i\} \cap E_{n+1}}. \end{aligned} \tag{11}$$

Since $E_n \subseteq \cup_{i=1}^n E_i$, we have

$$\mathbf{1}_{E_n \cap E_{n+1}} \leq \mathbf{1}_{\{\cup_{i=1}^n E_i\} \cap E_{n+1}}. \tag{12}$$

Using (12) and the induction hypothesis (i.e., the inequality in (9)) in (11) yields

$$\begin{aligned} &\mathbf{1}_{\cup_{i=1}^{n+1} E_i} \\ &\leq \mathbf{1}_{\cup_{i=1}^n E_i} + \mathbf{1}_{E_{n+1}} - \mathbf{1}_{E_n \cap E_{n+1}} \\ &\leq \sum_{i=1}^n \mathbf{1}_{E_i} - \sum_{i=1}^{n-1} \mathbf{1}_{E_i \cap E_{i+1}} + \mathbf{1}_{E_{n+1}} - \mathbf{1}_{E_n \cap E_{n+1}} \\ &= \sum_{i=1}^{n+1} \mathbf{1}_{E_i} - \sum_{i=1}^{n} \mathbf{1}_{E_i \cap E_{i+1}}. \end{aligned} \tag{13}$$

Taking expectation on both sides of (9) yields (10). ■

Lemma 2. *Under the constraints in (A1) and (A2),*

$$\mathsf{P}(X_1(t)=X_2(t))=\frac{1}{n}. \tag{14}$$

The proof of Lemma 2 is immediate and is left to the reader.

Lemma 3. *Under the constraints in (A1) and (A3),*

$$\mathsf{P}(X_1(s)=X_2(s),X_1(t)=X_2(t))\geq\frac{1}{n^2},\quad s\neq t. \tag{15}$$

Proof. Note that

$$\begin{aligned}&\mathsf{P}(X_1(s)=X_2(s),X_1(t)=X_2(t))\\&=\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\mathsf{P}(X_1(s)=X_2(s)=i,X_1(t)=X_2(t)=j)\\&=\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\mathsf{P}(X_1(s)=i,X_1(t)=j,X_2(s)=i,X_2(t)=j).\end{aligned} \tag{16}$$

From the independence constraint in (A1) and the identical distribution constraint in (A3), it follows that

$$\begin{aligned}&\mathsf{P}(X_1(s)=i,X_1(t)=j,X_2(s)=i,X_2(t)=j)\\&=\mathsf{P}(X_1(s)=i,X_1(t)=j)\mathsf{P}(X_2(s)=i,X_2(t)=j)\\&=(\mathsf{P}(X_1(s)=i,X_1(t)=j))^2.\end{aligned} \tag{17}$$

Using this in (16), we then have

$$\mathsf{P}(X_1(s)=X_2(s),X_1(t)=X_2(t))=\sum_{j=0}^{n-1}\sum_{i=0}^{n-1}(\mathsf{P}(X_1(s)=i,X_1(t)=j))^2. \tag{18}$$

Note from Jensen's inequality that for any $x_1,x_2,\ldots,x_m\in\mathcal{R}$

$$\sum_{i=1}^{m}\frac{1}{m}x_i^2\geq(\frac{1}{m}\sum_{i=1}^{m}x_i)^2. \tag{19}$$

Also, we know from the law of the total probability that

$$\sum_{j=0}^{n-1}\sum_{i=0}^{n-1}\mathsf{P}(X_1(s)=i,X_1(t)=j)=1. \tag{20}$$

It then follows from (19) and (20) that

$$\begin{aligned}&\mathsf{P}(X_1(s)=X_2(s),X_1(t)=X_2(t))\\&=n^2\sum_{j=0}^{n-1}\sum_{i=0}^{n-1}\frac{1}{n^2}(\mathsf{P}(X_1(s)=i,X_1(t)=j))^2\\&\geq n^2\Big(\frac{1}{n^2}\sum_{j=0}^{n-1}\sum_{i=0}^{n-1}\mathsf{P}(X_1(s)=i,X_1(t)=j)\Big)^2\\&=\frac{1}{n^2}.\end{aligned} \tag{21}$$

■

Proof. (Theorem 2) The event $\{T \geq \tau\}$ is equivalent to the event that these two users do not rendezvous before time τ. Thus,

$$\begin{aligned}\mathsf{P}(T \geq \tau) &= \mathsf{P}(X_1(s) \neq X_2(s), 1 \leq s \leq \tau - 1)\\ &= 1 - \mathsf{P}(\cup_{s=1}^{\tau-1}\{X_1(s) = X_2(s)\}).\end{aligned} \tag{22}$$

Clearly, we have

$$\mathsf{P}(T \geq 2) = 1 - \mathsf{P}(X_1(1) = X_2(1)) = 1 - \frac{1}{n}, \tag{23}$$

where we use (14) in Lemma 2 for the last identity. Using the generalized union bound in (10) of Lemma 1, we have for $\tau \geq 3$,

$$\begin{aligned}&\mathsf{P}(\cup_{s=1}^{\tau-1}\{X_1(s) = X_2(s)\})\\ &\leq \sum_{s=1}^{\tau-1}\mathsf{P}(X_1(s) = X_2(s)) - \sum_{s=1}^{\tau-2}\mathsf{P}(X_1(s) = X_2(s), X_1(s+1) = X_2(s+1)).\end{aligned} \tag{24}$$

It then follows from (14) in Lemma 2 and (15) in Lemma 3 that

$$\mathsf{P}(\cup_{s=1}^{\tau-1}\{X_1(s) = X_2(s)\}) \leq \frac{\tau-1}{n} - \frac{\tau-2}{n^2}. \tag{25}$$

In conjunction with (22) and (23), we then have for $\tau \geq 2$,

$$\mathsf{P}(T \geq \tau) \geq (1 - \frac{\tau-1}{n} + \frac{\tau-2}{n^2})^+. \tag{26}$$

In particular, choosing $\tau = n+1$, we have

$$\mathsf{P}(T \geq n+1) \geq \frac{1}{n}(1 - \frac{1}{n}) > 0. \tag{27}$$

Since $\mathsf{E}[T] = \sum_{\tau=1}^{\infty}\mathsf{P}(T \geq \tau)$ and $\mathsf{P}(T \geq 1) = 1$, we have from the inequality in (26) that

$$\begin{aligned}\mathsf{E}[T] &\geq 1 + \sum_{\tau=2}^{\infty}(1 - \frac{\tau-1}{n} + \frac{\tau-2}{n^2})^+\\ &\geq 1 + \sum_{\tau=2}^{n+1}(1 - \frac{\tau-1}{n} + \frac{\tau-2}{n^2}) = \frac{n+1}{2} + \frac{1}{2} - \frac{1}{2n}.\end{aligned} \tag{28}$$

■

3.2. Finite projective planes In this section, we show how one can construct random sequences that satisfy (A1), (A2) and (A3), and achieve the lower bounds in Theorem 2. In the following, we give a brief review of finite projective planes. Readers interested in this topic may find additional material in [30] and its references.

Definition 1. A finite projective plane of order n is a collection of n^2+n+1 lines and n^2+n+1 points such that

(P1) every line contains $n+1$ points,
(P2) every point is on $n+1$ lines,

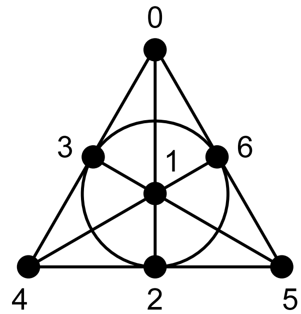

Figure 1. A finite projective plane of order 2

(P3) any two distinct lines intersect at exactly one point, and
(P4) any two distinct points lie on exactly one line.

The finite projective plane of order 1 is simply a triangle that consists of three points and three lines. In Figure 1, we show a finite projective plane of order 2, known as the Fano plane, where points are shown as dots and lines are shown as lines or circles. In this figure, there are 7 lines:

$$\begin{aligned} L_0 &= \{0,1,2\}, \\ L_1 &= \{0,3,4\}, \\ L_2 &= \{0,5,6\}, \\ L_3 &= \{1,3,5\}, \\ L_4 &= \{1,4,6\}, \\ L_5 &= \{2,3,6\}, \quad \text{and} \\ L_6 &= \{2,4,5\}. \end{aligned} \tag{29}$$

If n is a prime power, then there exists a systematic method to construct a finite projective plane of order n via projective geometry over the Galois field $GF(n)$ [14, 15]. However, a finite projective plane may not exist for arbitrary n. It was shown by Bose [14] that there is no projective plane of order 6. Moreover, a much more general theorem by Bruck and Ryser [16] provided a necessary condition for the existence of a finite projective plane of order n when $n = 4m+1$ or $4m+2$ for some nonnegative integer m. The necessary condition requires that n to be a sum of two integer squares. However, such a necessary condition is not sufficient. In particular, when $n = 10 = 1^2 + 3^2$, it was shown in [30] by computer calculation that there is no projective plane of order 10.

