Greedy Constructions of Optical Queues with a Limited Number of Recirculations

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***Abstract*—One of the main problems in all-optical packet-switched networks is the lack of optical buffers, and currently the only known feasible technology for the constructions of optical buffers is to use optical crossbar Switches and fiber Delay Lines (SDL). In this paper, we consider SDL constructions of optical queues with a limited number of recirculations through the optical switches and the fiber delay lines. Such a problem arises from practical feasibility considerations, such as crosstalk, power loss, amplified spontaneous emission (ASE) from the Erbium doped fiber amplifiers (EDFA), and the pattern effect of the optical switches.**

We first transform the design of the fiber delays in such SDL constructions into an equivalent integer representation problem. Specifically, given $1 \leq k \leq M$, we seek for an M-sequence $\mathbf{d}_M = (d_1, d_2, \ldots, d_M)$ of positive integers to maximize the number of consecutive integers (starting from 0) that can be represented by the $\mathcal{C}$-transform (a generalization of the well-known binary representation) with respect to $\mathbf{d}_M$ such that there are at most k 1-entries in their $\mathcal{C}$-transforms. Then we propose a class of greedy constructions of $\mathbf{d}_M$, in which $d_1, d_2, \ldots, d_M$ are obtained recursively in a greedy manner so that the number of representable consecutive integers by using $d_1, d_2, \ldots, d_i$ is larger than that by using $d_1, d_2, \ldots, d_{i-1}$ for all i. Finally, we show that every optimal construction (in the sense of maximizing the number of representable consecutive integers) must be a greedy construction. As a result, the complexity of searching for an optimal construction can be greatly reduced from exponential time to polynomial time by only considering the greedy constructions rather than performing an exhaustive search. The solution of such an integer representation problem can be applied to the constructions of optical 2-to-1 FIFO multiplexers with a limited number of recirculations. Similar results can be obtained for the constructions of optical linear compressors/decompressors with a limited number of recirculations.

***Index Terms*—Fiber delay lines, FIFO multiplexers, integer representation, linear compressors, linear decompressors, optical buffers, optical queues, optical switches.**

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I. Introduction

Due to the lack of optical buffers to resolve conflicts among packets competing for the same resources in the optical domain, current high-speed packet-switched networks suffer from the serious overheads incurred by the O-E-O (optical-electrical-optical) conversion and the accompanied signal processing. As a result, the design of optical buffers has become one of the most critically sought after optical technologies in all-optical packet-switched networks.

Currently, the only known way to "store" optical packets without converting them into other media is to direct them through a set of (bufferless) optical crossbar Switches and fiber Delay Lines (SDL) so that the optical packets can be routed to the right place at the right time. Recently, there has been a lot of attention in the literature [1]–[38] on the SDL constructions of optical queues, including output-buffered switches in [5]–[9], FIFO multiplexers in [5] and [9]–[15], FIFO queues in [15]–[18], LIFO queues in [18]–[19], priority queues in [20]–[24], time slot interchanges in [15] and [25], and linear compressors, linear decompressors, non-overtaking delay lines, and flexible delay lines in [15] and [26]–[29]. Furthermore, results on the fundamental complexity of SDL constructions of optical queues can be found in [30] and performance analysis for optical queues has been addressed in [31]–[32]. For review articles on SDL constructions of optical queues, we refer to [33]–[38] and the references therein.

In this paper, we address an important practical feasibility issue that is of great concern in the SDL constructions of optical queues: the constructions of optical queues with a limited number of recirculations through the optical switches and the fiber delay lines. As pointed out in [39]–[41], crosstalk due to power leakage from other optical links, power loss experienced during recirculations through the optical switches and the fiber delay lines, amplified spontaneous emission (ASE) from the Erbium doped fiber amplifiers (EDFA) that are used for boosting the signal power, and the pattern effect of the optical switches, among others, lead to a limitation on the number of times that an optical packet can be recirculated through the optical switches and the fiber delay lines. If such an issue is not taken into consideration during the design of optical queues, then for an optical packet recirculated through the optical switches and the fiber delay lines for a number of times exceeding a certain threshold, there is a good chance that it cannot be reliably recognized at the destined output port due to severe power loss and/or serious noise accumulation even if it appears at the right place at the right time.

For certain optical queues, including 2-to-1 FIFO multiplexers [11] and linear compressors/decompressors [27], the delay x of a packet is known upon its arrival and the routing of the packet is according to the $\mathcal{C}$-transform [11] $\mathcal{C}(x;\mathbf{d}_M) = (I_1(x;\mathbf{d}_M), I_2(x;\mathbf{d}_M), \dots, I_M(x;\mathbf{d}_M))$ (a generalization of the well-known binary representation) of the packet delay x with respect to the M-sequence $\mathbf{d}_M = (d_1, d_2, \dots, d_M)$ of the fiber delays in the SDL constructions of these queues. For these optical queues, there is a prominent *route-once* property which says that an optical packet can only be routed through each fiber *at most once*. Specifically, if $I_i(x;\mathbf{d}_M) = 1$ for some $1 \le i \le M$, then the packet will be routed through the i^{th} fiber with delay d_i once; otherwise, the packet will not be routed to the i^{th} fiber. Therefore, if $I_i(x;\mathbf{d}_M) = 1$ for all $i = 1, 2, \dots, M$, then the packet will be routed through each of the M fibers once.

The problem arises if there is a limitation on the number, say k, of recirculations through the M fibers due to the practical feasibility considerations mentioned above. If $k < M$, then a packet routed through more than k of the M fibers cannot be reliably recognized at the destined output port. It follows that in such situations the *effective* buffer size (for 2-to-1 FIFO multiplexers) or the *effective* maximum delay (for linear compressors/decompressors) is given by the *maximum representable integer* with respect to $\mathbf{d}_M$ and k, which is defined as the largest nonnegative integer such that all of the nonnegative integers not exceeding it have at most k 1-entries in their $\mathcal{C}$-transforms with respect to $\mathbf{d}_M$. Therefore, the problem of constructing the delays $d_1, d_2, \dots, d_M$ of the M fibers in these optical queues so that the effective buffer size/maximum delay is as large as possible under the constraint of recirculations through at most k of the M fibers is equivalent to the integer representation problem of constructing an M-sequence $\mathbf{d}_M = (d_1, d_2, \dots, d_M)$ of positive integers so that the maximum representable integer with respect to $\mathbf{d}_M$ and k is as large as possible.

In [13], a dynamic programming formulation obtained through a divide-and-conquer approach was proposed for SDL constructions of 2-to-1 FIFO multiplexers under the constraint of recirculations through at most k of the M fibers. However, the constructions in [13] are not optimal since they are designed to provide a guaranteed effective buffer size (so as to provide a guaranteed quality of service) and the fiber delays are limited to be integral multiples of powers of 2.

Our first contribution in this paper is to propose a class of greedy constructions of the M-sequences $\mathbf{d}_M = (d_1, d_2, \dots, d_M)$, in which $d_1, d_2, \dots, d_M$ are obtained recursively in a greedy manner so that the maximum representable integer is *increased* when d_i is added to the already determined $d_1, d_2, \dots, d_{i-1}$ for all i. For each M-sequence $\mathbf{d}_M$ given by the greedy constructions, we obtain an explicit recursive expression for $d_1, d_2, \dots, d_M$ so that d_i is expressed in terms of $d_1, d_2, \dots, d_{i-1}$ for all i, and we also obtain an explicit expression for the maximum representable integer with respect to $\mathbf{d}_M$ and k in terms of $d_1, d_2, \dots, d_M$, and k. Our second contribution is to show that every optimal M-sequence (in the sense of achieving the largest possible maximum representable integer) among all M-sequences satisfying the condition in (A2) (described in Section II) must be a greedy construction. This implies that every optimal construction (in the sense of achieving the largest possible effective buffer size) of an optical 2-to-1 multiplexer with a limited number of recirculations must be a greedy construction. Consequently, the complexity of searching for an optimal construction is greatly reduced by only considering the greedy constructions when compared to performing an exhaustive search (polynomial time vs. exponential time). Similar results can be obtained for the constructions of optical linear compressors/decompressors with a limited number of recirculations, but the algebra involved is more tedious so that they are not presented here due to space limit.

This paper is organized as follows. In Section II, we describe in detail the transformation of the constructions of certain types of optical queues, including optical 2-to-1 FIFO multiplexers and optical linear compressors/decompressors, into an equivalent integer representation problem. In Section III, we propose a class of greedy constructions for the M-sequence $\mathbf{d}_M$ in the equivalent integer representation problem. Furthermore, we obtain an explicit recursive expression for such an M-sequence $\mathbf{d}_M$, and obtain an explicit expression for the maximum representable integer with respect to $\mathbf{d}_M$ and k. In Section IV, we show that every optimal construction must be a greedy construction. Finally, we conclude this paper in Section V.

II. Transformation into an Equivalent Integer Representation Problem

As mention in Section I, the SDL constructions of optical 2-to-1 FIFO multiplexers in [11] and optical linear compressors/decompressors in [27] rely on the $\mathcal{C}$-transform (a generalization of the well-known binary representation) for the unique representation of nonnegative integers. We first recall the $\mathcal{C}$-transform and its unique representation property.

Definition 1 ***($\mathcal{C}$-Transform)*** *[11] Let $\mathbf{d}_M = (d_1, d_2, \dots, d_M)$ be an M-sequence of positive integers. The $\mathcal{C}$-transform $\mathcal{C}(x;\mathbf{d}_M)$ of a nonnegative integer x with respect to $\mathbf{d}_M$ is defined as the M-sequence $(I_1(x;\mathbf{d}_M), I_2(x;\mathbf{d}_M), \dots, I_M(x;\mathbf{d}_M))$, where $I_M(x;\mathbf{d}_M)$, $I_{M-1}(x;\mathbf{d}_M), \dots, I_1(x;\mathbf{d}_M)$, in that order, are given recursively by*

$$I_i(x;\mathbf{d}_M) = \begin{cases} 1, & \text{if } x - \sum_{j=i+1}^{M} I_j(x;\mathbf{d}_M) d_j \ge d_i, \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

with the convention that the sum in (1) is 0 if the upper index is smaller than its lower index. In other words, if $x \ge d_M$, then $I_M(x;\mathbf{d}_M) = 1$, and otherwise $I_M(x;\mathbf{d}_M) = 0$; if the remaining value $x - I_M(x;\mathbf{d}_M) d_M \ge d_{M-1}$, then

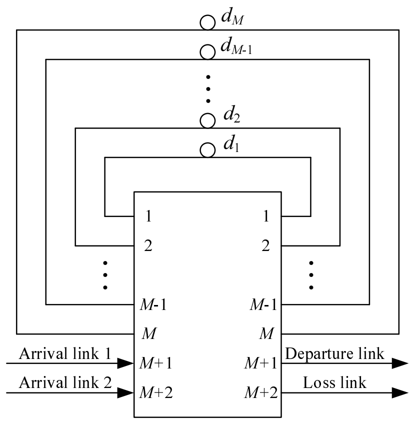
$I_{M-1}(x;\mathbf{d}_M) = 1$*, and otherwise* $I_{M-1}(x;\mathbf{d}_M) = 0$*; and so forth.*

Theorem 2 ***(Unique Representation Property)*** *[11] Let* $\mathbf{d}_M = (d_1, d_2, \ldots, d_M)$ *be an* M*-sequence of positive integers. The* $\mathcal{C}$*-transform* $\mathcal{C}(x;\mathbf{d}_M)$ *of a nonnegative integer* x *with respect to* $\mathbf{d}_M$ *is the unique representation of* x*, i.e.,* $x = \sum_{i=1}^{M} I_i(x;\mathbf{d}_M)d_i$*, for all* $x = 0, 1, \ldots, \sum_{i=1}^{M} d_i$ *if and only if* $d_1, d_2, \ldots, d_M$ *satisfy the following condition in (A1):*

(A1) $d_1 = 1$ *and* $1 \le d_{i+1} \le \sum_{j=1}^{i} d_j + 1$ *for* $i = 1, 2, \ldots, M-1$.

It is clear that if $d_i = 2^{i-1}$ for $i = 1, 2, \ldots, M$, then the $\mathcal{C}$-transform becomes the well-known binary representation for the unique representation of the nonnegative integers $0, 1, \ldots, 2^M - 1$.