Now we show how to construct the random sequence from a finite projective plane of order n. Without loss of generality, we index the n^2+n+1 points from 0 to n^2+n. Choose point 0 and select the $n+1$ lines that contains point 0. For each one of these $n+1$ lines, we form a set by removing point 0 in that line. This gives us $n+1$ sets $S_1, S_2, \ldots, S_{n+1}$, and each of them contains exactly n points. Suppose that S_i contains the n points $\{S_{i,0}, S_{i,1}, \ldots, S_{i,n-1}\}$ for $i = 1, 2, \ldots, n+1$. In the following lemma, we show several properties for the sets S_i, $i = 1, 2, \ldots, n+1$.

Lemma 4. *(i) For all $1 \le i < j \le n+1$, $S_i \cap S_j$ is an empty set and $\cup_{i=1}^{n+1} S_i$ is the set of points from 1 to n^2+n.*
(ii) Consider a point $S_{1,j_1} \in S_1$ and another point $S_{2,j_2} \in S_2$. There is a unique line, denoted by L_{j_1,j_2}, that contains these two points. Moreover, L_{j_1,j_2} is one of the n^2 lines that does not contain point 0, and it intersects with S_i at exactly one point for $i = 1, 2 \ldots, n+1$, i.e., $|L_{j_1,j_2} \cap S_i| = 1$.
(iii) Consider two lines L_{j_1,j_2} and $L_{j_1',j_2'}$. If either $j_1 \ne j_1'$ or $j_2 \ne j_2'$, there exists a unique point in $L_{j_1,j_2} \cap L_{j_1',j_2'}$, i.e., $|L_{j_1,j_2} \cap L_{j_1',j_2'}| = 1$.
(iv) For every point z in $\cup_{i=1}^{n+1} S_i$, there are exactly n lines in the n^2 lines, $L_{j_1,j_2}, 0 \le j_1, j_2 \le n-1$, that contain point z.

Proof. (i) As every two distinct lines intersect at exactly one point and any two of these $n+1$ lines containing point 0 have already intersected at point 0, we know that $S_i \cap S_j$ is an empty set for all $1 \le i < j \le n+1$. Thus, $\cup_{i=1}^{n+1} S_i$ is the set of points from 1 to n^2+n.

(ii) Since S_{1,j_1} and S_{2,j_2} are distinct and any two distinct points lie on exactly one line, these two points determines a line L_{j_1,j_2}. Now we argue by contradiction that line L_{j_1,j_2} does not contain point 0. Suppose that L_{j_1,j_2} contains point 0. Then L_{j_1,j_2} and the line formed by S_1 and point 0 intersect at two distinct points, i.e., point 0 and point S_{1,j_1}. This contradicts to (P3).

Recall that S_i is obtained from a line containing point 0 and line L_{j_1,j_2} is a line that does not contain point 0, there exists exactly one point in $L \cap S_i$ as any two distinct lines intersect at exactly one point.

(iii) This follows directly from (P4) for a finite projective plane.

(iv) Suppose that point z is in S_i for some $1 \le i \le n+1$. Since every point is on $n+1$ lines and point z is already on the line $\{0\} \cup S_i$, there are exactly n lines in the n^2 lines, $L_{j_1,j_2}, 0 \le j_1, j_2 \le n-1$, that contain point z. ∎

Now we propose an algorithm to construct a random hopping sequence from a finite projective plane of order n that satisfies the uniform load constraint in (A2). The key idea is to select a line *uniformly* and *independently* among the n^2 lines that do not contain point 0.

Algorithm 1. *Select a point in S_1 and another point in S_2* uniformly *and* independently. *Suppose that the line determined by these two points is line $L = \{S_{i,j_i}, i = 1, 2, \ldots, n+1\}$. Construct a periodic sequence $\{X(t), t \ge 1\}$ with period $n+1$ by assigning $X(t) = j_i$, where $i = ((t-1) \bmod (n+1)) + 1$.*

Lemma 5. *Suppose that a finite projective plane of order n exist. Then the random hopping sequence $\{X(t), t \ge 1\}$ constructed from Algorithm 1 satisfies the uniform load constraint in (A2), i.e., for all t and $i = 0, 1, 2, \ldots, n-1$,*

$$\mathsf{P}(X(t) = i) = \frac{1}{n}. \tag{30}$$

Proof. Note that there are n^2 lines that can be chosen from selecting a point in S_1 and another point in S_2. Thus, there are n^2 sample sequences in this construction. As we select a point in S_1 and another point in S_2 *uniformly* and *independently*, each sample sequence is selected with probability $1/n^2$. If $X(i) = j$ for some $1 \le i \le n+1$ and $0 \le j \le n-1$, then a line containing the point $S_{i,j}$ must be selected. Since there are n lines containing $S_{i,j}$ in these n^2 lines (Lemma 4(iv)), we conclude that for all $1 \le i \le n+1$ and $0 \le j \le n-1$,

$$\mathsf{P}(X(i) = j) = \frac{n}{n^2} = \frac{1}{n}. \tag{31}$$

Since $X(t)$ constructed from Algorithm 1 is periodic with period $n+1$, we know that (31) also hold for any $t \ge 1$ and the constraint in (A2) is satisfied. ∎

Example 2. $(n=2)$ *In this example, we consider the case with $n = 2$. As shown in (29), the three lines L_0, L_1 and L_2 contain point 0. By removing point 0 from these three lines, we form $S_1 = \{S_{1,0}, S_{1,1}\} = \{1, 2\}$, $S_2 = \{S_{2,0}, S_{2,1}\} = \{3, 4\}$ and $S_3 = \{S_{3,0}, S_{3,1}\} = \{5, 6\}$. For a point in S_1 and another point in S_2, we have the following four lines:*

$$\begin{aligned}
L_{1,1} &= \{1, 3, 5\} = L_3, \\
L_{1,2} &= \{1, 4, 6\} = L_4, \\
L_{2,1} &= \{2, 3, 6\} = L_5, \\
L_{2,2} &= \{2, 4, 5\} = L_6.
\end{aligned}$$

If $L_{1,1}$ is selected, then the corresponding sample sequence is $000000\dots$ *as* $1 = S_{1,0}$*,* $3 = S_{2,0}$ *and* $5 = S_{3,0}$*. The other three sequences corresponding to* $L_{1,2}$*,* $L_{2,1}$ *and* $L_{2,2}$ *are* $011011\dots$*,* $101101\dots$*, and* $110110\dots$ *Each of these four sequences is selected with probability* $1/4$ *and both 0 and 1 appear with probability* $1/2$ *at any time.*

Theorem 3. *Suppose that a finite projective plane of order n exist and that both users select their hopping sequences* independently *via Algorithm 1.*
(i) Both users rendezvous within $n+1$ steps, i.e., $T \le n+1$.
(ii) The tail distribution of the random variable T satisfies

$$\mathsf{P}(T \ge \tau) = 1 - \frac{\tau-1}{n} + \frac{\tau-2}{n^2}, \quad 2 \le \tau \le n+1. \tag{32}$$

As such,

$$\mathsf{P}(T = 1) = \frac{1}{n}, \tag{33}$$

$$\mathsf{P}(T = \tau) = \frac{1}{n}(1 - \frac{1}{n}), \tau = 2, \dots, n+1. \tag{34}$$

(iii) The expected number of steps for both users to rendezvous is $\frac{n+1}{2} + \frac{1}{2} - \frac{1}{2n}$, i.e.,

$$\mathsf{E}[T] = \frac{n+1}{2} + \frac{1}{2} - \frac{1}{2n}. \tag{35}$$

Proof. (i) If both users follow the same sample sequence, they rendezvous at time 1. Thus, we only need to consider the case that they follow two distinct sample sequences. As each sample sequence constructed from Algorithm 1 corresponds to one of the n^2 lines in a finite projective plane, we know from Lemma 4(iii) that any two distinct sample sequences constructed from Algorithm 1 intersect at a point. Since there are $n+1$ points in a line, it then follows that $T \le n+1$.

(ii) Clearly, $\mathsf{P}(T \ge 1) = 1$. On the other hand, in view of $T \le n+1$ in (1), we know that for $\tau \ge n+2$,

$$\mathsf{P}(T \ge \tau) = 0.$$

Thus, we only need to find $\mathsf{P}(T \ge \tau)$, $\tau = 2, \dots, n+1$.