(a)



(b)

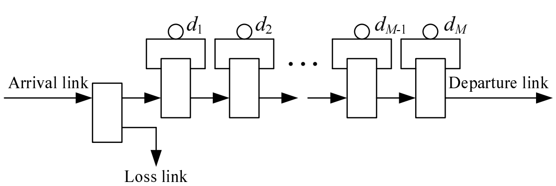


Fig. 1. (a) A construction of a 2-to-1 FIFO multiplexer with buffer size $\sum_{i=1}^{M} d_i$. (b) A construction of a linear compressor with maximum delay $\sum_{i=1}^{M} d_i$.

Now we briefly describe the SDL constructions of optical 2-to-1 FIFO multiplexers in [11] and optical linear compressors/decompressors in [27]. In [11], it was shown that the construction in Figure 1(a) consisting of an $(M+2)\times(M+2)$ optical crossbar switch and M fiber delay lines with delays $d_1, d_2, \ldots, d_M$ can be operated as a 2-to-1 FIFO multiplexer with buffer size $\sum_{i=1}^{M} d_i$ under a simple packet routing scheme if and only if $d_1, d_2, \ldots, d_M$ satisfy the following condition in (A2):

(A2) $d_1 = 1$ and $d_i \le d_{i+1} \le 2d_i$ for $i = 1, 2, \ldots, M-1$.

Furthermore, it was shown in [27] that the construction in Figure 1(b) consisting of a 1×2 optical crossbar switch, M 2×2 optical crossbar switches, and M fiber delay lines with delays $d_1, d_2, \ldots, d_M$ can be operated as a linear compressor with maximum delay $\sum_{i=1}^{M} d_i$ under a simple packet routing scheme if and only if $d_1, d_2, \ldots, d_M$ satisfy the condition in (A1). We note that a linear decompressor with maximum delay $\sum_{i=1}^{M} d_i$ can be similarly constructed since it is the mirror image of a linear compressor with maximum delay $\sum_{i=1}^{M} d_i$ [27]. It is to be noted that the condition in (A2) is stronger than that in (A1) as it has been shown in [11] that if $d_1, d_2, \ldots, d_M$ satisfy the condition in (A2), then they also satisfy the condition in (A1).

The simple packet routing scheme mentioned above is a *self-routing* scheme which is described as follows. Suppose that the delay of a packet arriving at time t is x. If $x > \sum_{i=1}^{M} d_i$, then the packet is routed to the loss link immediately so that the packet is lost. On the other hand, if $0 \le x \le \sum_{i=1}^{M} d_i$, then the packet is routed to the fiber with delay d_1 at time t if $I_1(x;\mathbf{d}_M) = 1$, to the fiber with delay d_2 at time $t + I_1(x;\mathbf{d}_M)d_1$ if $I_2(x;\mathbf{d}_M) = 1, \ldots$, to the fiber with delay d_M at time $t + \sum_{i=1}^{M-1} I_i(x;\mathbf{d}_M)d_i$ if $I_M(x;\mathbf{d}_M) = 1$, and finally to the departure link at time $t + \sum_{i=1}^{M} I_i(x;\mathbf{d}_M)d_i = t + x$. Therefore, the packet is routed to the right place at the right time.

In reality, there is a limitation on the number, say k, of recirculations through the M fibers in order to ensure that a packet can be reliably recognized at the destined output port. In such situations, the effective buffer size (for 2-to-1 FIFO multiplexers) or the effective maximum delay (for linear compressors/decompressors) is given by the largest nonnegative integer such that all of the nonnegative integers not exceeding this nonnegative integer have at most k 1-entries in their $\mathcal{C}$-transforms with respect to $\mathbf{d}_M$. This follows from the fact that if $k < M$, then a packet with delay equal to one more than this largest nonnegative integer has more than k 1-entries in the $\mathcal{C}$-transform of its delay with respect to $\mathbf{d}_M$, and hence under the self-routing scheme described above this packet is routed through more than k fibers so that it cannot be reliably recognized at the destined output port. We call such a largest nonnegative integer the maximum representable integer with respect to $\mathbf{d}_M$ and k, denoted $B(\mathbf{d}_M;k)$, i.e.,

$$B(\mathbf{d}_M;k) = \max\left\{ 0 \le y \le \sum_{i=1}^{M} d_i : \begin{array}{c} \sum_{i=1}^{M} I_i(x;\mathbf{d}_M) \le k \\ \text{for all } x = 0, 1, \ldots, y \end{array} \right\}. \quad (2)$$

For obvious reasons, we define $B(\mathbf{d}_M;k) = 0$ if $M = 0$ or $k = 0$. As each $I_i(x;\mathbf{d}_M)$ is equal to 0 or 1, it follows that $B(\mathbf{d}_M;k) = \sum_{i=1}^{M} d_i$ if $k \ge M$.

Since we are most interested in the constructions of these optical queues with as large effective buffer size/maximum delay as possible, the problem of finding optimal constructions of the fiber delays $d_1, d_2, \ldots, d_M$ that achieve the largest possible effective buffer size/maximum delay for these optical queues under the constraint of recirculations through at most k of the M fibers is equivalent to the integer representation problem of finding optimal constructions of the M-sequence $\mathbf{d}_M$ such that the maximum representable integer $B(\mathbf{d}_M;k)$ with respect to $\mathbf{d}_M$ and k is the largest possible.

In this paper, we focus on a class of greedy constructions of the M-sequence in the integer representation problem in

Section III, and show that every optimal construction of the M-sequence that achieves the largest possible maximum representable integer among all M-sequences satisfying the condition in (A2) must be a greedy construction in Section IV. Therefore, the results in this paper can be directly used for the constructions of optical 2-to-1 FIFO multiplexers with a limited number of recirculations through the fibers. We note that similar results can be obtained if the maximization of the maximum representable integer is over all M-sequences satisfying the condition in (A1), and these results can be directly used for the constructions of optical linear compressors/decompressors with a limited number of recirculations through the fibers.

III. A Class of Greedy Constructions

In this section, we propose a class of greedy constructions of the M-sequence $\mathbf{d}_M = (d_1, d_2, \ldots, d_M)$. In our proposed greedy constructions, $d_1, d_2, \ldots, d_M$ are obtained recursively and each d_i is obtained from $d_1, d_2, \ldots, d_{i-1}$ in a greedy manner so that the maximum representable integer by using $d_1, d_2, \ldots, d_i$ is larger than that by using $d_1, d_2, \ldots, d_{i-1}$ for all i. For convenience, we denote $\mathcal{A}_M$ as the set of all M-sequences $\mathbf{d}_M$ satisfying the condition in (A2).

Consider the case that $M = 6$ and $k = 2$. Suppose that $\mathbf{d}'_6 = (1, 2, 4, 8, 16, 32) \in \mathcal{A}_6$ (note that $\mathbf{d}'_6 = \arg\max_{\mathbf{d}_6 \in \mathcal{A}_6} \sum_{i=1}^{6} d_i$) and $\mathbf{d}''_6 = (1, 2, 3, 5, 6, 8) \in \mathcal{A}_6$. According to the unique representation property of the $\mathcal{C}$-transform in Theorem 2, the nonnegative integers $0, 1, \ldots, \sum_{i=1}^{6} d'_i = 63$ can be uniquely represented by their $\mathcal{C}$-transforms with respect to $\mathbf{d}'_6$ and the nonnegative integers $0, 1, \ldots, \sum_{i=1}^{6} d''_i = 25$ can be uniquely represented by their $\mathcal{C}$-transforms with respect to $\mathbf{d}''_6$. From the definition of the maximum representable integer in (2), we can see that the maximum representable integer with respect to $\mathbf{d}'_6$ and 2 is given by $B(\mathbf{d}'_6; 2) = 6$ (as $\sum_{i=1}^{6} I_i(x; \mathbf{d}'_6) \leq 2$ for $x = 0, 1, \ldots, 6$ and $\sum_{i=1}^{6} I_i(x; \mathbf{d}'_6) = 3 > 2$ for $x = 7$) and the maximum representable integer with respect to $\mathbf{d}''_6$ and 2 is given by $B(\mathbf{d}''_6; 2) = 11$ (as $\sum_{i=1}^{6} I_i(x; \mathbf{d}''_6) \leq 2$ for $x = 0, 1, \ldots, 11$ and $\sum_{i=1}^{6} I_i(x; \mathbf{d}''_6) = 3 > 2$ for $x = 12$). Although $\sum_{i=1}^{6} d''_i = 25$ is smaller than $\sum_{i=1}^{6} d'_i = 63$, the maximum representable integer $B(\mathbf{d}''_6; 2) = 11$ by using $\mathbf{d}''_6$ is larger than maximum representable integer $B(\mathbf{d}'_6; 2) = 6$ by using $\mathbf{d}'_6$. It follows that $\mathbf{d}''_6$ is a better choice than $\mathbf{d}'_6$ for our purpose as it gives rise to a larger maximum representable integer.

A natural question we would like to ask is then: can we do better and how to do that? In other words, are there any methods for choosing a sequence $\mathbf{d}_6$ in $\mathcal{A}_6$ such that $B(\mathbf{d}_6; 2) > B(\mathbf{d}''_6; 2)$. The answer is affirmative. A direct approach to choose a sequence $\mathbf{d}_6$ in $\mathcal{A}_6$ is to divide the choice into two parts, say the choice of d_1, d_2, d_3 and the choice of d_4, d_5, d_6, so that there is at most one 1-entry in $(I_1(x; \mathbf{d}_6), I_2(x; \mathbf{d}_6), I_3(x; \mathbf{d}_6))$ and there is at most one 1-entry in $(I_4(x; \mathbf{d}_6), I_5(x; \mathbf{d}_6), I_6(x; \mathbf{d}_6))$ (hence there are at most two 1-entries in $\mathcal{C}(x; \mathbf{d}_6)$) for as many consecutive nonnegative integers x as possible. For instance, we can first choose $d_1 = 1$, $d_2 = 2$, and $d_3 = 3$. Then we have $B(\mathbf{d}_3; 1) = 3$ and we can choose $d_4 = B(\mathbf{d}_3; 1) + 1 = 4$, $d_5 = (d_4 + B(\mathbf{d}_3; 1)) + 1 = 2(B(\mathbf{d}_3; 1) + 1) = 8$, and $d_6 = (d_5 + B(\mathbf{d}_3; 1)) + 1 = 3(B(\mathbf{d}_3; 1) + 1) = 12$. It is easy to see that $B(\mathbf{d}_6; 2) = 16$ (as $\sum_{i=1}^{6} I_i(x; \mathbf{d}_6) \leq 2$ for $x = 0, 1, \ldots, 16$ and $\sum_{i=1}^{6} I_i(x; \mathbf{d}_6) = 3 > 2$ for $x = 17$), which is greater than $B(\mathbf{d}''_6; 2) = 11$.

An even better approach, called a *greedy* approach in this paper, is described as follows. We still divide the choice of a sequence $\mathbf{d}_6$ in $\mathcal{A}_6$ into two parts, say the choice of d_1, d_2, d_3 and the choice of d_4, d_5, d_6, as in the direct approach above. First we choose d_1, d_2, d_3 recursively and each d_i is obtained from $d_1, d_2, \ldots, d_{i-1}$ in a greedy manner so that $B(\mathbf{d}_i; 1)$ is larger than $B(\mathbf{d}_{i-1}; 1)$, and this can be achieved by simply choosing $d_i = B(\mathbf{d}_{i-1}; 1) + 1$ for $i = 1, 2, 3$, i.e.,

$$\begin{aligned}
d_1 &= B(\mathbf{d}_0; 1) + 1 = 0 + 1 = 1, \\
d_2 &= B(\mathbf{d}_1; 1) + 1 = B((1); 1) + 1 = 1 + 1 = 2, \\
d_3 &= B(\mathbf{d}_2; 1) + 1 = B((1, 2); 1) + 1 = 2 + 1 = 3.
\end{aligned}$$

Then we choose d_4, d_5, d_6 recursively and each d_i is obtained from $d_1, d_2, \ldots, d_{i-1}$ in a greedy manner so that $B(\mathbf{d}_i; 2)$ is larger than $B(\mathbf{d}_{i-1}; 2)$, and this can be achieved by simply choosing $d_i = B(\mathbf{d}_{i-1}; 2) + 1$ for $i = 4, 5, 6$, i.e.,

$$\begin{aligned}
d_4 &= B(\mathbf{d}_3; 2) + 1 = B((1, 2, 3); 2) + 1 = 5 + 1 = 6, \\
d_5 &= B(\mathbf{d}_4; 2) + 1 = B((1, 2, 3, 6); 2) + 1 = 9 + 1 = 10, \\
d_6 &= B(\mathbf{d}_5; 2) + 1 = B((1, 2, 3, 6, 10); 2) + 1 = 13 + 1 = 14.
\end{aligned}$$

It is easy to see that $B(\mathbf{d}_6; 2) = 17$ (as $\sum_{i=1}^{6} I_i(x; \mathbf{d}_6) \leq 2$ for $x = 0, 1, \ldots, 17$ and $\sum_{i=1}^{6} I_i(x; \mathbf{d}_6) = 3 > 2$ for $x = 18$), which is larger than 16 in the direct approach above.

We are now in a position to describe our greedy constructions of a sequence $\mathbf{d}_M$ in $\mathcal{A}_M$ in a general setting. Suppose that $1 \leq k \leq M$. Let $\mathbf{n}_k = (n_1, n_2, \ldots, n_k)$ be a k-sequence of positive integers such that $\sum_{i=1}^{k} n_i = M$. Let $s_0 = 0$ and $s_i = \sum_{\ell=1}^{i} n_\ell$ for $i = 1, 2, \ldots, k$, and let $d_1, d_2, \ldots, d_M$ be recursively given by

$$\begin{aligned}
d_{s_i+j} &= B(\mathbf{d}_{s_i+j-1}; i + 1) + 1, \\
&\quad \text{for } i = 0, 1, \ldots, k-1 \text{ and } j = 1, 2, \ldots, n_{i+1}. \quad (3)
\end{aligned}$$

In other words, we divide the choice of a sequence $\mathbf{d}_M = (d_1, d_2, \ldots, d_M)$ in $\mathcal{A}_M$ into k parts, first the choice of $d_1, d_2, \ldots, d_{n_1} = d_{s_1}$, then the choice of $d_{s_1+1}, d_{s_1+2}, \ldots, d_{s_1+n_2} = d_{s_2}, \ldots$, and finally the choice of $d_{s_{k-1}+1}, d_{s_{k-1}+2}, \ldots, d_{s_{k-1}+n_k} = d_{s_k} = d_M$. In the $(i+1)^{\text{th}}$ part, where $0 \leq i \leq k-1$, d_{s_i+j} is obtained recursively by using $d_1, d_2, \ldots, d_{s_i+j-1}$ according to (3) for $j = 1, 2, \ldots, n_{i+1}$. For example, in Table I we show the M-sequence $\mathbf{d}_M$ given by (3) for the case that $M = 18$, $k = 6$, and $\mathbf{n}_k = (3, 4, 2, 5, 1, 3)$.

The reason why we choose $d_1, d_2, \ldots, d_M$ recursively according to (3) can be explained as follows. After $d_1, d_2, \ldots, d_{s_i+j-1}$ have been determined for some $0 \leq i \leq k-1$ and $1 \leq j \leq n_{i+1}$, the nonnegative integers $0, 1, \ldots, B(\mathbf{d}_{s_i+j-1}; i+1)$ are representable by using at most $i+1$ of the integers $d_1, d_2, \ldots, d_{s_i+j-1}$ according to the $\mathcal{C}$-transform. The key idea in our greedy construction is to

i	1	2	3	4	5	6	7	8	9
d_i	1	2	3	6	10	14	18	36	58

i	10	11	12	13	14	15	16	17	18
d_i	116	196	276	356	436	872	1744	3132	4520

TABLE I
THE SEQUENCE $\mathbf{d}_M$ GIVEN BY (3) FOR THE CASE THAT $M = 18$, $k = 6$, AND $\mathbf{n}_k = (3, 4, 2, 5, 1, 3)$.

choose d_{s_i+j} such that the maximum representable integer $B(\mathbf{d}_{s_i+j}; i+1)$ by using $d_1, d_2, \dots, d_{s_i+j}$ is greater than the maximum representable integer $B(\mathbf{d}_{s_i+j-1}; i+1)$ by using $d_1, d_2, \dots, d_{s_i+j-1}$.

If we choose $d_{s_i+j} > B(\mathbf{d}_{s_i+j-1}; i+1) + 1$, then the nonnegative integer $B(\mathbf{d}_{s_i+j-1}; i+1)+1$ is not representable by using at most $i+1$ of the integers $d_1, d_2, \dots, d_{s_i+j}$ according to the $\mathcal{C}$-transform. This is because the integer d_{s_i+j} is not used in the representation of $B(\mathbf{d}_{s_i+j-1}; i+1) + 1$ (as $d_{s_i+j} > B(\mathbf{d}_{s_i+j-1}; i+1) + 1$) and by definition $B(\mathbf{d}_{s_i+j-1}; i+1) + 1$ is not representable by using at most $i+1$ of the integers $d_1, d_2, \dots, d_{s_i+j-1}$. It follows that $B(\mathbf{d}_{s_i+j}; i+1) = B(\mathbf{d}_{s_i+j-1}; i+1)$ and hence such a choice has no use in increasing the maximum representable integer.

Therefore, we choose $1 \le d_{s_i+j} \le B(\mathbf{d}_{s_i+j-1}; i+1) + 1$. For such a choice, the nonnegative integers $0, 1, \dots, d_{s_i+j} - 1$ are representable by using at most $i+1$ of the integers $d_1, d_2, \dots, d_{s_i+j}$ according to the $\mathcal{C}$-transform. This is because the integer d_{s_i+j} is not used in their representations (as they are less than d_{s_i+j}) and by definition they are representable by using at most $i+1$ of the integers $d_1, d_2, \dots, d_{s_i+j-1}$ (as they are less than or equal to $B(\mathbf{d}_{s_i+j-1}; i+1)$). Furthermore, the nonnegative integers $d_{s_i+j}, d_{s_i+j}+1, \dots, d_{s_i+j} + B(\mathbf{d}_{s_i+j-1}; i)$ are representable by using at most $i+1$ of the integers $d_1, d_2, \dots, d_{s_i+j}$ because the integer d_{s_i+j} is used in their representations (as they are greater than or equal to d_{s_i+j}) and by definition the remaining values $0, 1, \dots, B(\mathbf{d}_{s_i+j-1}; i)$ of these nonnegative integers are representable by using at most i of the integers $d_1, d_2, \dots, d_{s_i+j-1}$ (as they are less than or equal to $B(\mathbf{d}_{s_i+j-1}; i)$). Finally, the nonnegative integer $d_{s_i+j} + B(\mathbf{d}_{s_i+j-1}; i) + 1$ is not representable by using at most $i+1$ of the integers $d_1, d_2, \dots, d_{s_i+j}$ because the integer d_{s_i+j} is used in its representation (as it is greater than or equal to d_{s_i+j}) and by definition the remaining value $B(\mathbf{d}_{s_i+j-1}; i)+1$ of this nonnegative integer is not representable by using at most i of the integers $d_1, d_2, \dots, d_{s_i+j-1}$. As a result, we have $B(\mathbf{d}_{s_i+j}; i+1) = d_{s_i+j} + B(\mathbf{d}_{s_i+j-1}; i)$.

Now it is clear that the best choice is $d_{s_i+j} = B(\mathbf{d}_{s_i+j-1}; i+1)+1$ as given by (3) since it gives rise to the largest maximum representable integer.

For convenience, we denote $\mathcal{G}_{M,k}$ as the set of all M-sequences $\mathbf{d}_M$ generated by k-sequences of positive integers $\mathbf{n}_k = (n_1, n_2, \dots, n_k)$ such that $\sum_{i=1}^k n_i = M$ by using (3). Note that if $k = M$, then we must have $n_1 = n_2 = \cdots = n_M = 1$ and $s_0 = 0, s_1 = 1, s_2 = 2, \dots, s_M = M$, and it follows from (3) that

$$\begin{aligned}
d_1 &= d_{s_0+1} = B(\mathbf{d}_{s_0}; 1) + 1 = B(\mathbf{d}_0; 1) + 1 \\
&= 0 + 1 = 1, \\
d_2 &= d_{s_1+1} = B(\mathbf{d}_{s_1}; 2) + 1 = B(\mathbf{d}_1; 2) + 1 \\
&= d_1 + 1 = 2, \\
d_3 &= d_{s_2+1} = B(\mathbf{d}_{s_2}; 3) + 1 = B(\mathbf{d}_2; 3) + 1 \\
&= d_1 + d_2 + 1 = 2^2, \\
&\vdots \\
d_M &= d_{s_{M-1}+1} = B(\mathbf{d}_{s_{M-1}}; M) + 1 = B(\mathbf{d}_{M-1}; M) + 1 \\
&= \sum_{\ell=1}^{M-1} d_\ell + 1 = 2^{M-1}.
\end{aligned}$$

It is clear that $d_1 = 1,\ d_2 = 2,\ d_3 = 2^2, \dots,\ d_M = 2^{M-1}$ satisfy the condition in (A2). Therefore, in the following we only consider the nontrivial case that $1 \le k \le M-1$. In such a nontrivial case, there must exist some $1 \le i \le k$ such that $n_i \ge 2$ as otherwise we have $n_1 = n_2 = \cdots = n_k = 1$ and $\sum_{i=1}^k n_i = k \le M-1$, contradicting to $\sum_{i=1}^k n_i = M$. Furthermore, by the following theorem, it suffices to consider only the case that $n_1 \ge 2$.

Theorem 3 *Suppose that $1 \le k \le M-1$. Let $n_1, n_2, \dots, n_k$ be positive integers such that $n_1 = 1$ and $\sum_{i=1}^k n_i = M$. Let $a = \min\{2 \le i \le k : \ n_i \ge 2\}$ (note that a is well defined as $n_1 = 1$ and hence there must exist some $2 \le i \le k$ such that $n_i \ge 2$), and let $n'_1 = n_1 + 1 = 2$, $n'_i = n_i = 1$ for $i = 2, 3, \dots, a-1$, $n'_a = n_a - 1$, and $n'_i = n_i$ for $i = a+1, a+2, \dots, k$ (note that $n'_1, n'_2, \dots, n'_k$ are positive integers such that $\sum_{i=1}^k n'_i = \sum_{i=1}^k n_i = M$). Suppose that $d_1, d_2, \dots, d_M$ are generated by $n_1, n_2, \dots, n_k$ by using (3), and $d'_1, d'_2, \dots, d'_M$ are generated by $n'_1, n'_2, \dots, n'_k$ by using (3), i.e.,*

$$\begin{aligned}
&d_{s_i+j} = B(\mathbf{d}_{s_i+j-1}; i+1) + 1, \\
&\qquad \textit{for } i = 0, 1, \dots, k-1 \textit{ and } j = 1, 2, \dots, n_{i+1}, \qquad (4) \\
&d'_{s'_i+j} = B(\mathbf{d'}_{s'_i+j-1}; i+1) + 1, \\
&\qquad \textit{for } i = 0, 1, \dots, k-1 \textit{ and } j = 1, 2, \dots, n'_{i+1}, \qquad (5)
\end{aligned}$$

where $s_0 = 0$ and $s_i = \sum_{\ell=1}^i n_\ell$ for $i = 1, 2, \dots, k$, and $s'_0 = 0$ and $s'_i = \sum_{\ell=1}^i n'_\ell$ for $i = 1, 2, \dots, k$, Then $d_\ell = d'_\ell$ for $\ell = 1, 2, \dots, M$.

Proof. See Appendix A. ■

We illustrate Theorem 3 by an example. Suppose that $M = 18$, $k = 7$, $\mathbf{n}_k = (1, 1, 1, 4, 2, 6, 3)$, and $\mathbf{n'}_k = (2, 1, 1, 3, 2, 6, 3)$. Let $\mathbf{d}_M$ be given by (4) and $\mathbf{d'}_M$ be given by (5). Since $n_1 = 1$, $\min\{2 \le i \le 7 : \ n_i \ge 2\} = 4$, $n'_1 = n_1 + 1 = 2$, $n'_i = n_i = 1$ for $i = 2, 3$, $n'_4 = n_4 - 1 = 3$, and $n'_i = n_i$ for $i = 5, 6, 7$, it follows from Theorem 3 that $d_\ell = d'_\ell$ for $\ell = 1, 2, \dots, 18$. Indeed, we see from Table II and Table III that $d_\ell = d'_\ell$ for $\ell = 1, 2, \dots, 18$.