For this, let us condition on the event $\{X_1(i) = j_i, i = 1, 2, \dots, n+1\}$. In other words, user 1 selects the line $L_{j_1,j_2} = \{S_{i,j_i}, i = 1, 2, \dots, n+1\}$. Now we compute the conditional probability for $\tau \ge 2$,

$$\mathsf{P}(T \ge \tau | \{X_1(i) = j_i, i = 1, 2, \dots, n+1\}) = \mathsf{P}(X_2(s) \ne j_s, 1 \le s \le \tau - 1).$$

Such a condition probability is thus the probability that user 2 selects a line that does not contain a point in $\{S_{1,j_1}, S_{2,j_2}, \dots, S_{\tau-1,j_{\tau-1}}\}$. Note that for each point z in $\cup_{i=1}^{n+1} S_i$, there are exactly n lines in the n^2 lines that contain that point (Lemma 4(iv)). Excluding line L_{j_1,j_2}, there are $n-1$ lines containing S_{i,j_i}. Thus, the total number of lines in the n^2 lines that contain a point in $\{S_{1,j_1}, S_{2,j_2}, \dots, S_{\tau-1,j_{\tau-1}}\}$ is $1 + (\tau-1)(n-1)$. As such, there are $n^2 - 1 - (\tau-1)(n-1)$ lines that do not contain any point in $\{S_{1,j_1}, S_{2,j_2}, \dots, S_{\tau-1,j_{\tau-1}}\}$. As every line is chosen with probability $1/n^2$, we then have

$$\begin{aligned}
&\mathsf{P}(T \ge \tau | \{X_1(i) = j_i, i = 1, 2, \dots, n+1\}) \\
&= \frac{n^2 - 1 - (\tau-1)(n-1)}{n^2} \\
&= 1 - \frac{\tau-1}{n} + \frac{\tau-2}{n^2}.
\end{aligned}$$

Unconditioning the event $\{X_1(i) = j_i, i = 1, 2, \ldots, n+1\}$ yields

$$\mathsf{P}(T \geq \tau) = 1 - \frac{\tau - 1}{n} + \frac{\tau - 2}{n^2}.$$

The probability distribution of T then follows from the fact that $\mathsf{P}(T \geq 1) = 1$ and

$$\mathsf{P}(T = \tau) = \mathsf{P}(T \geq \tau - 1) - \mathsf{P}(T \geq \tau).$$

We note that the results in (33) and (34) can also be argued by using the following insight: If both users select the same point on S_1, then they meet at time 1. If they select a different point on S_1, then their lines are different. Their lines intersect in one point. By symmetry, this point is uniformly distributed in $[2, \ldots, n+1]$.

(iii) Since $\mathsf{P}(T \geq 1) = 1$, one can use

$$\mathsf{E}[T] = \sum_{\tau=1}^{\infty} \mathsf{P}(T \geq \tau) = \sum_{\tau=1}^{n+1} \mathsf{P}(T \geq \tau)$$

to show (35). ∎

Recently, it was shown in [43] that the optimal expected time-to-rendezvous for symmetric rendezvous search on three *unlabelled* locations (channels) is 5/2 if the two users are placed at two distinct locations at the beginning. Such a result can be achieved by using the Anderson-Weber (AW) strategy in which each user stays at his initial location with probability 1/3 and tour the other locations in random order with probability 2/3 for each successive block of two steps. If we use the AW strategy for our *labelled* rendezvous problem for $n = 3$, then the expected time-to-rendezvous is $1 + (1/3)0 + (2/3)(5/2) = 8/3$ as the two users make an initial choice that may cause them to meet with probability 1/3. On the other hand, the expected time-to-rendezvous by using the finite projective plane in Algorithm 1 is $5/2 - 1/6 = 7/3$, which is smaller than 8/3. This is not surprising as the AW strategy does not need the locations to be labelled.

4. Channel blocking

In this section, we further take the channel blocking constraint in (A4) into account. If a channel is blocked at time t, then the two users will not rendezvous even though they hop to that channel at time t. Recall that $B(t)$ is the set of channels that are blocked at time t and $\mathsf{E}[|B(t)|] \geq b$ for all t.

4.1. Lower bounds

In the following theorem, we derive the lower bound for the tail distribution of the time-to-rendezvous under the constraints in (A1), (A2) and (A4).

THEOREM 4. *Under the constraints in (A1), (A2) and (A4),*

$$\mathsf{P}(T \geq \tau) \geq 1 - (\tau - 1)\frac{n-b}{n^2}, \quad \tau \geq 1. \tag{36}$$

Let

$$n_1^* = n + b + \lceil \frac{b^2}{n-b} \rceil, \tag{37}$$

where $\lceil x \rceil$ is the ceiling function that represents the smallest integer not less than x. It follows from (36) that

$$\mathsf{P}(T \geq n_1^*) > 0, \tag{38}$$

and that

$$\mathsf{E}[T] \geq n_1^* - (\frac{n-b}{n^2})(\frac{n_1^*(n_1^*-1)}{2}). \tag{39}$$

As a direct consequence, we know that under the constraints in (A1), (A2) and (A4), there does not exist two random sequences such that the time-to-rendezvous can be deterministically bounded by n_1^*-1 steps. Moreover, the expected time-to-rendezvous under the constraints (A1), (A2) and (A4) is not lower than $n_1^* - (\frac{n-b}{n^2})(\frac{n_1^*(n_1^*-1)}{2})$. We note that the results in Theorem 4 are more general than those in Theorem 1. If there are no blocked channels, i.e., $b=0$, then we have $n_1^*=n$ and the lower bound in (39) reduces to that in (4) of Theorem 1. Since adding an additional constraint increases the time-to-rendezvous, the lower bounds in (38) and (39) also hold under the constraints in (A1), (A2), (A3) and (A4). On the other hand, the lower bounds in Theorem 2 also hold under the constraints in (A1), (A2), (A3) and (A4).

Proof. Let E_t be the event that the two users hop to a common unblocked channel at time t, i.e., $E_t=\{X_1(t)=X_2(t)\notin B(t)\}$. Also, denote by E_t^c the complement of the event E_t. Thus, the event that the two users do not rendezvous before time τ, i.e., $\{T\ge\tau\}$, is the intersection of all the events E_t^c, $t=1,2,\dots,\tau-1$. Thus,

$$\mathsf{P}(T\ge\tau)=\mathsf{P}(\cap_{t=1}^{\tau-1}E_t^c)=1-\mathsf{P}(\cup_{t=1}^{\tau-1}E_t)\ge 1-\sum_{t=1}^{\tau-1}\mathsf{P}(E_t), \tag{40}$$

where we use the union bound in the last inequality.

Since we assume that the two random sequences $\{X_1(t),t\ge1\}$ and $\{X_2(t),t\ge1\}$ are independent of $B(t)$ in (A4), conditioning on the event that $\{B(t)=\Gamma\}$ for some subset Γ of $\{0,1,2,\dots,n-1\}$ yields

$$\begin{aligned}&\mathsf{P}(E_t)=\mathsf{P}(X_1(t)=X_2(t)\notin B(t))\\&=\sum_{\Gamma\subseteq\{0,1,2,\dots,n-1\}}\mathsf{P}(X_1(t)=X_2(t)\notin\Gamma)\mathsf{P}(B(t)=\Gamma).\end{aligned} \tag{41}$$

Note from the independence constraint in (A1), the uniform load constraint in (A2) and the channel blocking constraint in (A4) that

$$\begin{aligned}&\mathsf{P}(X_1(t)=X_2(t)\notin\Gamma)\\&=\sum_{i\notin\Gamma}\mathsf{P}(X_1(t)=X_2(t)=i)\\&=\sum_{i\notin\Gamma}\mathsf{P}(X_1(t)=i)\mathsf{P}(X_2(t)=i)\\&=\sum_{i\notin\Gamma}(\frac{1}{n})^2=\frac{n-|\Gamma|}{n^2}.\end{aligned} \tag{42}$$

Using this in (41) yields

$$\mathsf{P}(E_t)=\frac{n-\mathsf{E}[|B(t)|]}{n^2}. \tag{43}$$

Since we assume that $\mathsf{E}[|B(t)|]\ge b$ in (A4), it then follows that

$$\mathsf{P}(E_t)\le\frac{n-b}{n^2}. \tag{44}$$

In conjunction with (40), we then have

$$\mathsf{P}(T\ge\tau)\ge1-(\tau-1)\frac{n-b}{n^2}. \tag{45}$$

In particular, if we choose $\tau = n_1^*$, then

$$\begin{aligned}&\mathsf{P}(T \geq n_1^*)\\ &\geq 1-(n_1^*-1)\frac{n-b}{n^2}\\ &= \frac{1}{n^2}\Big(b^2+n-b-(n-b)\lceil\frac{b^2}{n-b}\rceil\Big).\end{aligned}$$

Since $\lceil x \rceil < x+1$, it then follows that

$$\mathsf{P}(T \geq n_1^*) > \frac{1}{n^2}\Big(b^2+n-b-(n-b)(\frac{b^2}{n-b}+1)\Big) = 0.$$

To show the lower bound for the expected time-to-rendezvous in (39), note that

$$\mathsf{E}[T] = \sum_{\tau=1}^{\infty}\mathsf{P}(T\geq\tau) \geq \sum_{\tau=1}^{n_1^*}\mathsf{P}(T\geq\tau), \tag{46}$$

and then use (36) in (46). ■

4.2. One permanently blocked channel In this section, we consider the special case that a particular channel is blocked all the time, i.e., $|B(t)| = b = 1$ for all t. Then n_1^* in (37) is $n+2$ and there does not exist random sequences such that the time-to-rendezvous can be deterministically bounded by $n+1$ steps. Moreover, we have from (39) that

$$\mathsf{E}[T] \geq \frac{n+1}{2} + \frac{n^2+n+2}{2n^2}. \tag{47}$$

As such, the best scenario is to look for random sequences such that the time-to-rendezvous can be deterministically bounded within $n+2$ steps and they have the expected time-to-rendezvous as the right hand side of (47). We will show this is possible if the two hopping sequences are allowed to have different joint distributions, i.e., without the constraint in (A3). Our idea for such an optimal construction is to use two orthogonal Latin squares. For this, we give a brief review of Latin squares and refer to [21, 44] for more references.