In Theorem 4 below, we derive an explicit recursive expression for the M-sequences $\mathbf{d}_M$ given by the greedy constructions, and derive an explicit expression for the maximum representable integer $B(\mathbf{d}_M; k)$. We also show that $d_1, d_2, \dots, d_M$ satisfy the condition in (A2) so that the feedback system in Figure 1(a) can be operated as a 2-to-1 FIFO multiplexer with effective buffer size $B(\mathbf{d}_M; k) = \sum_{i=1}^k d_{s_i}$ (see (10) below)

i	1	2	3	4	5	6	7	8	9
d_i	1	2	4	8	16	31	46	92	153
i	10	11	12	13	14	15	16	17	18
d_i	306	520	734	948	1162	1376	2752	4342	5932

TABLE II
THE SEQUENCE $\mathbf{d}_M$ GIVEN BY (4) FOR THE CASE THAT $M = 18$, $k = 7$, AND $\mathbf{n}_k = (1, 1, 1, 4, 2, 6, 3)$.

i	1	2	3	4	5	6	7	8	9
d_i	1	2	4	8	16	31	46	92	153
i	10	11	12	13	14	15	16	17	18
d_i	306	520	734	948	1162	1376	2752	4342	5932

TABLE III
THE SEQUENCE $\mathbf{d'}_M$ GIVEN BY (5) FOR THE CASE THAT $M = 18$, $k = 7$, AND $\mathbf{n'}_k = (2, 1, 1, 3, 2, 6, 3)$.

under the constraint that each packet can be routed through at most k of the M fibers by using the self-routing scheme described in Section II.

Theorem 4 *Suppose that $1 \leq k \leq M-1$. Let $n_1, n_2, \ldots, n_k$ be positive integers such that $n_1 \geq 2$ and $\sum_{i=1}^{k} n_i = M$. Let $d_1, d_2, \ldots, d_M$ be generated by $n_1, n_2, \ldots, n_k$ by using (3), i.e., $d_{s_i+j} = B(\mathbf{d}_{s_i+j-1}; i+1) + 1$ for $i = 0, 1, \ldots, k-1$ and $j = 1, 2, \ldots, n_{i+1}$, where $s_0 = 0$ and $s_i = \sum_{\ell=1}^{i} n_\ell$ for $i = 1, 2, \ldots, k$.*

(i) $d_1, d_2, \ldots, d_M$ can be recursively expressed as follows:

$$d_j = j, \text{ for } j = 1, 2, \ldots, s_1, \tag{6}$$

$$d_{s_i+j} = 2d_{s_i} + (j-1)(d_{s_1} + d_{s_2} + \cdots + d_{s_i} + 1), \quad \text{for } i = 1, 2, \ldots, k-1 \text{ and } j = 1, 2, \ldots, n_{i+1}, \tag{7}$$

(ii) $d_1, d_2, \ldots, d_\ell$ satisfy the condition in (A2) for $1 \leq \ell \leq M$. Therefore, $\mathbf{d}_M \in \mathcal{A}_M$ and hence $\mathcal{G}_{M,k} \subseteq \mathcal{A}_M$.

(iii) We have

$$B(\mathbf{d}_j; 1) = j, \text{ for } j = 1, 2, \ldots, s_1, \tag{8}$$

$$B(\mathbf{d}_{s_i+j}; i+1) = d_{s_i+j} + d_{s_1} + d_{s_2} + \cdots + d_{s_i}, \quad \text{for } i = 1, 2, \ldots, k-1, \ j = 1, 2, \ldots, n_{i+1}. \tag{9}$$

In particular, we have

$$B(\mathbf{d}_{s_i}; i) = d_{s_1} + d_{s_2} + \cdots + d_{s_i}, \text{ for } i = 1, 2, \ldots, k. \tag{10}$$

To prove Theorem 4, we need the following three lemmas, whose proofs are given in Appendix B, Appendix C, and Appendix D.

Lemma 5 *Suppose that $d_1, d_2, \ldots, d_m$ satisfy the condition in (A2) for some $1 \leq m \leq M$ and suppose that $1 \leq i \leq k$.*

(i) If $B(\mathbf{d}_m; i) < d_{\ell'+1}$ for some $1 \leq \ell' \leq m-1$ (note that as $d_1 = 1$ and $i \geq 1$, we have $B(\mathbf{d}_m; i) \geq 1 = d_1$ and hence ℓ' cannot be 0), then $B(\mathbf{d}_m; i) + 1 < d_{\ell'+1}$ and $B(\mathbf{d}_m; i) = B(\mathbf{d}_{m-1}; i) = \cdots = B(\mathbf{d}_{\ell'}; i)$.

(ii) Let $\ell' = \max\{1 \leq \ell \leq m : \ d_\ell \leq B(\mathbf{d}_m; i)\}$ (note that ℓ' is well defined as $B(\mathbf{d}_m; i) \geq 1 = d_1$). Then we have $B(\mathbf{d}_m; i) = B(\mathbf{d}_{m-1}; i) = \cdots = B(\mathbf{d}_{\ell'}; i) = d_{\ell'} + B(\mathbf{d}_{\ell'-1}; i-1)$.

We remark that the definition that $\ell' = \max\{1 \leq \ell \leq m : \ d_\ell \leq B(\mathbf{d}_m; i)\}$ is essential for the relation $B(\mathbf{d}_m; i) = d_{\ell'} + B(\mathbf{d}_{\ell'-1}; i-1)$ in Lemma 5(ii) to hold. This is because $B(\mathbf{d}_m; i) \geq d_{\ell'}$ does not always guarantee that $B(\mathbf{d}_m; i) = d_{\ell'} + B(\mathbf{d}_{\ell'-1}; i-1)$ unless $\ell' = \max\{1 \leq \ell \leq m : \ d_\ell \leq B(\mathbf{d}_m; i)\}$. We illustrate this by an example. If $\mathbf{d}_4 = (1, 2, 4, 8)$, then we can see that $B(\mathbf{d}_4; 2) = 6 \geq d_2$ and $B(\mathbf{d}_1; 1) = 1$, but $B(\mathbf{d}_4; 2) \neq d_2 + B(\mathbf{d}_1; 1)$. However, we have $d_3 \leq B(\mathbf{d}_4; 2) < d_4$ and $B(\mathbf{d}_2; 1) = 2$, and hence $B(\mathbf{d}_4; 2) = d_3 + B(\mathbf{d}_2; 1)$.

Lemma 6 *Suppose that $1 \leq k \leq M-1$. Let $n_1, n_2, \ldots, n_k$ be positive integers such that $n_1 \geq 2$ and $\sum_{i=1}^{k} n_i = M$, and let $s_0 = 0$ and $s_i = \sum_{\ell=1}^{i} n_\ell$ for $i = 1, 2, \ldots, k$. Let $d_1, d_2, \ldots, d_{s_i+j}$ be given by (3) for some $1 \leq i \leq k-1$ and $0 \leq j \leq n_{i+1}$. If $d_1, d_2, \ldots, d_{s_i+j}$ satisfy the condition in (A2), then we have $B(\mathbf{d}_{s_i+j}; i+1) = d_{s_i+j} + B(\mathbf{d}_{s_i+j-1}; i)$.*

Lemma 7 *Suppose that $1 \leq k \leq M-1$. Let $n_1, n_2, \ldots, n_k$ be positive integers such that $n_1 \geq 2$ and $\sum_{i=1}^{k} n_i = M$, and let $s_0 = 0$ and $s_i = \sum_{\ell=1}^{i} n_\ell$ for $i = 1, 2, \ldots, k$. Let $d_1, d_2, \ldots, d_{s_i+j}$ be given by (6) and (7) for some $1 \leq i \leq k-1$ and $0 \leq j \leq n_{i+1}$.*

(i) $d_1, d_2, \ldots, d_{s_i+j}$ satisfy the condition in (A2).

(ii) If $B(\mathbf{d}_{s_i}; i) = d_{s_1} + d_{s_2} + \cdots + d_{s_i}$, then we have $B(\mathbf{d}_{s_i+j}; i) = B(\mathbf{d}_{s_i}; i) = d_{s_1} + d_{s_2} + \cdots + d_{s_i}$.

Proof. (Proof of Theorem 4) From (3), we see that

$$\begin{aligned}
d_1 &= d_{s_0+1} = B(\mathbf{d}_{s_0}; 1) + 1 = B(\mathbf{d}_0; 1) + 1 \\
&= 0 + 1 = 1, \\
d_2 &= d_{s_0+2} = B(\mathbf{d}_{s_0+1}; 1) + 1 = B(\mathbf{d}_1; 1) + 1 \\
&= B((1); 1) + 1 = 1 + 1 = 2, \\
d_3 &= d_{s_0+3} = B(\mathbf{d}_{s_0+2}; 1) + 1 = B(\mathbf{d}_2; 1) + 1 \\
&= B((1, 2); 1) + 1 = 2 + 1 = 3, \\
&\vdots \\
d_{s_1} &= d_{s_0+n_1} = B(\mathbf{d}_{s_0+n_1-1}; 1) + 1 = B(\mathbf{d}_{n_1-1}; 1) + 1 \\
&= B((1, 2, \ldots, n_1 - 1); 1) + 1 = (n_1 - 1) + 1 = s_1.
\end{aligned}$$

Thus, we have proved (6). As it is easy to see from $\mathbf{d}_j = (1, 2, \ldots, j)$ that $B(\mathbf{d}_j; 1) = B((1, 2, \ldots, j); 1) = j$ for $j = 1, 2, \ldots, s_1$, we have proved (8). Furthermore, it is also clear that $d_1, d_2, \ldots, d_j$ satisfy the condition in (A2) for $j = 1, 2, \ldots, s_1$.

In the following, we show by induction that (7) and (9) hold and $d_1, d_2, \ldots, d_{s_i+j}$ satisfy the condition in (A2) for all $1 \leq i \leq k-1$ and $1 \leq j \leq n_{i+1}$. From (3), Lemma 6 (with $i = 1$ and $j = 0$ in Lemma 6), (8), and (6), we have

$$\begin{aligned}
d_{s_1+1} &= B(\mathbf{d}_{s_1}; 2) + 1 = d_{s_1} + B(\mathbf{d}_{s_1-1}; 1) + 1 \\
&= d_{s_1} + (s_1 - 1) + 1 = d_{s_1} + d_{s_1} = 2d_{s_1}.
\end{aligned}$$

Thus, (7) holds for $i = 1$ and $j = 1$. As such, we have from Lemma 7(i) that $d_1, d_2, \ldots, d_{s_1+1}$ satisfy the condition in (A2). It then follows from Lemma 6 (with $i = 1$ and $j = 1$ in Lemma 6), (8), and (6) that

$$\begin{aligned}
B(\mathbf{d}_{s_1+1}; 2) &= d_{s_1+1} + B(\mathbf{d}_{s_1}; 1) = d_{s_1+1} + s_1 \\
&= d_{s_1+1} + d_{s_1}.
\end{aligned}$$

Thus, (9) holds for $i = 1$ and $j = 1$.

Assume as the induction hypothesis that (7) and (9) hold and $d_1, d_2, \ldots, d_{s_i+j}$ satisfy the condition in (A2) up to some $1 \le i \le k-1$ and $1 \le j \le n_{i+1}$ such that $s_i + j < M$. We need to consider the following two cases.

Case 1: $1 \le j \le n_{i+1}-1$. In this case, we have $2 \le j+1 \le n_{i+1}$. It follows from (3) and the induction hypothesis that

$$\begin{aligned} d_{s_i+j+1} &= B(\mathbf{d}_{s_i+j}; i+1) + 1 \\ &= d_{s_i+j} + d_{s_1} + d_{s_2} + \cdots + d_{s_i} + 1 \\ &= 2d_{s_i} + (j-1)(d_{s_1} + d_{s_2} + \cdots + d_{s_i} + 1) \\ &\quad + d_{s_1} + d_{s_2} + \cdots + d_{s_i} + 1 \\ &= 2d_{s_i} + j(d_{s_1} + d_{s_2} + \cdots + d_{s_i} + 1). \end{aligned}$$

Thus, (7) holds for i and $j+1$. As such, we have from Lemma 7(i) that $d_1, d_2, \ldots, d_{s_i+j+1}$ satisfy the condition in (A2). Since it is easy to see from the induction hypothesis that $B(\mathbf{d}_{s_i}; i) = d_{s_1} + d_{s_2} + \cdots + d_{s_i}$, we see from Lemma 7(ii) that $B(\mathbf{d}_{s_i+j}; i) = d_{s_1} + d_{s_2} + \cdots + d_{s_i}$. It then follows from Lemma 6 that

$$\begin{aligned} B(\mathbf{d}_{s_i+j+1}; i+1) &= d_{s_i+j+1} + B(\mathbf{d}_{s_i+j}; i) \\ &= d_{s_i+j+1} + d_{s_1} + d_{s_2} + \cdots + d_{s_i}. \end{aligned}$$

Thus, (9) holds for i and $j+1$.

Case 2: $j = n_{i+1}$. In this case, we have $1 \le i \le k-2$ (as $s_i + j = s_{i+1} < M$ in the induction hypothesis) and $s_i + j + 1 = s_{i+1} + 1$. It follows from (3) and Lemma 6 that

$$\begin{aligned} d_{s_{i+1}+1} &= B(\mathbf{d}_{s_{i+1}}; i+2) + 1 \\ &= d_{s_{i+1}} + B(\mathbf{d}_{s_{i+1}-1}; i+1) + 1. \end{aligned} \tag{11}$$

If $n_{i+1} = 1$, then $s_{i+1} - 1 = s_i$, and it follows from Lemma 6, (3), and $d_{s_{i+1}} = d_{s_i+1} = 2d_{s_i}$ in the induction hypothesis that

$$\begin{aligned} B(\mathbf{d}_{s_{i+1}-1}; i+1) + 1 &= B(\mathbf{d}_{s_i}; i+1) + 1 \\ &= d_{s_i} + B(\mathbf{d}_{s_i-1}; i) + 1 \\ &= d_{s_i} + d_{s_i} = d_{s_{i+1}}. \end{aligned} \tag{12}$$

On the other hand, if $n_{i+1} \ge 2$, then it follows from the induction hypothesis that

$$\begin{aligned} & B(\mathbf{d}_{s_{i+1}-1}; i+1) + 1 \\ &= B(\mathbf{d}_{s_i+n_{i+1}-1}; i+1) + 1 \\ &= d_{s_i+n_{i+1}-1} + d_{s_1} + d_{s_2} + \cdots + d_{s_i} + 1 \\ &= 2d_{s_i} + (n_{i+1}-2)(d_{s_1} + d_{s_2} + \cdots + d_{s_i} + 1) \\ &\quad + d_{s_1} + d_{s_2} + \cdots + d_{s_i} + 1 \\ &= 2d_{s_i} + (n_{i+1}-1)(d_{s_1} + d_{s_2} + \cdots + d_{s_i} + 1) \\ &= d_{s_i+n_{i+1}} = d_{s_{i+1}}. \end{aligned} \tag{13}$$

Therefore, we have from (11)–(13) that

$$d_{s_{i+1}+1} = d_{s_{i+1}} + B(\mathbf{d}_{s_{i+1}-1}; i+1) + 1 = 2d_{s_{i+1}}.$$

Thus, (7) holds for $i+1$ and 1. As such, we have from Lemma 7(i) that $d_1, d_2, \ldots, d_{s_{i+1}+1}$ satisfy the condition in (A2). It then follows from Lemma 6 and $B(\mathbf{d}_{s_{i+1}}; i+1) = d_{s_1} + d_{s_2} + \cdots + d_{s_i} + d_{s_{i+1}}$ in the induction hypothesis that

$$\begin{aligned} B(\mathbf{d}_{s_{i+1}+1}; i+2) &= d_{s_{i+1}+1} + B(\mathbf{d}_{s_{i+1}}; i+1) \\ &= d_{s_{i+1}+1} + d_{s_1} + d_{s_2} + \cdots + d_{s_{i+1}}. \end{aligned}$$

Thus, (9) holds for $i+1$ and 1. ■

IV. Every Optimal Construction Must Be a Greedy Construction

Recall that the the problem of finding positive integers $d_1, d_2, \ldots, d_M$ satisfying the condition in (A2) such that the feedback system in Figure 1(a) can be operated as a 2-to-1 FIFO multiplexer with the largest possible effective buffer size under the constraint that each packet can be routed through at most k of the M fibers by using the self-routing scheme described in Section II is equivalent to the problem of finding a sequence $\mathbf{d}_M$ in $\mathcal{A}_M$ such that the maximum representable integer $B(\mathbf{d}_M; k)$ is the large possible, where $\mathcal{A}_M$ is the set of all M-sequences $\mathbf{d}_M$ satisfying the condition in (A2). We call a construction of a sequence $\mathbf{d}_M^*$ in $\mathcal{A}_M$ an optimal construction if $\mathbf{d}_M^* \in \arg\max_{\mathbf{d}_M \in \mathcal{A}_M} B(\mathbf{d}_M; k)$.

In this section, we show that every optimal construction must be a greedy construction, i.e., if $\mathbf{d}_M^* \in \arg\max_{\mathbf{d}_M \in \mathcal{A}_M} B(\mathbf{d}_M; k)$, then $\mathbf{d}_M^* \in \mathcal{G}_{M,k}$, where $\mathcal{G}_{M,k}$ is the set of all M-sequences $\mathbf{d}_M$ generated by k-sequences of positive integers $\mathbf{n}_k$ such that $\sum_{i=1}^k n_i = M$ by using (3). Since the size of $\mathcal{A}_M$ is $\Omega(2^M)$ [12] and the size of $\mathcal{G}_{M,k}$ is $\binom{M-1}{k-1} = O(M^k)$, the complexity of searching for an optimal construction can be greatly reduced from exponential time to polynomial time by only considering the set $\mathcal{G}_{M,k}$ rather than performing an exhaustive search through the set $\mathcal{A}_M$.

Note that for $k = M$, it is easy to see that $\mathbf{d}_M^* = (1, 2, 2^2, \ldots, 2^{M-1})$ is the only optimal construction. As it is also the only sequence in $\mathcal{G}_{M,k}$ as shown in Section III, it then follows that the optimal construction is also the greedy construction. Therefore, in Theorem 8 below we only consider the nontrivial case that $1 \le k \le M-1$, and show that every optimal construction must be a greedy construction.

Theorem 8 *Suppose that $1 \le k \le M-1$. If $\mathbf{d}_M^* \in \arg\max_{\mathbf{d}_M \in \mathcal{A}_M} B(\mathbf{d}_M; k)$, then $\mathbf{d}_M^* \in \mathcal{G}_{M,k}$. In other words, every optimal construction must be a greedy construction.*

To prove Theorem 8, we need the following lemma on the basic properties of an optimal construction, whose proof is given in Appendix E.

Lemma 9 *Suppose that $1 \le k \le M-1$ and $\mathbf{d}_M^* \in \arg\max_{\mathbf{d}_M \in \mathcal{A}_M} B(\mathbf{d}_M; k)$. Let $s_k, s_{k-1}, \ldots, s_1$, in that order, be recursively given by*

$$s_k = \max\{1 \le \ell \le M : d_\ell^* \le B(\mathbf{d}_M^*; k)\}, \tag{14}$$

$$s_i = \max\{1 \le \ell \le s_{i+1} - 1 : d_\ell^* \le B(\mathbf{d}_{s_{i+1}-1}^*; i)\}, \quad \text{for } i = k-1, k-2, \ldots, 1. \tag{15}$$

(i) $s_k = M$ and $B(\mathbf{d}_{s_k}^*; k) = d_{s_k}^* + B(\mathbf{d}_{s_k-1}^*; k-1)$.
(ii) $s_i \ge i+1$ for $i = 1, 2, \ldots, k$.
(iii) $B(\mathbf{d}_{s_{i+1}-1}^*; i) = B(\mathbf{d}_{s_{i+1}-2}^*; i) = \cdots = B(\mathbf{d}_{s_i}^*; i) = d_{s_i}^* + B(\mathbf{d}_{s_i-1}^*; i-1)$ for $i = 1, 2, \ldots, k-1$. Therefore, $B(\mathbf{d}_{s_i}^*; i) = d_{s_1}^* + d_{s_2}^* + \cdots + d_{s_i}^*$ for $i = 1, 2, \ldots, k$.

Proof. (Proof of Theorem 8) Suppose $\mathbf{d}_M^* \in \arg\max_{\mathbf{d}_M \in \mathcal{A}_M} B(\mathbf{d}_M; k)$. Let s_i, $i = 1, 2, \ldots, k$ be given by (14) and (15). Let $s_0 = 0$ and $n_i = s_i - s_{i-1}$ for $i = 1, 2, \dots, k$, and let $d_1, d_2, \dots, d_M$ be generated by $n_1, n_2, \dots, n_k$ by using (3). Clearly, $\mathbf{d}_M \in \mathcal{G}_{M,k}$.

(i) We first show by induction on ℓ that $d^*_\ell \le d_\ell$ for all $\ell = 1, 2, \dots, M$, and if $d^*_{\ell'} < d_{\ell'}$ for some $2 \le \ell' \le M$ (note that $d^*_1 = d_1 = 1$), then $d^*_\ell < d_\ell$ for all $\ell = \ell', \ell'+1, \dots, M$. From the definition of $B(\mathbf{d}^*_{s_2-1}; 1)$ in (2), we can see that $B(\mathbf{d}^*_{s_2-1}; 1) = d^*_{\ell_1}$, where

$$\begin{aligned}\ell_1 = \max\{2 \le \ell \le s_2 - 1:\ d^*_2 - d^*_1 \le 1,\ d^*_3 - d^*_2 \le 1, \\ \dots, d^*_\ell - d^*_{\ell-1} \le 1\}\end{aligned} \tag{16}$$

(note that ℓ_1 is well defined as we have from Lemma 9(ii) that $s_2 - 1 \ge 2$ and we also have from $\mathbf{d}^*_M \in \mathcal{A}_M$ that $d^*_2 - d^*_1 \le 2d^*_1 - d^*_1 = d^*_1 = 1$). If $2 \le \ell_1 \le s_2 - 2$, then $d^*_{\ell_1+1} - d^*_{\ell_1} \ge 2$ and hence $d^*_{\ell_1+1} \ge d^*_{\ell_1} + 2 = B(\mathbf{d}^*_{s_2-1}; 1) + 2 > B(\mathbf{d}^*_{s_2-1}; 1)$. In this case, we see from (15) that $s_1 = \ell_1$, On the other hand, if $\ell_1 = s_2 - 1$, then we have $B(\mathbf{d}^*_{s_2-1}; 1) = d^*_{\ell_1} = d^*_{s_2-1}$. In this case, we see from (15) that $s_1 = s_2 - 1 = \ell_1$. As such, it follows from (16), (6), and $s_1 = \ell_1$ that

$$\begin{aligned}
d^*_1 &= 1 = d_1, \\
d^*_2 &\le d^*_1 + 1 = 1 + 1 = 2 = d_2, \\
d^*_3 &\le d^*_2 + 1 \le 2 + 1 = 3 = d_3, \\
&\vdots \\
d^*_{s_1} &\le d^*_{s_1-1} + 1 \le (s_1 - 1) + 1 = s_1 = d_{s_1}.
\end{aligned}$$

Furthermore, if $d^*_{\ell'} < d_{\ell'} = \ell'$ for some $2 \le \ell' \le s_1 - 1$, then we have from (16) and (6) that

$$\begin{aligned}
d^*_{\ell'+1} &\le d^*_{\ell'} + 1 < \ell' + 1 = d_{\ell'+1}, \\
&\vdots \\
d^*_{s_1} &\le d^*_{s_1-1} + 1 < (s_1 - 1) + 1 = s_1 = d_{s_1}.
\end{aligned}$$

Assume as the induction hypothesis that $d^*_1 \le d_1$, $d^*_2 \le d_2, \dots$, $d^*_\ell \le d_\ell$ for some $s_1 \le \ell \le M - 1$, and if $d^*_{\ell'} < d_{\ell'}$ for some $2 \le \ell' \le \ell$, then $d^*_{\ell'} < d_{\ell'}$, $d^*_{\ell'+1} < d_{\ell'+1}, \dots$, $d^*_\ell < d_\ell$. We need to consider the following two cases.

Case 1: $\ell = s_i$, where $1 \le i \le k - 1$. In this case, we have from ${\mathbf{d}^*}_1^M \in \mathcal{A}_M$, the induction hypothesis, and (7) that $d^*_{\ell+1} \le 2d^*_\ell \le 2d_\ell = 2d_{s_i} = d_{s_i+1} = d_{\ell+1}$. Furthermore, if $d^*_{\ell'} < d_{\ell'}$ for some $2 \le \ell' \le \ell$, then we have from the induction hypothesis that $d^*_{\ell'} < d_{\ell'}$, $d^*_{\ell'+1} < d_{\ell'+1}, \dots$, $d^*_\ell < d_\ell$, and it follows from ${\mathbf{d}^*}_1^M \in \mathcal{A}_M$ and (7) that $d^*_{\ell+1} \le 2d^*_\ell < 2d_\ell = 2d_{s_i} = d_{s_i+1} = d_{\ell+1}$.