Definition 2. **(Latin square and Orthogonal Latin squares)** A Latin square with the set of symbols S is an $|S|\times|S|$ matrix such that every symbol appears exactly once in every row and every column. Two $n\times n$ Latin squares $A=(a_{i,j})$ and $B=(b_{i,j})$ are said to be *orthogonal* if the order pairs $(a_{i,j}, b_{i,j})$ are all different for all $i,j = 0,1,2,\ldots,n-1$.

In the following, we show two 4×4 orthogonal Latin squares with $S=\{0,1,2,3\}$.

$$\begin{bmatrix}0&1&2&3\\1&0&3&2\\2&3&0&1\\3&2&1&0\end{bmatrix}, \begin{bmatrix}0&1&2&3\\3&2&1&0\\1&0&3&2\\2&3&0&1\end{bmatrix}. \tag{48}$$

Remark 1. *The study of the existence of two orthogonal Latin squares (also known as the Graeco-Latin square) was first proposed by L. Euler in 1782 [24]. He was not able to construct two orthogonal Latin squares of order 6 (known as the 36 officers problem) and then conjectured that there does not exist two orthogonal Latin squares of order n for $n = 4m+2$, where m is a nonnegative integer. It was confirmed later by G. Tarry via exhaustive enumeration that there does not exist two orthogonal Latin squares of order 6. However, via extensive computer enumeration, two orthogonal Latin squares of order 10 and order 22 were found and it was later shown that Euler's conjecture is false for all $n\geq 10$. We now know that two orthogonal Latin squares exist for all $n\geq 3$ except $n=6$.*

As mentioned in the previous remark, two orthogonal Latin squares exist for all $n \geq 3$ except for $n = 6$. Let $A = (a_{i,j})$ and $B = (b_{i,j})$ (with $i, j = 0, 1, 2, \ldots, n-1$) be two $n \times n$ orthogonal Latin squares with the symbol set $\{0, 1, \ldots, n-1\}$. Consider a symbol $u \in \{0, 1, \ldots, n-1\}$. As every symbol appears exactly once in every row and every column in a Latin square, we know for every column j there is one *distinct* row index $F_j^a(u)$ such that the $(F_j^a(u), j)^{th}$ element in A is u, i.e., $a_{F_j^a(u),j} = u$. Similarly, for every column j there is one *distinct* row index $F_j^b(u)$ such that the $(F_j^b(u), j)^{th}$ element in B is u, i.e., $b_{F_j^b(u),j} = u$.

Algorithm 2. *Let U_1 and U_2 be two uniform random variables over $[0, 1, \ldots, n-1]$ that are independent of each other. We construct the two random sequences by assigning $\{X_1(t) = F_{t-1}^a(U_1), 1 \leq t \leq n\}$ and $\{X_2(t) = F_{t-1}^b(U_2), 1 \leq t \leq n\}$.*

To illustrate the construction of the two random hopping sequences, we consider the case $n = 4$ with the two orthogonal Latin squares in (48) (left for user 1 and right for user 2). Suppose that $U_1 = 0$ and $U_2 = 2$. Then we have $(X_1(1), X_1(2), X_1(3), X_1(4)) = (0, 1, 2, 3)$ and $(X_2(1), X_2(2), X_2(3), X_2(4)) = (3, 1, 0, 2)$. They both hop to channel 1 at time 2.

With such a construction, the two constraints (A1) and (A2) are satisfied (for $1 \leq t \leq n$). Moreover, since A and B are two orthogonal Latin squares, we know that these two users will hop to a common channel within n steps. As both U_1 and U_2 are uniformly distributed, we also know these two users will hop to a common channel with an equal probability $1/n^2$ at any channel in $[0, 1, \ldots, n-1]$ and at any time t in $[1, 2, \ldots, n]$.

Now suppose that channel 0 is blocked for all time $t \geq 1$ and this is known to both users. Since channel 0 is blocked, both users will not rendezvous even if they both hop to channel 0 at the same time. To compensate for the time that both users hop to channel 0, we construct the hopping sequences by modifying those from algorithm 2.

Algorithm 2′. *Suppose that channel 0 is blocked for all time $t \geq 1$ and this is known to both users. As in Algorithm 2, let U_1 and U_2 be two uniform random variables over $[0, 1, \ldots, n-1]$ that are independent of each other. We construct the two periodic random sequences with period $n+2$ by assigning $\{X_1(t) = F_{t-1}^a(U_1), 1 \leq t \leq n\}$ and $\{X_2(t) = F_{t-1}^b(U_2), 1 \leq t \leq n\}$. As each sequence is constructed from the appearance of a particular symbol in a Latin square, there is a unique time in $[1, n]$ for each user to hop to channel 0. Suppose that user 1 (resp. user 2) hops to channel 0 at time t_1 (resp. t_2), i.e., $X_1(t_1) = 0$ (resp. $X_2(t_2) = 0$). We set $X_1(n+1) = t_1 - 1$ and $X_1(n+2) = (t_1 \bmod n)$. Similarly, we also set $X_2(n+1) = t_2 - 1$ and $X_2(n+2) = (t_2 \bmod n)$.*

Note that $X_1(n+1)$ and $X_1(n+2)$ (resp. $X_2(n+1)$ and $X_2(n+2)$) are deterministic functions of $\{X_1(t), 1 \leq t \leq n\}$ (resp. $\{X_2(t), 1 \leq t \leq n\}$). Thus, the only randomness that determine $\{X_1(t), 1 \leq t \leq n+2\}$ (resp. $\{X_2(t), 1 \leq t \leq n+2\}$) is the random variable U_1 (resp. U_2). Since U_1 and U_2 are independent, we conclude that the independent constraint in (A1) is still satisfied. On the other hand, since U_1 is uniformly distributed in $[0, 1, \ldots, n-1]$ and $\{X_1(t), 1 \leq t \leq n\}$ is constructed from the appearance of a symbol in a Latin square, we know that t_1 is also uniformly distributed in $[1, 2, \ldots, n]$. Thus, both $X_1(n+1)$ and $X_1(n+2)$ are uniformly distributed in $[0, 1, \ldots, n-1]$. Similarly, both $X_2(n+1)$ and $X_2(n+2)$ are uniformly distributed in $[0, 1, \ldots, n-1]$ and the uniform load constraint in (A2) is satisfied.

THEOREM 5. *For $n \geq 3$ and $n \neq 6$, suppose that channel 0 is blocked for all time $t \geq 1$ and this is known to both users. If both users select their hopping sequences* independently *via Algorithm 2′, then both users rendezvous within $n+2$ steps, i.e., $T \leq n+2$. Moreover,*

$$\mathsf{E}[T] = \frac{n+1}{2} + \frac{n^2+n+2}{2n^2}. \tag{49}$$

Proof. Since A and B are two orthogonal Latin squares, we know that these two users will hop to a common channel within n steps. As both U_1 and U_2 are uniformly distributed, we also know

these two users will hop to a common channel with an equal probability $1/n^2$ at any channel in $[0,1,\ldots,n-1]$ and at any time t in $[1,2,\ldots,n]$. We consider the following three cases:
Case 1. Both users hop to a common channel at time $1\le t\le n$ and this channel is not channel 0:
In this case, we have $T=t$ and this happens with probability $\frac{n-1}{n^2}$. Thus, for $1\le t\le n$,

$$\mathsf{P}(T=t)=\frac{n-1}{n^2}. \tag{50}$$

Case 2. Both users hop to channel 0 at time $2\le t\le n$:
In this case, we have $t_1=t_2=t$ and $X_1(n+1)=X_2(n+1)=t-1$. Since $1\le t-1\le n-1$, these two users rendezvous at channel $t-1$ and at time $n+1$. This also happens with probability $\frac{n-1}{n^2}$. Thus,

$$\mathsf{P}(T=n+1)=\frac{n-1}{n^2}. \tag{51}$$

Case 3. Both users hop to channel 0 at time $t=1$:
In this case, we have $t_1=t_2=1$, $X_1(n+1)=X_2(n+1)=0$ and $X_1(n+2)=X_2(n+2)=1$. Thus, these two users rendezvous at channel 1 and at time $n+2$. This happens with probability $\frac{1}{n^2}$ and thus

$$\mathsf{P}(T=n+2)=\frac{1}{n^2}. \tag{52}$$

From these cases, we know that both users rendezvous within $n+2$ steps. Moreover, for $1\le\tau\le n+2$,

$$\mathsf{P}(T\ge\tau)=1-(\tau-1)\frac{n-1}{n^2}, \tag{53}$$

and the lower bound in (36) is achieved. As such, we also have

$$\mathsf{E}[T]=\frac{n+1}{2}+\frac{n^2+n+2}{2n^2}. \tag{54}$$

■

Remark 2. *When the permanently blocked channel is known to both users, it seems that the problem could be viewed as the rendezvous problem with $n-1$ channels. However, there is the uniform load constraint in (A2) and that constraint requires the blocked channel to be visited with an equal probability. With this constraint, the problem with one permanently blocked channel is not the same as the problem with $n-1$ channels. In a cognitive radio network, the "permanently" blocked channel might be a channel assigned to a primary (licensed) user and it is still possible for secondary (unlicensed) users to access that channel when the primary user is not using it. As such, it might still be beneficial for secondary users to probe that channel. Of course, it may not be optimal to probe it with an equal probability.*

4.3. Multiple permanently blocked channels In this section, we consider the case that there are a subset of channels Γ that are blocked all the time, i.e., $|B(t)|=|\Gamma|=b\ge 1$. For $b=1$, we have shown in Theorem 5 how one can construct two random sequences that satisfy (A1), (A2) and (A4) and achieve the lower bound in Theorem 4. However, such a construction cannot be extended to achieve the lower bound in Theorem 4 for $b>1$. Instead, we propose in this section a suboptimal solution that not only satisfy (A1), (A2) and (A4) but also has a deterministic upper bounded on the time-to-rendezvous. This is done without the knowledge of the set of blocked channels.

Algorithm 3. *Use Algorithm 1 to construct a periodic sequence $\{X(t),t\ge 1\}$ with period $n+1$. Construct the* rotated *sequence $\{\tilde{X}(t),t\ge 1\}$ by assigning*

$$\tilde{X}(t)=((X(t)-1+\lceil t/(n+1)\rceil)\ mod\,(n+1)). \tag{55}$$

Clearly, we have $\tilde{X}(t) = X(t)$ for $t = 1, 2 \ldots, n+1$, $\tilde{X}(t) = ((X(t)+1) \bmod (n+1))$ for $t = n+2, n+3, \ldots, 2(n+1)$, $\tilde{X}(t) = ((X(t)+2) \bmod (n+1))$ for $t = 2(n+1)+1, 2(n+1)+2, \ldots, 3(n+1)$, $\ldots$. In other words, if we view every $n+1$ steps as a frame, then the values of $\tilde{X}(t)$ in a frame are simply the rotation of $X(t)$ in that frame. Since we have shown in Lemma 5 that the random hopping sequence $\{X(t), t \geq 1\}$ constructed from Algorithm 1 satisfies the uniform load constraint in (A2), the random hopping sequence $\{\tilde{X}(t), t \geq 1\}$, as a rotated version of $\{X(t), t \geq 1\}$, also satisfies the uniform load constraint in (A2).

THEOREM 6. *Suppose that a finite projective plane of order n exist and that both users select their hopping sequences* independently *via Algorithm 3. If there are $b < n$ permanently blocked channels, then both users rendezvous within $(n+1)(b+1)$ steps, i.e.,*

$$T \leq (n+1)(b+1). \tag{56}$$

Proof. As shown in Theorem 3(i), if we use Algorithm 1 to construct the two random sequences, then both users will hop to a common channel every $n+1$ steps and this rendezvous channel stays the same for every $n+1$ steps. Since the two sequences constructed from Algorithm 3 are rotated versions of their counterparts from Algorithm 1, we know that both users will still hop to a common channel every $n+1$ steps but this rendezvous channel will rotate for every $n+1$ steps. As such, they will rendezvous at $b+1$ distinct channels after $(n+1)(b+1)$ steps. Since there are only b permanently blocked channels, they must rendezvous within $(n+1)(b+1)$ steps and thus $T \leq (n+1)(b+1)$. ■

5. Time synchronization

In this section, we further take (A5) into account. As both random sequences are *stationary*, the distribution of the time-to-rendezvous is *invariant* for an arbitrary time shift of any one of the two random sequences. In practice, time synchronization is not needed under (A5) as it makes no difference for the performance in terms of the time-to-rendezvous. One particular random sequence that satisfies (A1), (A2), (A3) and (A5) is to select a channel *uniformly* and *independently* at any time t. We call such a random sequence the *coin-tossing* sequence. If both users use two independent coin-tossing random sequences, then it is well-known that $\mathsf{E}[T] = n$ (see e.g., [42]).

One drawback of using the coin-tossing random sequences is that there is no upper bound for the time-to-rendezvous [42]. To provide a deterministic upper bound for the time-to-rendezvous, a sequence-based approach using permutations was proposed in [20] and it was shown to have a deterministic upper bound of $O(n^2)$. It was further shown in [33] by using the Jump-Stay algorithm that there is a deterministic upper bound of $3n$ steps. Our main result in this section is to construct random sequences by using sawtooth sequences in [45] so that they satisfy (A1-3) and (A5) and have a deterministic upper bound of $2n+1$ steps. Moreover, the expected time-to-rendezvous is less than n and thus better than using the coin-tossing random sequences. For the setting with a single permanently blocked channels, we propose interleaving a sawtooth sequence and an inverted sawtooth sequence in time so that any two sequences are guaranteed to rendezvous within $4n+2$ steps.

5.1. Sawtooth sequences

In the following, we propose using sawtooth sequences in [45] for our constructions of *stationary* hopping sequences. Our construction is inspired by the ideal square dot matrix in [29] that is closely related to difference sets in [48, 26].

Definition 3. **(Sawtooth sequence [45])** A deterministic sequence $\{f(t), t \geq 1\}$ is called a sawtooth sequence of order n if

$$f(t) = \begin{cases} t-1, & \text{for } t = 1, 2, \ldots, n \\ 2n - t, & \text{for } t = n+1, n+1, \ldots, 2n \end{cases}, \tag{57}$$

and

$$f(t) = f(t \bmod (2n+1)). \tag{58}$$

for $t > 2n+1$ and t is not an integer multiple of $2n+1$.
Note that $f(t)$ is not specified for any t that is an integer multiple of $2n+1$. In view of (58), it is periodic with period $2n+1$ for the rest of time t. Also, every number i, $i = 0, 1, \ldots, n-1$, appears exactly twice for $t = 1, 2, \ldots, 2n$.

For example, a sawtooth sequence of order 4 is

$$01233210x01233210x\ldots,$$

where x denotes an unspecified value. One key property of a sawtooth sequence of order n is that for any time shift d, there exists some $1 \leq \tau \leq 2n$ such $f(\tau) = f(\tau + d)$. This is formally stated in the following proposition. The proof of Proposition 1 is quite straightforward and thus omitted.

Proposition 1. *Suppose $f(t)$ is a sawtooth sequence of order n. Consider two sequences $x_1(t) = f(t)$ and $x_2(t) = f(t+d)$ for some $d \geq 0$. Let*

$$d_0 = (d \mod (2n+1)).$$

(i) Suppose $d_0 = 0$. Then $x_1(1) = x_2(1) = 0$.
(ii) Suppose that d_0 is a nonzero even number. Let $\tau_e = 2n + 1 - \frac{d_0}{2}$. Then $n + 1 \leq \tau_e \leq 2n$ and

$$x_1(\tau_e) = x_2(\tau_e) = 2n - \tau_e.$$

(iii) Suppose that d_0 is an odd number. Let $\tau_o = \frac{2n - d_0 + 1}{2}$. Then $1 \leq \tau_o \leq n$ and

$$x_1(\tau_o) = x_2(\tau_o) = \tau_o - 1.$$

In all these three cases,

$$x_1(\tau + (2n+1)m) = x_2(\tau + (2n+1)m)$$

for some $1 \leq \tau \leq 2n$ and all nonnegative integer m.

In Table 3, we illustrate the results in Proposition 1 by considering the case for $n = 4$. Here we show all the time shifted sequences $f(t+d)$, $d = 0, 1, 2, \ldots, 8$. In the first row, we show a sawtooth sequence of order 4 (the case with $d = 0$). Those numbers (channels) marked with an $*$ in all the subsequent rows are the numbers (channels) that match their counterparts in the first row.