Case 2: $\ell = s_i + j$, where $1 \le i \le k - 1$ and $1 \le j \le n_{i+1} - 1$. We first show that

$$d^*_{s_i+j+1} \le d^*_{s_i+j} + d^*_{s_1} + d^*_{s_2} + \dots + d^*_{s_i} + 1. \tag{17}$$

It is clear from ${\mathbf{d}^*}_1^M \in \mathcal{A}_M$ that $d^*_{s_i+j+1} \ge d^*_{s_i+j}$. If $d^*_{s_i+j+1} = d^*_{s_i+j}$, then (17) holds trivially. So in the following we assume that $d^*_{s_i+j+1} > d^*_{s_i+j}$. Since $s_i + j + 1 \le s_i + n_{i+1} = s_{i+1}$ in this case, we have from ${\mathbf{d}^*}_1^M \in \mathcal{A}_M$ that $d^*_{s_i+j+1} \le d^*_{s_{i+1}}$, and it then follows from Lemma 9(iii) that $d^*_{s_i+j+1} - 1 \le d^*_{s_{i+1}} - 1 < d^*_{s_1} + d^*_{s_2} + \dots + d^*_{s_{i+1}} = B(\mathbf{d}^*_{s_{i+1}}; i+1)$. By definition, the nonnegative integers $d^*_{s_i+j}, d^*_{s_i+j}+1, \dots, d^*_{s_i+j+1}-1$ are representable by using at most $i+1$ of the integers $d^*_1, d^*_2, \dots, d^*_{s_{i+1}}$ (as they are less than or equal to $B(\mathbf{d}^*_{s_{i+1}}; i+1)$) according to the $\mathcal{C}$-transform. As the integers $d^*_{s_i+j+1}, d^*_{s_i+j+2}, \dots, d^*_{s_{i+1}}$ are not used, but the integer $d^*_{s_i+j}$ is used, in the representations of these nonnegative integers (as they are less than $d^*_{s_i+j+1}, d^*_{s_i+j+2}, \dots, d^*_{s_{i+1}}$, but greater than or equal to $d^*_{s_i+j}$), it is clear the remaining values $0, 1, \dots, d^*_{s_i+j+1} - d^*_{s_i+j} - 1$ of these nonnegative integers are representable by using at most i of the integers $d^*_1, d^*_2, \dots, d^*_{s_i+j-1}$. Therefore, we have $d^*_{s_i+j+1} - d^*_{s_i+j} - 1 \le B(\mathbf{d}^*_{s_i+j-1}; i)$. By using Lemma 9(iii), we have $d^*_{s_i+j+1} \le d^*_{s_i+j} + B(\mathbf{d}^*_{s_i+j-1}; i) + 1 = d^*_{s_i+j} + d^*_{s_1} + d^*_{s_2} + \dots + d^*_{s_i} + 1$, and hence (17) is proved.

From (17), the induction hypothesis, and (7), we have

$$\begin{aligned}
d^*_{\ell+1} &= d^*_{s_i+j+1} \\
&\le d^*_{s_i+j} + d^*_{s_1} + d^*_{s_2} + \dots + d^*_{s_i} + 1 \\
&\le d_{s_i+j} + d_{s_1} + d_{s_2} + \dots + d_{s_i} + 1 \\
&= 2d_{s_i} + (j-1)(d_{s_1} + d_{s_2} + \dots + d_{s_i} + 1) \\
&\quad + d_{s_1} + d_{s_2} + \dots + d_{s_i} + 1 \\
&= 2d_{s_i} + j(d_{s_1} + d_{s_2} + \dots + d_{s_i} + 1) \\
&= d_{s_i+j+1} = d_{\ell+1}.
\end{aligned} \tag{18}$$

Furthermore, if $d^*_{\ell'} < d_{\ell'}$ for some $2 \le \ell' \le \ell$, then we have from the induction hypothesis that $d^*_{\ell'} < d_{\ell'}$, $d^*_{\ell'+1} < d_{\ell'+1}, \dots$, $d^*_\ell < d_\ell$. Therefore, the inequality in (18) becomes a strict inequality and we have $d^*_{\ell+1} < d_{\ell+1}$.

(ii) Now we show that $d^*_\ell = d_\ell$ for all $\ell = 1, 2, \dots, M$. From $\mathbf{d}_M \in \mathcal{A}_M$ in Theorem 4(ii), Theorem 4(iii), $d^*_\ell \le d_\ell$ for $\ell = 1, 2, \dots, M$, and Lemma 9(iii), we have

$$\begin{aligned}
B(\mathbf{d}^*_M; k) &= \max_{\mathbf{d}'_M \in \mathcal{A}_M} B(\mathbf{d}'_M; k) \\
&\ge B(\mathbf{d}_M; k) = B(\mathbf{d}_{s_k}; k) \\
&= d_{s_1} + d_{s_2} + \dots + d_{s_k} \\
&\ge d^*_{s_1} + d^*_{s_2} + \dots + d^*_{s_k} \\
&= B(\mathbf{d}^*_{s_k}; k) = B(\mathbf{d}^*_M; k).
\end{aligned} \tag{19}$$

As such, the two inequalities in (19) hold with equality, and it is easy to deduce from $d^*_{s_i} \le d_{s_i}$ for all $i = 1, 2, \dots, k$ that $d^*_{s_i} = d_{s_i}$ for all $i = 1, 2, \dots, k$.

We show by contradiction that $d^*_\ell = d_\ell$ for all $\ell = 1, 2, \dots, M$. Assume on the contrary that $d^*_{\ell'} < d_{\ell'}$ for some $2 \le \ell' \le M$. Then we see from (i) that $d^*_\ell < d_\ell$ for all $\ell = \ell', \ell'+1, \dots, M$. In particular, we have $d^*_{s_k} = d^*_M < d_M = d_{s_k}$, and a contradiction is reached. ■

V. Conclusion

In this paper, we considered an important problem arising from practical feasibility considerations in the SDL constructions of optical queues: the constructions of optical queues with a limited number of recirculations through the optical switches and the fiber delay lines. We first transformed the design of the fiber delays in the SDL constructions of certain types of optical queues into an equivalent integer representation problem. We then proposed a class of greedy constructions for such an equivalent integer representation problem, and showed that every optimal construction that achieves the largest possible maximum representable integer must be a greedy construction. Therefore, the complexity of searching for an optimal construction can be greatly reduced from exponential time to polynomial time by only considering the greedy constructions instead of performing an exhaustive search. The results in this paper can be applied to the constructions of optical 2-to-1 FIFO multiplexers with a limited number of recirculations. Similar results can be obtained for the constructions of optical linear compressors/decompressors with a limited number of recirculations and will be reported in a follow-up work by the first author and Dr. Xuan-Chao Huang, in which a simple algorithm is also proposed to obtain the optimal constructions.

APPENDIX A
PROOF OF THEOREM 3

We show by induction on ℓ that $d_\ell = d'_\ell$ for $\ell = 1, 2, \ldots, M$. From $n_1 = 1$ and $a = \min\{2 \le i \le k : \ n_i \ge 2\}$, we can see that $n_i = 1$ for $i = 1, 2, \ldots, a-1$ and $n_a \ge 2$. It then follows from $s_i = \sum_{\ell=1}^{i} n_\ell$ for $i = 1, 2, \ldots, k$ that $s_i = i$ for $i = 1, 2, \ldots, a-1$ and $s_a \ge a+1$. As such, we have from (4) that

$$\begin{aligned}
d_1 &= d_{s_0+1} = B(\mathbf{d}_{s_0}; 1) + 1 = B(\mathbf{d}_0; 1) + 1 \\
&= 0 + 1 = 1, \\
d_2 &= d_{s_1+1} = B(\mathbf{d}_{s_1}; 2) + 1 = B(\mathbf{d}_1; 2) + 1 \\
&= d_1 + 1 = 2, \\
d_3 &= d_{s_2+1} = B(\mathbf{d}_{s_2}; 3) + 1 = B(\mathbf{d}_2; 3) + 1 \\
&= d_1 + d_2 + 1 = 2^2, \\
&\vdots \\
d_a &= d_{s_{a-1}+1} = B(\mathbf{d}_{s_{a-1}}; a) + 1 = B(\mathbf{d}_{a-1}; a) + 1 \\
&= \sum_{\ell=1}^{a-1} d_\ell + 1 = 2^{a-1}, \\
d_{a+1} &= d_{s_{a-1}+2} = B(\mathbf{d}_{s_{a-1}+1}; a) + 1 = B(\mathbf{d}_a; a) + 1 \\
&= \sum_{\ell=1}^{a} d_\ell + 1 = 2^a.
\end{aligned}$$

Furthermore, from $n'_1 = n_1 + 1 = 2$, $n'_i = n_i = 1$ for $i = 2, 3, \ldots, a-1$, $n'_a = n_a - 1$, $n'_i = n_i$ for $i = a+1, a+2, \ldots, k$, and $s'_i = \sum_{\ell=1}^{i} n'_\ell$ for $i = 1, 2, \ldots, k$, we can see that $s'_i = i + 1 = s_i + 1$ for $i = 1, 2, \ldots, a-1$, and $s'_i = s_i$ for $i = a, a+1, \ldots, k$. As such, we have from (5) that

$$\begin{aligned}
d'_1 &= d'_{s'_0+1} = B(\mathbf{d'}_{s'_0}; 1) + 1 = B(\mathbf{d'}_0; 1) + 1 \\
&= 0 + 1 = 1, \\
d'_2 &= d'_{s'_0+2} = B(\mathbf{d'}_{s'_0+1}; 1) + 1 = B(\mathbf{d'}_1; 1) + 1 \\
&= d'_1 + 1 = 2, \\
d'_3 &= d'_{s'_1+1} = B(\mathbf{d'}_{s'_1}; 2) + 1 = B(\mathbf{d'}_2; 2) + 1 \\
&= \sum_{\ell=1}^{2} d'_\ell + 1 = 2^2, \\
&\vdots \\
d'_a &= d'_{s'_{a-2}+1} = B(\mathbf{d'}_{s'_{a-2}}; a-1) + 1 \\
&= B(\mathbf{d'}_{a-1}; a-1) + 1 = \sum_{\ell=1}^{a-1} d'_\ell + 1 = 2^{a-1}, \\
d'_{a+1} &= d'_{s'_{a-1}+1} = B(\mathbf{d'}_{s'_{a-1}}; a) + 1 = B(\mathbf{d'}_a; a) + 1 \\
&= \sum_{\ell=1}^{a} d'_\ell + 1 = 2^a.
\end{aligned}$$

Therefore, we have $d_1 = d'_1, d_2 = d'_2, \ldots, d_{s_{a-1}+2} = d'_{s_{a-1}+2}$ (note that $a + 1 = s_{a-1} + 2$).

Assume as the induction hypothesis that $d_1 = d'_1, d_2 = d'_2, \ldots, d_\ell = d'_\ell$ for some $s_{a-1} + 2 \le \ell \le M - 1$. To complete the induction, we have to show that $d_{\ell+1} = d'_{\ell+1}$. We need to consider the following two cases.

Case 1: $\ell = s_{a-1} + j$, where $2 \le j \le n_a - 1$. In this case, we have

$$\begin{aligned}
d_{\ell+1} &= d_{s_{a-1}+j+1} = B(\mathbf{d}_{s_{a-1}+j}; a) + 1 \\
&= B(\mathbf{d'}_{s_{a-1}+j}; a) + 1 = B(\mathbf{d'}_{s'_{a-1}+j-1}; a) + 1 \\
&= d'_{s'_{a-1}+j} = d'_{s_{a-1}+j+1} = d'_{\ell+1},
\end{aligned}$$

where the second equality follows from (4) and $3 \le j + 1 \le n_a$, the third equality follows from $\mathbf{d'}_{s_{a-1}+j} = \mathbf{d}_{s_{a-1}+j}$ in the induction hypothesis, and the fifth equality follows from (5) and $2 \le j \le n_a - 1 = n'_a$.

Case 2: $\ell = s_i + j$, where $a \le i \le k-1$ and $0 \le j \le n_{i+1} - 1$. In this case, we have

$$\begin{aligned}
d_{\ell+1} &= d_{s_i+j+1} = B(\mathbf{d}_{s_i+j}; i+1) + 1 \\
&= B(\mathbf{d'}_{s_i+j}; i+1) + 1 = B(\mathbf{d'}_{s'_i+j}; i+1) + 1 \\
&= d'_{s'_i+j+1} = d'_{s_i+j+1} = d'_{\ell+1},
\end{aligned}$$

where the second equality follows from (4) and $1 \le j + 1 \le n_{i+1}$, the third equality follows from $\mathbf{d'}_{s_i+j} = \mathbf{d}_{s_i+j}$ in the induction hypothesis, and the fifth equality follows from (5) and $1 \le j + 1 \le n_{i+1} = n'_{i+1}$.

APPENDIX B
PROOF OF LEMMA 5

Note that since $d_1, d_2, \ldots, d_m$ satisfy the condition in (A2), we have $1 = d_1 \le d_2 \le \cdots \le d_m$.

(i) Suppose $B(\mathbf{d}_m; i) < d_{\ell'+1}$ for some $1 \le \ell' \le m - 1$. We first show that $B(\mathbf{d}_m; i) + 1 < d_{\ell'+1}$. Assume on the contrary that $B(\mathbf{d}_m; i) + 1 = d_{\ell'+1}$. Then the nonnegative integer $B(\mathbf{d}_m; i) + 1$ is representable by using exactly one of the integers $d_1, d_2, \ldots, d_m$, namely, $d_{\ell''}$, where $\ell'' = \max\{\ell' + 1 \le \ell \le m : d_\ell = d_{\ell'+1}\}$, according to the $\mathcal{C}$-transform. However, by definition, the nonnegative integer $B(\mathbf{d}_m; i) + 1$ is not representable by using at most i of the integers $d_1, d_2, \ldots, d_m$, and we have reached a contradiction.

By definition, the nonnegative integers $0, 1, \ldots, B(\mathbf{d}_m; i)$ are representable by using at most i of the integers $d_1, d_2, \ldots, d_m$ (as they are less than or equal to $B(\mathbf{d}_m; i)$), but the nonnegative integers $B(\mathbf{d}_m; i) + 1$ is not representable by using at most i of the integers $d_1, d_2, \ldots, d_m$ according to the $\mathcal{C}$-transform. Since the integers $d_{\ell'+1}, d_{\ell'+2}, \ldots, d_m$ are not used in the representations of the nonnegative integers $0, 1, \ldots,\ B(\mathbf{d}_m; i), B(\mathbf{d}_m; i) + 1$ (as these nonnegative integers are less than $d_{\ell'+1}, d_{\ell'+2}, \ldots, d_m$), it follows that the nonnegative integers $0, 1, \dots, B(\mathbf{d}_m; i)$ are representable by using at most i of the integers $d_1, d_2, \dots, d_{m'}$, but the nonnegative integers $B(\mathbf{d}_m; i) + 1$ is not representable by using at most i of the integers $d_1, d_2, \dots, d_{m'}$ according to the $\mathcal{C}$-transform, where $\ell' \leq m' \leq m$. Therefore, we have $B(\mathbf{d}_m; i) = B(\mathbf{d}_{m-1}; i) = \cdots = B(\mathbf{d}_{\ell'}; i)$.

(ii) We consider the two cases $\ell' = m$ and $1 \leq \ell' \leq m-1$ separately.

Case 1: $\ell' = m$. In this case, we have $d_m \leq B(\mathbf{d}_m; i)$ and we need to show that $B(\mathbf{d}_m; i) = d_m + B(\mathbf{d}_{m-1}; i-1)$. If $m = 1$, then we have

$$\begin{aligned} B(\mathbf{d}_m; i) &= B(\mathbf{d}_1; i) = d_1 = d_1 + 0 = d_1 + B(\mathbf{d}_0; i-1) \\ &= d_m + B(\mathbf{d}_{m-1}; i-1). \end{aligned}$$

If $i = 1$, then we have $B(\mathbf{d}_m; 1) = B(\mathbf{d}_m; i) \geq d_m$. As it is easy to see from the definition of $B(\mathbf{d}_m; 1)$ in (2) that $B(\mathbf{d}_m; 1) \leq \max\{d_1, d_2, \dots, d_m\} = d_m$, it then follows that $B(\mathbf{d}_m; 1) = d_m$. As a result, we have

$$\begin{aligned} B(\mathbf{d}_m; i) &= B(\mathbf{d}_m; 1) = d_m = d_m + 0 = d_m + B(\mathbf{d}_{m-1}; 0) \\ &= d_m + B(\mathbf{d}_{m-1}; i-1). \end{aligned}$$

Therefore, we assume that $2 \leq m \leq M$ and $2 \leq i \leq k$ in the rest of the proof.

We first show that $B(\mathbf{d}_m; i) \leq d_m + B(\mathbf{d}_{m-1}; i-1)$. By definition, the nonnegative integers $d_m, d_m+1, \dots, B(\mathbf{d}_m; i)$ are representable by using at most i of the integers $d_1, d_2, \dots, d_m$ (as they are less than or equal to $B(\mathbf{d}_m; i)$) according to the $\mathcal{C}$-transform. Since the integer d_m is used in their representations (as they are all greater than or equal to d_m), it follows that the remaining values $0, 1, \dots, B(\mathbf{d}_m; i) - d_m$ of these nonnegative integers are representable by using at most $i-1$ of the integers $d_1, d_2, \dots, d_{m-1}$. Therefore, we have $B(\mathbf{d}_{m-1}; i-1) \geq B(\mathbf{d}_m; i) - d_m$.

Now we show that $B(\mathbf{d}_m; i) \geq d_m + B(\mathbf{d}_{m-1}; i-1)$. Since in this case we have $B(\mathbf{d}_m; i) \geq d_m$, it remains to show that the nonnegative integers $d_m+1, d_m+2, \dots, d_m + B(\mathbf{d}_{m-1}; i-1)$ are representable by using at most i of the integers $d_1, d_2, \dots, d_m$ according to the $\mathcal{C}$-transform. As the integer d_m is used in their representations (as they are all greater than d_m) and by definition the remaining values $1, 2, \dots, B(\mathbf{d}_{m-1}; i-1)$ of these nonnegative integers are representable by using at most $i-1$ of the integers $d_1, d_2, \dots, d_{m-1}$ (as they are less than or equal to $B(\mathbf{d}_{m-1}; i-1)$), the proof is completed.

Case 2: $1 \leq \ell' \leq m-1$. In this case, we have $d_{\ell'} \leq B(\mathbf{d}_m; i) < d_{\ell'+1}$. As $B(\mathbf{d}_m; i) < d_{\ell'+1}$, we have from (i) that $B(\mathbf{d}_m; i) = B(\mathbf{d}_{m-1}; i) = \cdots = B(\mathbf{d}_{\ell'}; i)$. Therefore, we have $d_{\ell'} \leq B(\mathbf{d}_m; i) = B(\mathbf{d}_{\ell'}; i)$, and it follows from Case 1 above that $B(\mathbf{d}_{\ell'}; i) = d_{\ell'} + B(\mathbf{d}_{\ell'-1}; i-1)$.

APPENDIX C
PROOF OF LEMMA 6

Suppose that $d_1, d_2, \dots, d_{s_i+j}$ satisfy the condition in (A2). We consider the two cases $j = 0$ and $1 \leq j \leq n_{i+1}$ separately.

Case 1: $j = 0$. From (3), we have $d_{s_i} = B(\mathbf{d}_{s_i-1}; i) + 1$. It follows that the nonnegative integers $0, 1, \dots, d_{s_i} - 1 = B(\mathbf{d}_{s_i-1}; i)$ are representable by using at most i of the integers $d_1, d_2, \dots, d_{s_i}$ according to the $\mathcal{C}$-transform because the integer d_{s_i} is not used in their representations (as they are less than d_{s_i}) and by definition they are representable by using at most i of the integers $d_1, d_2, \dots, d_{s_i-1}$ (as they are less than or equal to $B(\mathbf{d}_{s_i-1}; i)$). Furthermore, the nonnegative integer d_{s_i} is representable by using exactly one of the integers $d_1, d_2, \dots, d_{s_i}$, namely, d_{s_i} itself, according to the $\mathcal{C}$-transform. As a result, the nonnegative integers $0, 1, \dots, d_{s_i}$ are representable by using at most $i+1$ of the integers $d_1, d_2, \dots, d_{s_i}$, and hence we have $B(\mathbf{d}_{s_i}; i+1) \geq d_{s_i}$. By Lemma 5(ii) (with $\ell' = s_i$ in Lemma 5(ii)), we obtain

$$\begin{aligned} B(\mathbf{d}_{s_i+j}; i+1) &= B(\mathbf{d}_{s_i}; i+1) = d_{s_i} + B(\mathbf{d}_{s_i-1}; i) \\ &= d_{s_i+j} + B(\mathbf{d}_{s_i+j-1}; i). \end{aligned}$$

Case 2: $1 \leq j \leq n_{i+1}$. From (3), we have $d_{s_i+j} = B(\mathbf{d}_{s_i+j-1}; i+1) + 1$ (note that this equality does not reduce to $d_{s_i} = B(\mathbf{d}_{s_i-1}; i) + 1$ when $j = 0$ as in Case 1 above, and that is why we need to discuss the two cases $j = 0$ and $1 \leq j \leq n_{i+1}$ separately). By the same argument as in Case 1, we also have $B(\mathbf{d}_{s_i+j}; i+1) \geq d_{s_i+j}$, and it follows from Lemma 5(ii) (with $\ell' = s_i + j$ in Lemma 5(ii)) that $B(\mathbf{d}_{s_i+j}; i+1) = d_{s_i+j} + B(\mathbf{d}_{s_i+j-1}; i)$.

APPENDIX D
PROOF OF LEMMA 7

(i) From (6), we have $d_\ell = \ell$ for $\ell = 1, 2, \dots, s_1$, and it is easy to see that $d_1 = 1$ and $d_\ell \leq d_{\ell+1} \leq 2d_\ell$ for $\ell = 1, 2, \dots, s_1 - 1$. To show that $d_1, d_2, \dots, d_{s_i+j}$ satisfy the condition in (A2), we also need to show that $d_\ell \leq d_{\ell+1} \leq 2d_\ell$ for $\ell = s_1, s_1+1, \dots, s_i + j - 1$. We consider the following two cases.

Case 1: $\ell = s_{i'}$, where $1 \leq i' \leq i-1$, or $i' = i$ and $j \geq 1$. In this case, we have from (7) that $d_{\ell+1} = d_{s_{i'}+1} = 2d_{s_{i'}} = 2d_\ell$. Clearly, $d_\ell \leq d_{\ell+1} \leq 2d_\ell$.