Algorithm 4. *Suppose $f(t)$ is a sawtooth sequence of order n. Construct a random sequence $\{Y(t), t \geq 1\}$ by letting (i) $Y(t) = f(t)$ when t is* not *an integer multiple of $2n+1$, and (ii) $Y(t) = U_1$ for some uniform random variable U_1 over $[0, 1, \ldots, n-1]$ when t is an integer multiple of $2n+1$. Choose a uniform random variable U_2 over $[0, 1, \ldots, 2n]$ that is independent of U_1. Construct the random sequence $\{X(t), t \geq 1\}$ by letting $X(t) = Y(t + U_2)$.*

Observe that $\{Y(t), t \geq 1\}$ is now periodic with period $2n+1$ and thus $\{X(t), t \geq 1\}$, as a random start in a period, is stationary and the constraint in (A5) is satisfied. Moreover, we have for all t and for $0 \leq i \leq n-1$,

$$\begin{aligned} &\mathsf{P}(X(t) = i) = \mathsf{P}(X(1) = i) = \mathsf{P}(Y(1 + U_2) = i) \\ &= \mathsf{P}(U_2 = i) + \mathsf{P}(U_2 = 2n - (i+1)) + \mathsf{P}(U_2 = 2n, U_1 = i) \\ &= \frac{2}{2n+1} + \frac{1}{2n+1}\frac{1}{n} = \frac{1}{n}. \end{aligned} \tag{59}$$

Thus, the constraint in (A2) is satisfied.

t	1	2	3	4	5	6	7	8	9
$f(t)$	0	1	2	3	3	2	1	0	x
$f(t+1)$	1	2	3	3^*	2	1	0	x	0
$f(t+2)$	2	3	3	2	1	0	x	0^*	1
$f(t+3)$	3	3	2^*	1	0	x	0	1	2
$f(t+4)$	3	2	1	0	x	0	1^*	2	3
$f(t+5)$	2	1^*	0	x	0	1	2	3	3
$f(t+6)$	1	0	x	0	1	2^*	3	3	2
$f(t+7)$	0^*	x	0	1	2	3	3	2	1
$f(t+8)$	x	0	1	2	3^*	3	2	1	0

TABLE 3. A sawtooth sequence of order 4

THEOREM 7. *Suppose that both users select their hopping sequences* independently *via Algorithm 4 and that there are no blocked channels. Then both users rendezvous within $2n+1$ steps, i.e., $T \leq 2n+1$.*

Proof. This is a direct consequence of Proposition 1. ■

Moreover, it can be shown by a lengthy and detailed calculation that for $n \geq 2$,

$$\mathsf{E}[T] = n - \frac{1}{6} + \frac{2n^2 + 11n - 4}{6n(2n+1)^2} < n,$$

if both users select their hopping sequences *independently* via Algorithm 4. As such, the expected time-to-rendezvous is also better than using the coin-tossing sequences.

5.2. Time-interleaving sawtooth sequences The sawtooth sequence [45] has the smallest deterministic upper bound for the time-to-rendezvous among all the works in the literature when there are no blocked channels (see e.g., Table 1 of [17] for the comparison of various existing algorithms). In this section, we extend the result for the setting with a single permanently blocked channel. The key idea is based on time-interleaving a sawtooth sequence and an inverted sawtooth sequence defined below.

DEFINITION 4. (**Inverted Sawtooth Sequence**) A deterministic sequence $\{f(t),\, t \geq 1\}$ is called an inverted sawtooth sequence of order n if

$$f(t) = \begin{cases} n - t, & \textit{for } t = 1, 2 \ldots, n, \\ t - (n+1), & \textit{for } t = n+1, \ldots, 2n, \end{cases} \tag{60}$$

and

$$f(t) = f(t \; mod \; (2n+1)), \tag{61}$$

for $t > 2n+1$ and t is not an integer multiple of $2n+1$.
For example, an inverted sawtooth sequence of order 4 is

$$32100123x32100123x\ldots \tag{62}$$

where x denotes an unspecified value.

As the inverted sawtooth sequence is simply a mapping of the channel indices of a sawtooth sequence, the inverted sawtooth sequence possesses the same property as the original sawtooth sequence in Proposition 1. This is formally stated in the following corollary.

Corollary 1. *Suppose $f(t)$ is an inverted sawtooth sequence of order n. Consider two sequences $x_1(t) = f(t)$ and $x_2(t) = f(t+d)$ for some $d \geq 0$. Let*

$$d_0 = d\, mod\, (2n+1). \tag{63}$$

(i) Suppose $d_0 = 0$. Then $x_1(1) = x_2(1) = n-1$.
(ii) Suppose that d_0 is a nonzero even number. Let $\tau_e = 2n+1-\frac{d_0}{2}$. Then $n+1 \leq \tau_e \leq 2n$ and

$$x_1(\tau_e) = x_2(\tau_e) = \tau_e - (n+1). \tag{64}$$

(iii) Suppose that d_0 is an odd number. Let $\tau_o = \frac{2n-d_0+1}{2}$. Then $1 \leq \tau_o \leq n$ and

$$x_1(\tau_o) = x_2(\tau_o) = n - \tau_o. \tag{65}$$

In all these three cases,

$$x_1(\tau + (2n+1)m) = x_2(\tau + (2n+1)m) \tag{66}$$

for some $1 \leq \tau \leq 2n$ and all nonnegative integer m.

We now interleave an original sawtooth sequence of order n and another inverted sawtooth sequence of order n in time to construct a sequence $g(t)$.

Definition 5. (**Interleaved Sawtooth Sequence**) Let $f_o(t)$ and $f_e(t)$ be sawtooth and inverted sawtooth sequences of order n, respectively. Let t_0 equals t by modulo $(4n+2)$, namely, $t_0 = t\, mod\, (4n+2)$. Then, a deterministic sequence $\{g(t), t \geq 1\}$ is called an interleaved sawtooth sequence of order n if

$$g(t) = \begin{cases} f_o(s), \textit{ for } t_0 = 2s-1,\ s = 1,2,\ldots,2n, \\ f_e(s), \textit{ for } t_0 = 2s,\ s = 1,2,\ldots,2n, \end{cases} \tag{67}$$

and $g(t)$ is not specified if $t_0 = 4n+1$ and 0.
For example, an interleaved sawtooth sequence of order 4 is

$$0312213030211203xx0312213030211203xx\ldots \tag{68}$$

where x denotes an unspecified value.

In the following theorem, we state the main result for the interleaved sawtooth sequence. For the ease of our presentation, the proof of Theorem 8 is deferred to the end of this section.

Theorem 8. *Suppose that $\{g(t), t \geq 1\}$ is an interleaved sawtooth sequence of order n and n is an even number. Consider two sequences $x_1(t) = g(t)$ and $x_2(t) = g(t+d)$ for some $d \geq 0$. Then for any $d \geq 0$, there exists $1 \leq \tau_1, \tau_2 \leq 4n+2$ such that*

$$x_1(\tau_1) = x_2(\tau_1), \quad \textit{and}\, x_1(\tau_2) = x_2(\tau_2), \tag{69}$$

and

$$x_1(\tau_1) \neq x_2(\tau_2). \tag{70}$$

In view of Theorem 8, one can use an interleaved sawtooth sequence for constructing hopping sequences such that the time-to-rendezvous can be deterministically bounded by $4n+2$ steps even when there is a single permanently blocked channel. The algorithm is stated formally in Algorithm 5 below.

Algorithm 5. *Suppose $g(t)$ is an interleaved sawtooth sequence of order n. Choose three independent random variables U_1, U_2 and U_3, where U_1 and U_2 are uniformly distributed over* $[0, 1, \ldots, n-1]$

and U_3 is uniformly distributed over $[0,1,\dots,4n+1]$. Construct a random sequence $\{Y(t), t \geq 1\}$ by letting (i) $Y(t) = g(t)$ when $(t \; mod \, 4n+2) \neq 0$ or $(t \; mod \, 4n+2) \neq 4n+1$, (ii) $Y(t) = U_1$ when $(t \; mod \, 4n+2) = 0$, and (iii) $Y(t) = U_2$ when when $(t \; mod \, 4n+2) = 4n+1$ Construct the random sequence $\{X(t), t \geq 1\}$ by letting $X(t) = Y(t+U_3)$.

Observe that $\{Y(t), t \geq 1\}$ is now periodic with period $4n+2$ and thus $\{X(t), t \geq 1\}$, as a random start in a period, is stationary and the constraint in (A5) is satisfied. It is also easy to see that the constraint in (A2) is satisfied.

Corollary 2. *Suppose that (i) n is an even number, (ii) both users select their hopping sequences* independently *via Algorithm 5, and (iii) there is at most one permanently blocked channel. Then both users rendezvous within $4n+2$ steps, i.e., $T \leq 4n+2$.*

Proof. This is a direct consequence of Theorem 8. ■

Since $g(t)$ is the time-interleaving sequence of the two sequences $f_o(t)$ and $f_e(t)$, it is not difficult to see from Proposition 1 and Corollary 1 that (69) holds for any *even* time shift d (the detailed proof will be given in Lemma 6 later). The hard part is to show that (69) holds when d is odd.

From the definitions of sawtooth sequences and inverted sawtooth sequences, we note that for $j = 0, 1, \dots, n-1$,

$$\begin{aligned} j &= f_o(j+1) = f_o(2n-j) \\ &= f_e(n-j) = f_e(n+1+j). \end{aligned}$$

In view of the definition of the interleaved sawtooth sequence $g(t)$ in (67), we then have

$$\begin{aligned} j &= g(2j+1) = g(4n-2j-1) \\ &= g(2n-2j) = g(2n+2+2j). \end{aligned} \tag{71}$$

Also, by replacing j by $(n-1)-j$ in (71), we have that

$$\begin{aligned} &(n-1)-j \\ &= g(4n-2((n-1)-j)-1) = g(2n+2j+1) \\ &= g(2n+2+2((n-1)-j)) = g(4n-2j), \end{aligned} \tag{72}$$

for all $j = 0, 1, \dots, n-1$.