Case 2: $\ell = s_{i'} + j'$, where $1 \leq i' \leq i-1$ and $1 \leq j' \leq n_{i'+1} - 1$, or $i' = i$ and $1 \leq j' \leq j-1$. In this case, we have $1 \leq i' \leq i-1$ and $1 \leq j' < j'+1 \leq n_{i'+1}$, or $i' = i$ and $1 \leq j' < j'+1 \leq j \leq n_{i+1}$, and it follows from (7) that

$$\begin{aligned} d_{\ell+1} - d_\ell &= d_{s_{i'}+j'+1} - d_{s_{i'}+j'} \\ &= d_{s_1} + d_{s_2} + \cdots + d_{s_{i'}} + 1 > 0 \end{aligned}$$

and

$$\begin{aligned} & d_{\ell+1} - 2d_\ell \\ &= d_{s_{i'}+j'+1} - 2d_{s_{i'}+j'} \\ &= -2d_{s_{i'}} - (j'-2)(d_{s_1} + d_{s_2} + \cdots + d_{s_{i'}} + 1). \end{aligned} \tag{20}$$

If $j' \geq 2$, then we have from (20) that $d_{\ell+1} - 2d_\ell = -2d_{s_{i'}} - (j'-2)(d_{s_1} + d_{s_2} + \cdots + d_{s_{i'}} + 1) \leq -2d_{s_{i'}} < 0$. On the other hand, if $j' = 1$, then let $a = \max\{1 \leq a' \leq i' : n_{a'} \geq 2\}$ (note that a is well defined as $n_1 \geq 2$) so that $n_{a+1} = n_{a+2} = \cdots = n_{i'} = 1$ and $n_a \geq 2$, and we have from (20), (7), and

(6) that

$$\begin{aligned}
&d_{\ell+1} - 2d_\ell \\
&= -2d_{s_{i'}} - (j'-2)(d_{s_1} + d_{s_2} + \cdots + d_{s_{i'}} + 1) \\
&= d_{s_1} + d_{s_2} + \cdots + d_{s_{i'-1}} + 1 - d_{s_{i'}} \\
&= -2d_{s_{i'-1}} - (n_{i'}-2)(d_{s_1} + d_{s_2} + \cdots + d_{s_{i'-1}} + 1) \\
&= d_{s_1} + d_{s_2} + \cdots + d_{s_{i'-2}} + 1 - d_{s_{i'-1}} \\
&\vdots \\
&= -2d_{s_a} - (n_{a+1}-2)(d_{s_1} + d_{s_2} + \cdots + d_{s_a} + 1) \\
&= d_{s_1} + d_{s_2} + \cdots + d_{s_{a-1}} + 1 - d_{s_a} \\
&= \begin{cases} 1 - d_{s_1} = 1 - s_1 = 1 - n_1 < 0, \text{ if } a = 1, \\ -2d_{s_{a-1}} - (n_a - 2)(d_{s_1} + d_{s_2} + \cdots + d_{s_{a-1}} + 1) \\ \qquad\qquad \leq -2d_{s_{a-1}} < 0, \text{ if } a \geq 2. \end{cases} \qquad (21)
\end{aligned}$$

Therefore, we also have $d_\ell \leq d_{\ell+1} \leq 2d_\ell$ in this case.

(ii) Suppose that$B(\mathbf{d}_{s_i}; i) = d_{s_1} + d_{s_2} + \cdots + d_{s_i}$. If $j = 0$, then there is nothing to prove. So we assume that $1 \leq j \leq n_{i+1}$ in the rest of the proof.

From (7), we have

$$\begin{aligned}
B(\mathbf{d}_{s_i}; i) + 1 - d_{s_i+1} &= d_{s_1} + d_{s_2} + \cdots + d_{s_i} + 1 - 2d_{s_i} \\
&= d_{s_1} + d_{s_2} + \cdots + d_{s_{i-1}} + 1 - d_{s_i}.
\end{aligned}$$

By the same argument that leads to (21), we can see that $B(\mathbf{d}_{s_i}; i) + 1 - d_{s_i+1} < 0$. It then follows from (i) that $B(\mathbf{d}_{s_i}; i) + 1 < d_{s_i+1} \leq d_{s_i+2} \leq \cdots \leq d_{s_i+j}$. Thus, the nonnegative integers $0, 1, \ldots, B(\mathbf{d}_{s_i}; i)$ are representable by using at most i of the integers $d_1, d_2, \ldots, d_{s_i+j}$ according to the $\mathcal{C}$-transform because the integers $d_{s_i+1}, d_{s_i+2}, \ldots, d_{s_i+j}$ are not used in their representations (as they are less than $d_{s_i+1}, d_{s_i+2}, \ldots, d_{s_i+j}$) and by definition they are representable by using at most i of the integers $d_1, d_2, \ldots, d_{s_i}$ (as they are less than or equal to $B(\mathbf{d}_{s_i}; i)$). However, the nonnegative integer $B(\mathbf{d}_{s_i}; i)+1$ is not representable by using at most i of the integers $d_1, d_2, \ldots, d_{s_i+j}$ according to the $\mathcal{C}$-transform because the integers $d_{s_i+1}, d_{s_i+2}, \ldots, d_{s_i+j}$ are not used in its representation (as it is less than $d_{s_i+1}, d_{s_i+2}, \ldots, d_{s_i+j}$) and by definition it is not representable by using at most i of the integers $d_1, d_2, \ldots, d_{s_i}$. This shows that $B(\mathbf{d}_{s_i+j}; i) = B(\mathbf{d}_{s_i}; i)$ and the proof is completed.

Appendix E
Proof of Lemma 9

Note that since $\mathbf{d}^*_M \in \arg\max_{\mathbf{d}_M \in \mathcal{A}_M} B(\mathbf{d}_M; k)$, we have $\mathbf{d}^*_M \in \mathcal{A}_M$.

(i) We will show that $B(\mathbf{d}^*_M; k) \geq d^*_M$. If this can be done, then we see from (14) that s_k is well defined and $s_k = M$, and it follows from Lemma 5(ii) (note that $\mathbf{d}^*_M \in \mathcal{A}_M$) that $B(\mathbf{d}^*_{s_k}; k) = B(\mathbf{d}^*_M; k) = d^*_M + B(\mathbf{d}^*_{M-1}; k-1) = d^*_{s_k} + B(\mathbf{d}^*_{s_k-1}; k-1)$, and the proof is completed.

Assume on the contrary that $B(\mathbf{d}^*_M; k) < d^*_M$. As $\mathbf{d}^*_M \in \mathcal{A}_M$, we have $d^*_1 = 1$ and hence $B(\mathbf{d}^*_M; k) \geq 1 = d^*_1$. Let $\ell' = \max\{1 \leq \ell \leq M : d^*_\ell \leq B(\mathbf{d}^*_M; k)\}$. Then ℓ' is well defined, and we have $1 \leq \ell' \leq M-1$ and $d^*_{\ell'} \leq B(\mathbf{d}^*_M; k) < d^*_{\ell'+1}$. Let $d'_\ell = d^*_\ell$ for $\ell = 1, 2, \ldots, \ell'$, and $d'_\ell = B(\mathbf{d}^*_M; k)+\ell-\ell'$ for $\ell = \ell'+1, \ell'+2, \ldots, M$. In the following, we show that $\mathbf{d}'_M \in \mathcal{A}_M$ and $B(\mathbf{d}'_M; k) > B(\mathbf{d}^*_M; k)$. Then we have from $\mathbf{d}'_M \in \mathcal{A}_M$ that $B(\mathbf{d}'_M; k) \leq \max_{\mathbf{d}_M \in \mathcal{A}_M} B(\mathbf{d}_M; k) = B(\mathbf{d}^*_M; k)$, and a contradiction is reached.

Since $\mathbf{d}^*_M \in \mathcal{A}_M$, we have $d^*_1 = 1$ and $d^*_\ell \leq d^*_{\ell+1} \leq 2d^*_\ell$ for $\ell = 1, 2, \ldots, M-1$. Thus, it follows from $d'_\ell = d^*_\ell$ for $\ell = 1, 2, \ldots, \ell'$ that $d'_1 = 1$ and $d'_\ell \leq d'_{\ell+1} \leq 2d'_\ell$ for $\ell = 1, 2, \ldots, \ell'-1$. Furthermore, from $d^*_{\ell'} \leq B(\mathbf{d}^*_M; k) < d^*_{\ell'+1}$, we can see that $d'_{\ell'+1} = B(\mathbf{d}^*_M; k) + 1 \geq d^*_{\ell'} + 1 > d^*_{\ell'} = d'_{\ell'}$ and $d'_{\ell'+1} = B(\mathbf{d}^*_M; k) + 1 \leq d^*_{\ell'+1} \leq 2d^*_{\ell'} = 2d'_{\ell'}$. Finally, it is clear from $d'_\ell = B(\mathbf{d}^*_M; k) + \ell - \ell'$ for $\ell = \ell'+1, \ell'+2, \ldots, M$ that $d'_{\ell+1} = d'_\ell + 1$ for $\ell = \ell'+1, \ell'+2, \ldots, M-1$, and hence $d'_\ell \leq d'_{\ell+1} \leq 2d'_\ell$ for $\ell = \ell'+1, \ell'+2, \ldots, M-1$. Therefore, we have proved $\mathbf{d}'_M \in \mathcal{A}_M$.

From Lemma 5(ii) and $d'_\ell = d^*_\ell$ for $\ell = 1, 2, \ldots, \ell'$, we have $B(\mathbf{d}^*_M; k) = B(\mathbf{d}^*_{\ell'}; k) = B(\mathbf{d}'_{\ell'}; k)$. It follows that the nonnegative integers $0, 1, 2, \ldots, B(\mathbf{d}^*_M; k)$ are representable by using at most k of the integers $d'_1, d'_2, \ldots, d'_M$ according to the $\mathcal{C}$-transform because the integers $d'_{\ell'+1}, d'_{\ell'+2}, \ldots, d'_M$ are not used in their representations (as $B(\mathbf{d}^*_M; k) < d'_{\ell'+1} < d'_{\ell'+2} < \cdots < d'_M$) and by definition they are representable by using at most k of the integers $d'_1, d'_2, \ldots, d'_{\ell'}$ (as they are less than or equal to $B(\mathbf{d}^*_M; k) = B(\mathbf{d}'_{\ell'}; k)$). Furthermore, the nonnegative integer $B(\mathbf{d}^*_M; k)+1 = d'_{\ell'+1}$ (resp., $B(\mathbf{d}^*_M; k)+2 = d'_{\ell'+2}, \ldots, B(\mathbf{d}^*_M; k)+M-\ell' = d'_M$) is representable by using exactly one of the integers $d'_1, d'_2, \ldots, d'_M$, namely, $d'_{\ell'+1}$ (resp., $d'_{\ell'+2}, \ldots, d'_M$) itself, according to the $\mathcal{C}$-transform. Therefore, we have $B(\mathbf{d}'_M; k) \geq B(\mathbf{d}^*_M; k) + M - \ell' > B(\mathbf{d}^*_M; k)$.

(ii) We show by induction on i that s_i is well defined and $s_i \geq i+1$ for $i = k, k-1, \ldots, 1$.

From (i), we know that s_k is well defined and $s_k = M \geq k+1$. Assume as the induction hypothesis that $s_k, s_{k-1}, \ldots, s_{i+1}$ are well defined and $s_k \geq k+1, s_{k-1} \geq k, \ldots, s_{i+1} \geq i+2$ for some $1 \leq i \leq k-1$. Since ${\mathbf{d}^*}^M_1 \in \mathcal{A}_M$, we see from Lemma 5(ii) that s_i is well defined. To complete the induction, we show that $s_i \geq i+1$ by contradiction. So assume on the contrary that $s_i \leq i$. As $s_i < i+1 \leq s_{i+1}-1$, we see from the definition of s_i in (15) that $d^*_{i+1} > B(\mathbf{d}^*_{s_{i+1}-1}; i)$, and it then follows from Lemma 5(i) that

$$d^*_{i+1} > B(\mathbf{d}^*_{s_{i+1}-1}; i) + 1 \qquad (22)$$

and

$$\begin{aligned}
B(\mathbf{d}^*_{s_{i+1}-1}; i) &= B(\mathbf{d}^*_{s_{i+1}-2}; i) = \cdots = B(\mathbf{d}^*_{i+1}; i) \\
&= B(\mathbf{d}^*_i; i) = \sum_{\ell=1}^{i} d^*_\ell. \qquad (23)
\end{aligned}$$

From $\mathbf{d}^*_M \in \mathcal{A}_M$ and (23), we obtain $d^*_{i+1} \leq 2d^*_i \leq d^*_i + 2d^*_{i-1} \leq \cdots \leq d^*_i + d^*_{i-1} + \cdots + d^*_2 + 2d^*_1 = \sum_{\ell=1}^{i} d^*_\ell + 1 = B(\mathbf{d}^*_{s_{i+1}-1}; i) + 1$, which contradicts to $d^*_{i+1} > B(\mathbf{d}^*_{s_{i+1}-1}; i) + 1$ in (22).

(iii) From the definition of s_i in (15) and Lemma 5(ii) (note that $\mathbf{d}^*_M \in \mathcal{A}_M$), we have $B(\mathbf{d}^*_{s_{i+1}-1}; i) = B(\mathbf{d}^*_{s_{i+1}-2}; i) = \cdots = B(\mathbf{d}^*_{s_i}; i) = d^*_{s_i} + B(\mathbf{d}_{s_i-1^*}; i-1)$ for $i = 1, 2, \ldots, k-1$. Together with $B(\mathbf{d}^*_{s_k}; k) = d^*_{s_k} + B(\mathbf{d}^*_{s_k-1}; k-1)$ in (i), it is easy to see that $B(\mathbf{d}^*_{s_i}; i) = d^*_{s_1} + d^*_{s_2} + \cdots + d^*_{s_i}$ for $i = 1, 2, \ldots, k$.

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