The proof of Theorem 8 is based on the following lemma that specifically identifies both the time and the channel index for two users to rendezvous.

Lemma 6. *Assume that $\{g(t), t \geq 1\}$ is an interleaved sawtooth sequence of order n. Consider two sequences $x_1(t) = g(t)$ and $x_2(t) = g(t+d)$ for some $d \geq 0$. Let*

$$d_0 = (d \; mod \, (4n+2)). \tag{73}$$

(i) Suppose that $d_0 = 0$. Then, $x_1(t) = x_2(t)$ for all $1 \leq t \leq 4n$.

(ii) Suppose that $d_0 = 2n+1$. Then, $x_1(t) = x_2(t)$ for all odd t with $1 \leq t \leq 2n$ and all even t with $2n+2 \leq t \leq 4n$.

(iii) Suppose that d_0 is a nonzero even number and $d_0/2$ is also an even number. Let $\tau_{e,1} = (2n+1) - (d_0/4)$. Then, $n+1 \leq \tau_{e,1} \leq 2n$, $x_1(2\tau_{e,1}-1) = x_2(2\tau_{e,1}-1) = 2n - \tau_{e,1}$ and $x_1(2\tau_{e,1}) = x_2(2\tau_{e,1}) = \tau_{e,1} - (n+1)$.

(iv) Suppose that d_0 is a nonzero even number and $d_0/2$ is an odd number. Let $\tau_{e,2} = (2n - (d_0/2)+1)/2$. Then, $1 \leq \tau_{e,2} \leq n$, $x_1(2\tau_{e,2}-1) = x_2(2\tau_{e,2}-1) = \tau_{e,2} - 1$, and $x_1(2\tau_{e,2}) = x_2(2\tau_{e,2}) = n - \tau_{e,2}$.

(v) Suppose that d_0 is an odd number and $d_0 = 2n-1-4j_0$ for some $0 \le j_0 \le \lfloor (2n-1)/4 \rfloor$. Let $\tau_{o,1} = n - \frac{d_0-1}{2}$. Then, $1 \le \tau_{o,1} \le n$, $x_1(\tau_{o,1}) = x_2(\tau_{o,1}) = \frac{\tau_{o,1}-1}{2}$, and $x_1(\tau_{o,1}+2n) = x_2(\tau_{o,1}+2n) = (n-1) - \frac{\tau_{o,1}-1}{2}$. In particular, for $d_0 = 2n-1$, $x_1(t) = x_2(t)$ for all even t with $2 \le t \le 2n$.

(vi) Suppose that d_0 is an odd number and $d_0 = 2n-3-4j_0$ for some $0 \le j_0 \le \lfloor (2n-3)/4 \rfloor$. Let $\tau_{o,1} = n - \frac{d_0-1}{2}$. Then, $1 \le \tau_{o,1} \le n$, $x_1(\tau_{o,1}) = x_2(\tau_{o,1}) = n - \frac{\tau_{o,1}}{2}$, and $x_1(\tau_{o,1}+2n) = x_2(\tau_{o,1}+2n) = \frac{\tau_{o,1}}{2} - 1$.

(vii) Suppose that d_0 is an odd number and $d_0 = 2n+3+4j_0$ for some $0 \le j_0 \le \lfloor (2n-1)/4 \rfloor$. Let $\tau_{o,2} = (2n+1) + n - \frac{d_0-1}{2}$. Then, $n+1 \le \tau_{o,2} \le 2n$, $x_1(\tau_{o,2}) = x_2(\tau_{o,2}) = n - \frac{\tau_{o,2}}{2}$, and $x_1(\tau_{o,2}+2n) = x_2(\tau_{o,2}+2n) = \frac{\tau_{o,2}}{2} - 1$. In particular, for $d_0 = 2n+3$, $x_1(t) = x_2(t)$ for all odd t with $2n+1 \le t \le 4n-1$.

(viii) Suppose that is d_0 an odd number and $d_0 = 2n+5+4j_0$ for some $0 \le j_0 \le \lfloor (2n-3)/4 \rfloor$. Let $\tau_{o,2} = (2n+1) + n - \frac{d_0-1}{2}$. Then, $n+1 \le \tau_{o,2} \le 2n$, $x_1(\tau_{o,2}) = x_2(\tau_{o,2}) = \frac{\tau_{o,2}-1}{2}$, and $x_1(\tau_{o,2}+2n) = x_2(\tau_{o,2}+2n) = (n-1) - \frac{\tau_{o,2}-1}{2}$.

Proof. (i) For $d_0 = 0$, we have from (67) that

$$x_1(t) = g(t) = g(t+d_0) = x_2(t) \tag{74}$$

for all $1 \le t \le 4n$.

(ii) $d_0 = 2n+1$. From (71), for all $j = 0, 1, \ldots, n-1$, we have that

$$x_1(2j+1) = g(2j+1) = g((2j+1)+(2n+1)) = x_2(2j+1),$$

and

$$\begin{aligned} x_1(2n+2+2j) &= g(2n+2+2j) = g(2j+1) = g(2j+1+4n+2) \\ &= g((2n+2+2j)+(2n+1)) = x_2(2n+2+2j). \end{aligned}$$

(iii) Since $d_0/2$ is an even number, we have from Proposition 1(ii) and Corollary 1(ii) that $f_o(\tau_{e,1}) = f_o(\tau_{e,1}+d_0/2) = 2n - \tau_{e,1}$ and $f_e(\tau_{e,1}) = f_e(\tau_{e,1}+d_0/2) = \tau_{e,1} - (n+1)$ with $\tau_{e,1} = 2n+1-d_0/4$. Thus, it follows from (67) that

$$x_1(2\tau_{e,1}-1) = g(2\tau_{e,1}-1) = f_o(\tau_{e,1}) = 2n - \tau_{e,1}.$$

On the other hand,

$$x_2(2\tau_{e,1}-1) = g(2\tau_{e,1}-1+d_0) = f_o(\tau_{e,1}+d_0/2) = 2n - \tau_{e,1}.$$

Also, from (67), we have that

$$x_1(2\tau_{e,1}) = g(2\tau_{e,1}) = f_e(\tau_{e,1}) = \tau_{e,1} - (n+1),$$

and

$$x_2(2\tau_{e,1}) = g(2\tau_{e,1}+d_0) = f_e(\tau_{e,1}+d_0/2) = \tau_{e,1} - (n+1).$$

(iv) Since $d_0/2$ is an odd number, we have from Proposition 1(iii) and Corollary 1(iii) that $f_o(\tau_{e,2}) = f_o(\tau_{e,2}+d_0/2) = \tau_{e,2} - 1$ and $f_e(\tau_{e,2}) = f_e(\tau_{e,2}+d_0/2) = n - \tau_{e,2}$ with $\tau_{e,2} = (2n-(d_0/2)+1)/2$. Thus, it follows from (67) that

$$x_1(2\tau_{e,2}-1) = g(2\tau_{e,2}-1) = f_o(\tau_{e,2}) = \tau_{e,2} - 1.$$

On the other hand,

$$x_2(2\tau_{e,2}-1)=g(2\tau_{e,2}-1+d_0)=f_o(\tau_{e,2}+d_0/2)=\tau_{e,2}-1.$$

Also, from (67), we have that

$$x_1(2\tau_{e,2})=g(2\tau_{e,2})=f_e(\tau_{e,2})=n-\tau_{e,2},$$

and

$$x_2(2\tau_{e,2})=g(2\tau_{e,2}+d_0)=f_e(\tau_{e,2}+d_0/2)=n-\tau_{e,2}.$$

(v) Since $d_0=2n-1-4j_0$, we have

$$\tau_{o,1}=n-\frac{d_0-1}{2}=2j_0+1. \tag{75}$$

It then follows from (71) that

$$\begin{aligned}&x_1(\tau_{o,1})=g(2j_0+1)=j_0=g(2n-2j_0)\\&=g(n-\frac{d_0-1}{2}+d_0)=x_2(\tau_{o,1}).\end{aligned}$$

In conjunction with (75), we have

$$x_1(\tau_{o,1})=x_2(\tau_{o,1})=\frac{\tau_{o,1}-1}{2}.$$

On the other hand, we have from (72) that

$$\begin{aligned}&x_1(\tau_{o,1}+2n)=g(2n+2j_0+1)=(n-1)-j_0=g(4n-2j_0)\\&=g(2n+2j_0+1+d_0)=x_2(\tau_{o,1}+2n).\end{aligned}$$

Thus,

$$x_1(\tau_{o,1}+2n)=x_2(\tau_{o,1}+2n)=(n-1)-\frac{\tau_{o,1}-1}{2}.$$

In particular, for the case that $d_0=2n-1$, we have from (75) that $\tau_{o,1}=1$ and $j_0=0$. Thus

$$x_1(1)=x_2(1)=0, \tag{76}$$

and

$$x_1(2n+1)=x_2(2n+1)=n-1. \tag{77}$$

Moreover, from (71), we also have that

$$\begin{aligned}&x_1(2n-2j)=g(2n-2j)\\&=g(4n-2j-1)\\&=g((2n-2j)+(2n-1))\\&=x_2(2n-2j)=j,\end{aligned} \tag{78}$$

for all $j=0,1,\dots,n-1$. In other words, for $d_0=2n-1$, we have that $x_1(t)=x_2(t)$ for all even t with $2\le t\le 2n$.

(vi) Let $k_0 = (n-1) - j_0$. Since $0 \le j_0 \le \lfloor 2n - 3/4 \rfloor$, it is easy to see that $0 \le k_0 \le n-1$. Since $d_0 = 2n - 3 - 4j_0$, we have

$$\tau_{o,1} = n - \frac{d_0 - 1}{2} = 2j_0 + 2 = 2n - 2k_0. \tag{79}$$

It then follows from (71) that

$$\begin{aligned}&x_1(\tau_{o,1}) = g(2n - 2k_0) = k_0 = g(2k_0 + 1)\\&= g(n - \frac{d_0 - 1}{2} + d_0) = x_2(\tau_{o,1}).\end{aligned}$$

In conjunction with (79), we have

$$x_1(\tau_{o,1}) = x_2(\tau_{o,1}) = k_0 = n - \frac{\tau_{o,1}}{2}.$$

On the other hand, we have from (72) that

$$\begin{aligned}&x_1(\tau_{o,1} + 2n) = g(4n - 2k_0) = (n-1) - k_0 = g(2n + 2k_0 + 1)\\&= g(4n - 2k_0 + d_0) = x_2(\tau_{o,1} + 2n).\end{aligned}$$

Thus,

$$x_1(\tau_{o,1} + 2n) = x_2(\tau_{o,1} + 2n) = (n-1) - k_0 = \frac{\tau_{o,1}}{2} - 1.$$

(vii) Since $d_0 = 2n + 3 + 4j_0$, we have

$$\tau_{o,2} = (2n+1) + n - \frac{d_0 - 1}{2} = 2n - 2j_0. \tag{80}$$

It then follows from (71) and the periodic property of $g(t)$ that

$$\begin{aligned}&x_1(\tau_{o,2}) = g(2n - 2j_0) = j_0 = g(2j_0 + 1)\\&= g(4n + 2 + 2j_0 + 1) = g((2n+1) + n - \frac{d_0 - 1}{2} + d_0) = x_2(\tau_{o,2}).\end{aligned}$$

In conjunction with (80), we have

$$x_1(\tau_{o,2}) = x_2(\tau_{o,2}) = n - \frac{\tau_{o,2}}{2}.$$

On the other hand, we have from (72) and the periodic property of $g(t)$ that

$$\begin{aligned}&x_1(\tau_{o,2} + 2n) = g(4n - 2j_0) = (n-1) - j_0 = g(2n + 2j_0 + 1)\\&= g(4n + 2 + 2n + 2j_0 + 1) = g(4n - 2j_0 + d_0) = x_2(\tau_{o,2} + 2n).\end{aligned}$$

Thus,

$$x_1(\tau_{o,2} + 2n) = x_2(\tau_{o,2} + 2n) = \frac{\tau_{o,2}}{2} - 1.$$

In particular, for the case that $d_0 = 2n + 3$, we have from (80) that $\tau_{o,2} = 2n$ and $j_0 = 0$. Thus

$$x_1(2n) = x_2(2n) = 0, \tag{81}$$

and

$$x_1(4n) = x_2(4n) = n - 1. \tag{82}$$

Moreover, from (71) and the periodic property of $g(t)$, we also have that

$$\begin{aligned} & x_1(4n-2j-1) = g(4n-2j-1) \\ & = g(2n-2j) \\ & = g(4n+2+2n-2j) \\ & = g((4n-2j-1)+(2n+3)) \\ & = x_2(4n-2j-1) = j, \end{aligned} \tag{83}$$

for all $j = 0, 1, \ldots, n-1$. In other words, for $d_0 = 2n+3$, we have that $x_1(t) = x_2(t)$ for all odd t with $2n+1 \le t \le 4n-1$.

(viii) Let $k_0 = (n-1) - j_0$. Since $d_0 = 2n+5+4j_0$, we have

$$\tau_{o,2} = 2n+1+n-\frac{d_0-1}{2} = 2k_0+1. \tag{84}$$

It then follows from (71) and the periodic property of $g(t)$ that

$$\begin{aligned} & x_1(\tau_{o,2}) = g(2k_0+1) = k_0 = g(2n-2k_0) \\ & = g(4n+2+2n-2k_0) = g(\tau_{o,2}+d_0) = x_2(\tau_{o,2}). \end{aligned}$$

In conjunction with (84), we have

$$x_1(\tau_{o,2}) = x_2(\tau_{o,2}) = \frac{\tau_{o,2}-1}{2}.$$

On the other hand, we have from (72) and the periodic property of $g(t)$ that

$$\begin{aligned} & x_1(\tau_{o,2}+2n) = g(2n+2k_0+1) = (n-1)-k_0 = g(4n-2k_0) \\ & = g(4n+2+4n-2k_0) = g(2n+2k_0+1+d_0) = x_2(\tau_{o,1}+2n). \end{aligned}$$

Thus,

$$x_1(\tau_{o,2}+2n) = x_2(\tau_{o,2}+2n) = (n-1) - \frac{\tau_{o,2}-1}{2}.$$

■

Proof. (Theorem 8) Note that the results in Lemma 6(iii) and (iv) address the case when d_0 is a nonzero even number. It is clear that (70) holds when n is a nonzero even number. Similarly, the results in Lemma 6(v) and (vi) address the case when d_0 is an odd number and $1 \le d_0 \le 2n-1$, and the results in Lemma 6(vii) and (viii) address the case when d_0 is an odd number and $2n+3 \le d_0 \le 4n+1$, respectively. It is also clear that (70) holds for all these four cases when n is an even number. Finally, the two trivial cases with $d_0 = 0$ and $d_0 = 2n+1$ are shown in Lemma 6(i) and (ii). ■

6. Conclusion In this paper, we formulated the multichannel rendezvous problem as an optimization problem that minimizes the expected (resp. maximum) time-to-rendezvous among all the random hopping sequences that satisfy certain constraints. We also proposed several new constructions of hopping sequences by using the mathematical theories of finite projective planes, orthogonal Latin squares, and sawtooth sequences. Though we are able to show some of them are indeed optimal or better than other existing schemes in the literature, there are still many unanswered questions that require further study. Here we address some possible extensions of our works.

(i) Relaxation of the load constraint: in this paper, we assume the uniform load constraint in (A2). One possible extension is to use a less stringent assumption on the load such as

$$\sup_{t\geq 1} \max_{0\leq i\leq n-1} \max[\mathsf{P}(X_1(t)=i), \mathsf{P}(X_2(t)=i)] \leq \rho$$

for some $\frac{1}{n} \leq \rho \leq 1$. In particular, when $\rho = 1/k$ for some integer $1 \leq k \leq n$, one can apply the majorization theory [37] to extend the result in Lemma 2 as follows:

$$\frac{1}{n} \leq \mathsf{P}(X_1(t) = X_2(t)) \leq \frac{1}{k}.$$

Following the same argument as in the proof of Theorem 2, one can show there does not exist random sequences such that the time-to-rendezvous can be deterministically upper bounded by k steps. Clearly, if a finite projective plane of order k exists, then one can select k channels out of the n channels and then use Algorithm 1 to construct random sequences such that the two users are guaranteed to rendezvous within $k+1$ steps.
(ii) Degree of overlapping: in addition to time-to-rendezvous, another important performance metric is the degree of overlapping that counts the number of distinct channels for the two users to rendezvous [13, 47]. For instance, we showed in Theorem 6 that the degree of overlapping for the sequences constructed by Algorithm 3 is n (under the condition of time synchronization), i.e., all the channels can be used for the two users to rendezvous. On the other hand, we showed in Theorem 8 that the interleaved sawtooth sequence constructed by Algorithm 5 has the degree of overlapping 2 (without the need for time synchronization). It was shown in [40, 33] that there are sequences with the degree of overlapping n. In particular, when there are $n-1$ blocked channels, it was shown in [40] that the two users are still guaranteed to rendezvous within $n(3n-1)$ steps. It would be of interest to see if it is possible to construct sawtooth sequences (by interleaving) that can have a better deterministic upper bound for the time-to-rendezvous when there are $n-1$ blocked channels.
(iii) Tighter lower bounds under the stationary constraint: as pointed out in [42], it is of interest to note that the objective of constructing optical orthogonal codes is to construct channel hopping sequences that minimize the interference (in terms of the number of overlapping time intervals over space and time) while the construction of hopping sequences in the multichannel rendezvous problem is to maximum the number of rendezvous. It seems that techniques for obtaining upper bounds in optical orthogonal codes for periodic sequences (see e.g., [23]) might also be able to used for obtaining lower bounds for the multichannel rendezvous problem.

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